THEORY OF AFFINE SHELLS: SECOND ORDER ESTIMATES OF THE STRAIN AND STRESS TENSORS TREATED BY P.D.E. METHODS

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Abstract: This theory has been developed by different authors based, from the geometrical point of view, on the classical theory of surfaces in three-dimensional space, particularly with respect to the invariants of the Euclidean group, ASO $(3,\mathbb{R})$. In this article we present an alternative foundation of the theory, invariant under the action of the Unimodular Affine group, ASL $(3,\mathbb{R})$. Here, we use firstly the integrability conditions, from the affine geometry of surfaces, in order to settle bidimensional compatibility conditions for each case of an affine shell.
Secondly, we establish equations of equilibrium for a solid shell, in the affine sense, after reducing three-dimensional equations to the corresponding bidimensional ones of the middle surface. Thirdly, appear the basic inequalities of the theory and estimates for the strain and stress tensors, as well as for their second order covariant derivatives within the framework of the Theory of Affine Shells.

Key Words: *Affine Shells, Compatibility Conditions, Stress-Strain Relations, Basic Inequalities, Estimates.*

1. Introduction

The Theory of Shells is a topic of Mathematics with a rich history and many, diverse applications to the real world: Engineering, Industry, Avionics, and so on. The usual viewpoint of presentation makes use of classical, Euclidean geometry of surfaces in three-dimensional space, particularly with regards to the invariants of the Euclidean group, ASO $(3,\mathbb{R})$, i.e., the group of transformations generated by translations and rotations of the space [7, 8, 9, 10, 12]. Hence, for example, what it is called "normal" is the Euclidean one, and the "distance" is the measure with respect to the norm induced by the usual scalar product of vectors (positive definite), which is the main, fundamental invariant in Euclidean geometry.

We have been working recently on an alternative foundation and development of the theory of shells which is invariant under the action of the unimodular affine group, ASL $(3,\mathbb{R})$. Thus, for the case in treatment, this gives rise to the so called affine geometry of surfaces. For a given surface in the three-dimensional space we use, within this context, concepts such as "affine normal" and "affine distance", corresponding to the above mentioned ones in Euclidean geometry. See [4, 5, and 6] for full details.

We introduce, in Section 2 of the present article, an abbreviated version of the concept of Affine Shell, already developed in previous articles [4, 5, 6]. The treatment of Compatibility Conditions occupies Section 3, while the Basic Inequalities of the Theory are treated in Section 4. The further development of the Theory consists in the presentation of the Strain-Stress Relations in Affine Shells which is taken care of in Section 5. Finally, we come to conclude this exposition by treating the Estimates for the L_2-Norms of Second Order Derivatives, in Section 6.

2. Affine Shells [4, 5, 6]

We consider the middle surface of a (solid) shell in its original (undeformed) state, denoted by M_0, parametrized locally by a vector function $X_0 : U \to \mathbb{R}^3$, where $U \subset \mathbb{R}^2$, which is assumed to be enough smooth. Coordinates in the domain are denoted by (u^1, u^2). Thus, we can write locally $M_0 = X_0(u^1, u^2)$ and assume besides, as it is usually done, that X_0 is a topological immersion (embedding). Particles in the original state have curvilinear Lagrange coordinates (U^1, U^2, U^3) that for our present purposes shall be chosen in a special way: $(U^1, U^2, U^3) = (u^1, u^2, u)$ if we represent them by equation $X(u^1, u^2, u) = X_t(u^1, u^2) = X_0(u^1, u^2) + u\,\vec{\mathbf{n}}$, where we have obviously extended the previous function to $X : U \times (-h, h) \to \mathbb{R}^3$, and $\vec{\mathbf{n}}$ is the vector field normal to the middle surface. This normal can be the Euclidean normal, N_{eu}, of the classical, Euclidean Theory of Surfaces, or the Unimodular Affine normal, N_{ua}, of our own, current development. In each case, we shall clarify the situation when we deal with one or the other.

In the Euclidean case we shall use the following notations regarding the main geometrical objects, defined on the middle surface prior to deformation, that take part in the formulation of the theory [7, 8, 11]:

$$I_{eu} = \sum_{\alpha,\beta} a_{\alpha\beta} du^\alpha du^\beta \quad \text{with} \quad a_{\alpha\beta} = \frac{\partial X_0}{\partial u^\alpha} \cdot \frac{\partial X_0}{\partial u^\beta}$$

denotes the Euclidean first fundamental form, while with the expression

$$II_{eu} = \sum_{\alpha,\beta} L_{\alpha\beta} du^\alpha du^\beta \quad \text{where} \quad L_{\alpha\beta} = N_{eu} \cdot \frac{\partial^2 X_0}{\partial u^\beta \partial u^\alpha},$$

we represent the second fundamental form, and with

$$III_{eu} = \sum_{\alpha,\beta} M_{\alpha\beta} du^\alpha du^\beta, \quad \text{where} \quad M_{\alpha\beta} = \sum_{\lambda} L_{\alpha\lambda} L^\lambda_\beta = \sum_{\gamma\lambda} a^{\gamma\lambda} L_{\alpha\lambda} L_{\beta\gamma},$$

the Euclidean third fundamental form.

In the state previous to deformation the border of the shell is made up of two "faces", which are surfaces parallel to the middle surface M_0 at respective distance h, measured along the Euclidean normal N_{eu}, and of the "border" constituted by segments normal to the faces. Therefore, along the normal to M_0 coordinates U^1, U^2 remain constant while $U^3 := u$ measures the signed distance from M_0. Faces can be represented, then, by equations $U^3 = u = \pm h$ while the middle surface is given by $U^3 = u = 0$.

Now if $a_{\alpha\beta}$, $L_{\alpha\beta}$, $M_{\alpha\beta}$, are respectively the coefficients of the first, second and third Euclidean fundamental forms of the middle surface M_0, the Euclidean structure of the ambient space induces a Riemannian structure on the shell and we can obtain, by means of a straightforward computation, the following expressions in normal coordinates $(U^1, U^2, U^3) = (U^1, U^2, u)$:

$$A_{\alpha\beta} = \frac{\partial X}{\partial u^\alpha} \cdot \frac{\partial X}{\partial u^\beta} = a_{\alpha\beta} - 2u\,L_{\alpha\beta} + u^2 M_{\alpha\beta},$$

$$A_{\alpha 3} = A_{3\alpha} = \frac{\partial X}{\partial u^\alpha} \cdot \frac{\partial X}{\partial t} = \frac{\partial X}{\partial u^\alpha} \cdot N_{eu} = 0,$$

$$A_{33} = \frac{\partial X}{\partial t} \cdot \frac{\partial X}{\partial t} = N_{eu} \cdot N_{eu} = 1.$$

Corresponding to the shell, and its middle surface, in the state previous to deformation, we can consider the geometrical objects belonging to the shell in the deformed state that we shall denote with an upper right asterisk. Thus, for example, $X_0^*: U \to \mathbb{R}^3$, where $U \subset \mathbb{R}^2$, represents the parametrization of the deformed middle surface $M_0^* = X_0^*(u^1, u^2)$, and we remark that the domain of definition of this immersion, $U \subset \mathbb{R}^2$, and the parameters (u^1, u^2) used in it, are the same as those belonging to the middle surface of the shell in the original state, previous to deformation.

Consequently, the rest of geometrical objects change from one state to the other and the problem is to determine the nature and extension of such changes for every one of them reducing, under appropriate hypotheses, the obtainable information to both middle surfaces. One such hypothesis is the one concerning the comparison of the thickness parameter h, which it is usually assumed to be small with respect to the other dimensions of the shell. This introduces in the theory the concept of "thin" shell which has important uses and applications.

Considering now the Unimodular Affine Geometry of Surfaces, we need to assume defined, in the ambient space $\mathbb{R}^3$ an exterior 3-form, or non-trivial determinant function, denoted by the symbol $[\ ,\ ,\ ] = \det$. Then, given the same previous mean surface, we represent the objects of that geometry by the following expressions:

In order to construct the Unimodular first fundamental form we define, firstly

$$h_{\alpha\beta} = \left[\frac{\partial X_0}{\partial u^1}, \frac{\partial X_0}{\partial u^2}, \frac{\partial^2 X_0}{\partial u^\alpha \partial u^\beta} \right],$$

then, if we assume that the surface is non-degenerate, i.e., $H = \det(h_{\alpha\beta}) \neq 0$, we can write $g_{\alpha\beta} = |H|^{-1/4} h_{\alpha\beta}$, obtaining the Unimodular Affine First Fundamental Form expressed by equation

$$I_{ua} = \sum_{\alpha,\beta} g_{\alpha\beta} du^\alpha du^\beta$$

that turns out to be a semi-Riemannian structure, [1, 2, 3, 13]. The Unimodular Affine Normal is defined now by the expression

$$N_{ua} = \tfrac{1}{2} \Delta(X_0)$$

where Δ is the Laplacian operator with respect to the pseudometric I_{ua}, i.e.:

$$\Delta X_0 = \frac{1}{\sqrt{|g|}} \sum_{\alpha=1}^{2} \frac{\partial}{\partial u^\alpha} \left(\sqrt{|g|} \sum_{\beta=1}^{2} g^{\alpha\beta} \frac{\partial X_0}{\partial u^\beta} \right) \quad \text{with} \quad g = \det\left(g_{\alpha\beta}\right).$$

From the above we obtain three connections:

1) The Levi-Civita connection with respect to the Euclidean metric I_{eu}, that we shall label here as ∇_{eu} and which coincides with the projection over M_0 of the usual, flat connection D of $\mathbb{R}^3$ in the direction of the classical Euclidean normal N_{eu}.

2) The Levi-Civita connection with respect to pseudometric I_{ua}: $\tilde{\nabla}$.

3) The ***affine normal induced*** connection: ∇, i.e., the projection of D in the direction of N_{ua}:

$$\nabla_{X_p} Y = proy_{N_{ua}}\left(D_{X_p} Y\right).$$

We define next the Unimodular Affine Second Fundamental Form [1, 2, 3]: $\nabla(I_{ua}) = II_{ua}$ that we also represent in local coordinates by:

$$II_{ua} = \sum_{\alpha\beta\gamma} g_{\alpha\beta\gamma} du^{\alpha} du^{\beta} du^{\gamma} ,$$

with the coefficients $g_{\alpha\beta\gamma}$ totally symmetric in their indices. Some authors prefer to refer to the latter as the ***Cubic Form*** [13].

Finally, we consider the ***Affine Third Fundamental Form*** that we can describe in the following way: similar to the Euclidean case regarding the Weingarten equation, it turns out too in affine geometry of surfaces that the local derivatives of the affine normal belong to the tangent plane of the surface at each point, i.e., we can write

$$\frac{\partial N_{ua}}{\partial u^{\alpha}} = -\sum_{\beta} B_{\alpha}^{\beta} \frac{\partial X_0}{\partial u^{\beta}} = -B_{\alpha}^{1} \frac{\partial X_0}{\partial u^{1}} - B_{\alpha}^{2} \frac{\partial X_0}{\partial u^{2}} ,$$

and define the ***Affine Third Fundamental Form*** by the expression:

$$III_{ua} = B_{\alpha\beta} du^{\alpha} du^{\beta} \quad \text{with} \quad B_{\alpha\beta} = \sum_{\gamma} g_{\alpha\gamma} B_{\beta}^{\gamma} .$$

As we have previously seen, the definition of shell as a three-dimensional body and, in particular, the Riemannian structure induced on that object by the ambient space metric is generated in a natural fashion. In the present case of Unimodular Affine Geometry that extension is not at all that immediate. However, as we shall see, it can also be realized in a canonical way. We start from the affine invariant pseudometric I_{ua}, defined on the middle surface M_0:

$$g_{\alpha\beta} = I_{\text{ua}}\left(\frac{\partial X_0}{\partial u^{\alpha}}, \frac{\partial X_0}{\partial u^{\beta}}\right).$$

In the present context we define on the shell a pseudo-metric, which is a Unimodular Affine invariant, to be denoted by

$$G = \sum G_{ij} du^i du^j ,$$

i.e.,

$$G_{ij} := G\left(\frac{\partial X}{\partial u^i}, \frac{\partial X}{\partial u^j}\right).$$

Observe that, since bilinearity must be preserved, we have to write in affine normal coordinates of the shell

$$
\begin{aligned}
G_{\alpha\beta} &= G\left(\frac{\partial X}{\partial u^{\alpha}}, \frac{\partial X}{\partial u^{\beta}}\right) \\
&= G\left(\frac{\partial X_0}{\partial u^{\alpha}} + u\,\frac{\partial N_{\mathrm{ua}}}{\partial u^{\alpha}}, \frac{\partial X_0}{\partial u^{\beta}} + u\,\frac{\partial N_{\mathrm{ua}}}{\partial u^{\beta}}\right)
\end{aligned}.
$$

when, by definition

$$
G_{\alpha\beta} := g_{\alpha\beta} - 2uB_{\alpha\beta} + u^2 \sum_{\lambda} B_{\alpha}^{\lambda} B_{\beta\lambda} \ .
$$

and where, as stated previously, Greek indices run from 1 to 2. Thus, in order to extend that definition to the third index, we also write:

$$
G_{3\alpha} = G_{\alpha 3} = G\left(\frac{\partial X}{\partial u^{\alpha}}, \frac{\partial X}{\partial u^{3}}\right) = G\left(X_{\alpha}, N_{ua}\right) := 0
$$

and, finally,

$$
G_{33} = G\left(\frac{\partial X}{\partial u^{3}}, \frac{\partial X}{\partial u^{3}}\right) = G\left(N_{ua}, N_{ua}\right) := 1 .
$$

It is easy to see that, for $u = u^3$ enough small, it holds:

$$
\det\left(G_{ij}\right) \neq 0
$$

and, consequently, the latter is a pseudo-Riemannian, Unimodular affine invariant metric defined on the shell, as it was our purpose to construct.

3. COMPATIBILITY CONDITIONS [4]

One of the main aspects in the theory of shells is the determination of compatibility conditions. These are conditions obtained on the behavior of the various difference tensors that can be defined by comparing the two states of the shell. The natural tool here is represented by the integrability conditions that must be satisfied, in all cases, by both middle surfaces. These conditions are very well known in the case of Euclidean shells, see, for example [7, 8, 9, 10, 12], and can be described, in our notation, as follows:

For the tensor with components defined by $\varepsilon_{\alpha\beta} = \frac{1}{2}\left(\overset{*}{a}_{\alpha\beta} - a_{\alpha\beta}\right)$, it is proven that

$$
\varepsilon_{\beta,\delta}^{\beta,\delta} - \varepsilon_{\beta,\delta}^{\delta,\beta} = L_{\beta}^{*\delta} L_{\delta}^{*\beta} - L_{\delta}^{*\delta} L_{\beta}^{*\beta} + g_{\mu}^{*\beta}\left(L_{\delta}^{\delta} L_{\beta}^{\mu} - L_{\beta}^{\delta} L_{\delta}^{\mu}\right) - \overset{*}{g}_{\mu\nu} g^{\beta\alpha} g^{\delta\gamma} (C_{\alpha\beta}^{\mu} C_{\gamma\delta}^{\nu} - C_{\alpha\delta}^{\mu} C_{\beta\gamma}^{\nu})
$$

while for the difference tensor $w_{\alpha\beta} := \overset{*}{L}_{\alpha\beta} - L_{\alpha\beta}$ it holds

$$
w_{\beta,\gamma}^{\alpha} - w_{\gamma,\beta}^{\alpha} = g^{\alpha\rho} (w_{\rho\beta,\gamma} - w_{\rho\gamma,\beta}) = g^{\alpha\rho} (C_{\rho\gamma}^{\mu} \overset{*}{L}_{\mu\beta} - C_{\rho\beta}^{\mu} \overset{*}{L}_{\mu\gamma}) \, .
$$

In both equations the symbol $C^{\mu}_{\rho\beta}$ represent the components of the difference tensor between the Levi-Civita connection of $\overset{*}{M}_0$ and that of M_0.

Now, for the case of affine shells the corresponding compatibility conditions were obtained in our previous article [4], and can be summarized as follows.

For the difference tensors defined by the various expressions that establish comparisons between the first, second and third fundamental forms, i.e.,

$$\varepsilon_{\alpha\beta} = \tfrac{1}{2}\left(\overset{*}{g}_{\alpha\beta} - g_{\alpha\beta}\right), \qquad \sigma_{\alpha\beta\gamma} := \overset{*}{g}_{\alpha\beta\gamma} - g_{\alpha\beta\gamma}\ , \qquad w_{\alpha\beta} := \overset{*}{B}_{\alpha\beta} - B_{\alpha\beta}\ ,$$

and the tensor defined by comparison between the corresponding Levi-Civita connections, represented by equation: $\tilde{\Gamma}^{*\mu}_{\alpha\beta} = C^{\mu}_{\alpha\beta} + \tilde{\Gamma}^{\mu}_{\alpha\beta}$ there hold the following conditions:

1) ***Affine Gauss condition***

$$\begin{aligned}\varepsilon^{\beta,\delta}_{\beta,\delta} - \varepsilon^{\delta,\beta}_{\beta,\delta} = \tfrac{1}{2}\left(B^{*\beta}_{\beta} g^{*\delta}_{\delta} - B^{*\beta}_{\delta} g^{*\delta}_{\beta} + B^{*\delta}_{\delta} g^{*\beta}_{\beta} - B^{*\delta}_{\beta} g^{*\beta}_{\delta}\right) - \tfrac{1}{2} g^{\beta\alpha} g^{\delta\gamma}\left(A^{*\eta}_{\gamma\beta}\cdot \overset{*}{g}_{\alpha\eta\delta} - A^{*\eta}_{\gamma\delta}\cdot \overset{*}{g}_{\alpha\eta\beta}\right) - \\ + \tfrac{1}{2} g^{*\beta}_{\mu}\left(-B^{\delta}_{\delta}\delta^{\mu}_{\beta} + A^{\delta\eta}_{\beta}\cdot A^{\mu}_{\eta\delta}\right) - \overset{*}{g}_{\lambda\mu} g^{\beta\alpha} g^{\delta\gamma}\left(C^{\lambda}_{\alpha\beta} C^{\mu}_{\gamma\delta} - C^{\lambda}_{\alpha\delta} C^{\mu}_{\beta\gamma}\right)\end{aligned}$$

2) ***Affine Mainardi-Codazzi condition***

$$\begin{aligned}\sigma_{\alpha\beta\gamma,\delta} - \sigma_{\alpha\beta\delta,\gamma} = {} & \overset{*}{g}_{\mu\beta\gamma} C^{\mu}_{\alpha\delta} + \overset{*}{g}_{\mu\alpha\gamma} C^{\mu}_{\beta\delta} - \overset{*}{g}_{\mu\beta\delta} C^{\mu}_{\alpha\gamma} - \overset{*}{g}_{\mu\alpha\delta} C^{\mu}_{\beta\gamma} + \\ & + \overset{*}{B}_{\alpha\delta}\overset{*}{g}_{\beta\gamma} + \overset{*}{B}_{\beta\delta}\overset{*}{g}_{\alpha\gamma} - \overset{*}{B}_{\alpha\gamma}\overset{*}{g}_{\beta\delta} - \overset{*}{B}_{\beta\gamma}\overset{*}{g}_{\alpha\delta} - \\ & - B_{\alpha\delta} g_{\beta\gamma} - B_{\beta\delta} g_{\alpha\gamma} + B_{\alpha\gamma} g_{\beta\delta} + B_{\beta\gamma} g_{\alpha\delta}\end{aligned}$$

3) ***Codazzi condition for the affine shape operators***

$$w^{\alpha}_{\beta,\alpha} - w^{\alpha}_{\alpha,\beta} = g^{\alpha\rho}\left[\overset{*}{B}_{\beta\mu}(C^{\mu}_{\rho\alpha} + A^{*\mu}_{\rho\alpha}) - \overset{*}{B}_{\alpha\mu}(C^{\mu}_{\rho\lambda} + A^{*\mu}_{\rho\beta}) + B_{\alpha\mu} A^{\mu}_{\rho\beta}\right]$$

4. BASIC INEQUALITIES FOR AFFINE SHELLS

The following basic inequalities, involving the geometrical objects treated before, were previously obtained in [6]. When represented in the form of Monge's, i.e., as a graph, the middle surface of the shell M_0 has all of its geometrical properties related to a given function f assumed to be enough differentiable and, in the present context of affine geometry, satisfying a partial differential equation of Monge-Ampère type:

$$\det\left(\partial_{\alpha\beta} f\right) = \pm F$$

and for such a kind of equations, with boundary conditions as in the present case, there hold bounds for the function f and its derivatives. Also, since the function F is strictly positive in the complete domain where f is defined, there exist lower an upper bounds for F as well.

As a consequence, we can also assume that the second derivatives of f are bounded, i.e., the components of the Hessian matrix $\left(\partial_{\alpha\beta} f\right)$, the components of the inverse matrix of the latter, denoted by $\left(f^{\alpha\beta}\right)$, and the components of the pseudometric tensor, covariant as well as contravariant, i.e., $g_{\alpha\beta}$ and $g^{\alpha\beta}$. These facts being expressed in the following inequalities:

$$\left|\partial_{\alpha\beta} f\right| < K \ , \ \left|f^{\alpha\beta}\right| < K \ , \ \left|g_{\alpha\beta}\right| < K \ , \ \left|g^{\alpha\beta}\right| < K \ .$$

Besides, since the higher order derivatives are also bounded, and in order to unify notation, we shall assume that there exists a ***generalized affine upper bound of curvature***, intimately related to the upper bound for the affine principal curvatures of the middle surface M_0, that we shall also denote by R, and that for the present, affine case, remains specified by the conditions that:

$$\left|\partial_{\alpha\beta\gamma} f\right| < \frac{1}{R^{1/2}}$$

and for the successive derivatives,

$$\left|\partial_{\alpha\beta\gamma\eta} f\right| \leq R^{-1}$$

$$\left|\partial_{\alpha\beta\gamma\eta\lambda} f\right| \leq \left(R^{-1/2}\right)^3 \ ; \ . \ . \ .$$

be satisfied for as high order of derivatives as needed in the development of the theory.

By using these hypotheses one obtains the corresponding bounds for the components of the tensor representing the third fundamental form:

$$B_{\alpha\beta} = -\frac{1}{4}\left(\partial_{\alpha\beta}\left(\log F\right) + \frac{1}{4}\partial_{\alpha}\left(\log F\right)\partial_{\beta}\left(\log F\right) - \sum_{\sigma,\lambda} f^{\sigma\lambda}\partial_{\alpha\beta\sigma} f \partial_{\lambda}\left(\log F\right)\right),$$

if we have in mind, besides, the two following, well-known identities:

$$\partial_{\alpha} \log F = \sum_{\rho,\sigma} f^{\rho\sigma}\partial_{\alpha\rho\sigma} f \ ,$$

$$\partial_{\alpha\beta} \log F = \sum_{\rho,\sigma} f^{\rho\sigma}\partial_{\alpha\beta\rho\sigma} f - \sum_{\rho,\sigma} f^{\rho\theta} f^{\tau\sigma}\partial_{\theta\tau\alpha} f \partial_{\rho\sigma\beta} f \ ,$$

with which it turns out that:

$$\left|B_{\alpha\beta}\right| \leq \frac{1}{4}\frac{1}{R}\left(4K + 32K^2\right) = \left(K + 8K^2\right)\frac{1}{R} .$$

We compute next the partial derivatives of these components

$$\partial_\gamma B_{\alpha\beta}=-\frac{1}{4}\begin{pmatrix}\partial_\gamma f^{\rho\tau}\partial_{\alpha\beta\rho\tau}f+f^{\rho\tau}\partial_{\alpha\beta\rho\tau\gamma}f-(\partial_\gamma f^{\rho\eta}\partial_{\eta\mu\alpha}f)(f^{\mu\tau}\partial_{\beta\rho\tau}f)-\\ -(f^{\rho\eta}\partial_{\eta\mu\alpha\gamma}f)(f^{\mu\tau}\partial_{\beta\rho\tau}f)-(f^{\rho\eta}\partial_{\eta\mu\alpha}f)(\partial_\gamma f^{\mu\tau}\partial_{\beta\rho\tau}f)-\\ -(f^{\rho\eta}\partial_{\eta\mu\alpha}f)(f^{\mu\tau}\partial_{\beta\rho\tau\gamma}f)\end{pmatrix}+$$
$$+\frac{1}{16}\begin{pmatrix}(\partial_\gamma f^{\rho\tau}\partial_{\tau\rho\alpha}f)(f^{\sigma\lambda}\partial_{\beta\sigma\lambda}f)+(f^{\rho\tau}\partial_{\tau\rho\alpha\gamma}f)(f^{\sigma\lambda}\partial_{\beta\sigma\lambda}f)+\\ (f^{\rho\tau}\partial_{\tau\rho\alpha}f)(\partial_\gamma f^{\sigma\lambda}\partial_{\beta\sigma\lambda}f)+(f^{\rho\tau}\partial_{\tau\rho\alpha}f)(f^{\sigma\lambda}\partial_{\beta\sigma\lambda\lambda}f)\end{pmatrix}-$$
$$-\frac{1}{4}\begin{pmatrix}(\partial_\gamma f^{\sigma\lambda}\partial_{\alpha\beta\sigma}f)(f^{\rho\tau}\partial_{\lambda\rho\tau}f)+(f^{\sigma\lambda}\partial_{\alpha\beta\sigma\gamma}f)(f^{\rho\tau}\partial_{\lambda\rho\tau}f)+\\ +(f^{\sigma\lambda}\partial_{\alpha\beta\sigma}f)(\partial_\gamma f^{\rho\tau}\partial_{\lambda\rho\tau}f)+(f^{\sigma\lambda}\partial_{\alpha\beta\sigma}f)(f^{\rho\tau}\partial_{\lambda\rho\tau\gamma}f)\end{pmatrix}$$

Then, by using the identity

$$\sum_\lambda f^{\lambda\sigma}\partial_{\lambda\mu}f=\delta_\mu^\sigma,$$

from which it follows that

$$\partial_\alpha f^{\sigma\gamma}=-\sum_\lambda f^{\lambda\gamma}f^{\mu\sigma}\partial_{\alpha\lambda\mu}f,$$

we find by direct computation the following estimate

$$\left|\partial\gamma B_{\alpha\beta}\right|\leq\frac{1}{R^{3/2}}\left(K+19K^2+24K^3\right).$$

With the development done so far, we can also obtain estimates for the components of the pseudometric, i.e., the components of the pseudo-Riemannian tensor $G=\sum_{i,j}G_{ij}du^i du^j$ of the shell in the undeformed state, and its successive derivatives, partial as well as covariant. For example, from

$$G_{\alpha\beta}:=g_{\alpha\beta}-2uB_{\alpha\beta}+u^2\sum_\lambda B_\alpha^\lambda B_{\beta\lambda},$$

we obtain, firstly, that

$$G_{\alpha\beta}=g_{\alpha\beta}-2uB_{\alpha\beta}+u^2\sum_{\lambda,\mu}g^{\lambda\mu}B_{\alpha\mu}B_{\beta\lambda},$$

and, consequently

$$\left|G_{\alpha\beta}\right|\leq K+2K\left(1+8K\right)\frac{h}{R}+4K^2\left(1+8K\right)\left(\frac{h}{R}\right)^2.$$

5. STRAIN-STRESS RELATIONS IN AFFINE SHELLS

For the present case of affine shells, the contravariant components of the stress tensor, t^{mk} , are connected with the components of the strain tensor, ε_{mk} , by means of the stress-strain relations

$$t^{mk} := \sqrt{\frac{G}{G^*}} \frac{\partial W}{\partial \varepsilon_{mk}}$$

defined in a similar fashion as to the Euclidean case, introduced by F. John, where is the strain energy density of the given material.

The same expression, in terms of the corresponding (1,1)-tensors is

$$t_i^m = \sum_k G_{ik} t^{mk} = \sqrt{\frac{G}{G^*}} \frac{\partial W}{\partial \varepsilon_m^i} .$$

Next, we introduce the components of the ***"pseudo-stress tensor"*** defined by

$$T_j^m := \sqrt{\frac{G^*}{G}}\ t_j^m - \delta_j^m W$$

and we may also write

$$\begin{aligned} T_i^m = &\left(W_{s_1} - W\right)\delta_i^m + \\ &+\left(2W_{s_1} + 2W_{s_2}\right)\varepsilon_i^m + \\ &+\left(4W_{s_2} + 3W_{s_3}\right)\sum_k \varepsilon_i^k \varepsilon_k^m + \\ &+\ 6\ W_{s_3} \sum_{s,k} \varepsilon_i^s \varepsilon_s^k \varepsilon_k^m \end{aligned}$$

where

$$s_1 = \sum_i \varepsilon_i^i \qquad s_2 = \sum_{i,j} \varepsilon_j^i \varepsilon_i^j \qquad s_3 = \sum_{i,j,k} \varepsilon_j^i \varepsilon_k^j \varepsilon_i^k$$

The equations of equilibrium can be written

$$t^{ij}{}_{,,j} + c_{hj}^i\, t^{hj} + c_{hj}^h\, t^{ij} = 0$$

where

$$c_{jk}^i = \tfrac{1}{2} \overline{G}^{*ir} \left(G_{rj,,k}^* + G_{rk,,j}^* - G_{jk,,r}^* \right)$$

and where we also have, as a consequence, that

$$\sum_m T^m_{i;m} = \left(\sqrt{\frac{G^*}{G}}\right) \sum_{m,r,s} \begin{pmatrix} c^r_{mr} t^{ms} G^*_{si} - \\ -t^{rs} c^m_{rm} G^*_{si} - \\ -t^{mr} c^s_{rm} G^*_{si} + \\ +t^{ms} G^*_{si;m} - \\ -\frac{1}{2} t^{ms} G^*_{sm;i} \end{pmatrix} = 0$$

$$\sum_m T^m_{i;m} = \left(\sqrt{\frac{G^*}{G}}\right) \sum_{m,r,s} \left(c^r_{mr} t^{ms} G^*_{si} - t^{rs} c^m_{rm} G^*_{si} - t^{mr} c^s_{rm} G^*_{si} + t^{ms} G^*_{si;m} - \tfrac{1}{2} t^{ms} G^*_{sm;i} \right) = 0$$

Additional notations are needed in order to compare components of stress and strain tensors, even those belonging to different spaces of definition. For example, and very particularly, in order to compare components of type (0,2) tensors t^{ij}, with those with components of type (1,1) t^j_i. Thus, we follow in this respect the kind of notation previously introduced by Fritz John in [7, 8]. In particular the so-called "general form of an expression" like

$$F\left(p,q\right)\left(u+v+w\right)$$

representing a vector, in a suitable space, where u, v, w, p, q are vectors themselves. The notation indicates that each of the components of $F\left(p,q\right)\left(u+v+\omega\right)$ is a sum of a linear form in the components of u, a linear form in the components of v, and a linear form in the components of w. The coefficients of these linear forms are functions of the components of the vectors p and q defined and differentiable as often as needed for all sufficiently small "lengths" $|p|$ and $|q|$. The letter F stands for a different expression in every equation to be considered . Thus, for example, we can write, for the components of the stress and strain tensors, of type (1,1)

$$t = \left(t^i_k\right) \text{ and } \varepsilon = \left(\varepsilon^i_k\right)$$

and in terms of the Lamé coefficients λ, μ, the following equation

$$t^m_i = \lambda \sum_j \varepsilon^j_j \delta^m_i + 2\mu\varepsilon^m_i + F\left(\varepsilon\right)\varepsilon^2$$

since such coefficients are defined by the relation

$$W := \frac{\lambda}{2}\left(s_1\right)^2 + \mu s_2 + F\left(\varepsilon\right)\varepsilon^3 .$$

where, as before

$$s_1 = \sum_i \varepsilon^i_i\ ,\ \ s_2 = \sum_{i,j} \varepsilon^i_j \varepsilon^j_i\ ,\ \ s_3 = \sum_{i,j,k} \varepsilon^i_j \varepsilon^j_k \varepsilon^k_i\ ,$$

and where we observe that the first two terms, on the right-hand side, are quadratic in terms of the strain tensor (operator) $\varepsilon = \left(\varepsilon_k^i\right)$, while the third term involves all of those components of order higher than two, representing otherwise the "remainder", of paramount importance when coming to the corresponding numerical estimates.

From now on we establish that in the same sense have to be interpreted all of the expressions to follow. Hence, by taking partial derivatives, we can write

$$W_{s_1} = \frac{\partial W}{\partial s_1} = \partial_{s_1} W = \lambda \; s_1 \quad , \quad W_{s_2} = \frac{\partial W}{\partial s_2} = \partial_{s_2} W = \mu \; .$$

From the latter we obtain, successively:

$$2W_{s_1} + 2W_{s_2} = 2\mu + 2\lambda \; s_1$$

$$4W_{s_2} + 3W_{s_3} = 2\mu + F\left(t\right)t^3 ,$$

$$W_{s_1} - W = \lambda \; s_1 - \frac{1}{2}\lambda\left(s_1\right)^2 - \frac{1}{2}\mu s_2 + F\left(\varepsilon\right)\varepsilon^3 .$$

Then, by using the Taylor´s series development

$$\left(1+x\right)^{-1/2} = 1 + \left(-\frac{1}{2}\right)x + \frac{3/4}{2}x^2 + \ldots ,$$

we can express

$$\sqrt{\frac{G}{G^*}} = \left(\frac{G}{G^*}\right)^{1/2} = \left(\frac{G^*}{G}\right)^{-1/2} = 1 - \frac{1}{2}s_1 + \frac{3}{8}\left(s_1\right)^2 + \ldots$$

and

$$t_i^m = \sqrt{\frac{G}{G^*}} \; \left(W_{s_1}\delta_i^m + 2W_{s_2}\varepsilon_i^m + 3W_{s_3}\sum_j \varepsilon_j^m \varepsilon_i^j\right)$$

becomes, first

$$t_i^m = \left(1 - \frac{1}{2}s_1 + \frac{3}{8}\left(s_1\right)^2 + \ldots\right)\left(W_{s_1}\delta_i^m + 2W_{s_2}\varepsilon_i^m + 3W_{s_3}\sum_j \varepsilon_j^m \varepsilon_i^j\right)$$

and, afterwards

$$t_i^m = W_{s_1}\delta_i^m + 2W_{s_2}\varepsilon_i^m + 3W_{s_3}\sum_j \varepsilon_j^m \varepsilon_i^j$$
$$= \frac{\lambda}{2}\, s_1\, \delta_i^m + 2\mu\, \varepsilon_i^m + 3W_{s_3}\sum_j \varepsilon_j^m \varepsilon_i^j$$
$$= \frac{\lambda}{2}\sum_j \varepsilon_j^j \delta_i^m + 2\mu\varepsilon_i^m + F(\varepsilon)\varepsilon^2$$

i.e.,

$$t_i^m = \frac{\lambda}{2}\sum_j \varepsilon_j^j \delta_i^m + 2\mu\varepsilon_i^m + F(\varepsilon)\varepsilon^2 .$$

From the latter, the trace of the stress tensor (operator) can be written

$$\sum_j t_j^j = \left(\frac{3}{2}\lambda + 2\mu\right)\sum_j \varepsilon_j^j + F(\varepsilon)\varepsilon^2$$

where $\sum_j \varepsilon_j^j$ itself represents the trace of the strain tensor (operator).

Hence, from the above we can also write

$$\varepsilon_i^m = \frac{1}{2\mu} t_i^m - \frac{\lambda}{2\mu} s_1 \delta_i^m + F(t)t^2$$

or, also,

$$\varepsilon_i^m = \frac{1}{2\mu} t_i^m - \frac{1-2\mu}{2\mu}\sum_j t_j^j\, \delta_i^m + F(t)t^2 .$$

Then, the expression for the components of the pseudo-stress tensor is

$$T_i^m = t_i^m + \frac{2}{\mu}\sum_i t_i^m t_i^s + \left(\frac{5\mu-2}{\mu}\right)\sum_j t_j^j\, t_i^s -$$
$$-\tfrac{1}{2}\left(\frac{1}{2\mu}\sum_{r,s} t_s^r t_r^s - \left(\frac{1-2\mu}{2\mu}\right)\sum_r t_r^r \sum_s t_s^s\right)\delta_i^m + F(t)t^3$$

We introduce next the ***"vector"*** $\eta = \left(\eta_k^i\right)$ by means of the relation:

$$G_{ik} = \delta_k^i + \eta_k^i\ .$$

This measures the difference between the metric matrix and that corresponding to the identity. Then, we obtain the following estimate for the components of the corresponding inverse matrix $\left(G^{ik}\right) := \left(G_{ik}\right)^{-1}$:

$$G^{ik} = \delta_k^i + \eta_k^i + F\left(\eta\right)\left(\eta^2\right) .$$

A straightforward computation shows that the Christoffel symbols satisfy the following estimate

$$\Gamma_{kr}^i = F\left(\eta\right)\left(\eta'\right).$$

Then, it also holds the following estimate

$$t_{ik} = t_k^i + F\left(\eta, t\right)\left(\eta\ t\right).$$

For the metric tensor in the deformed "strained" state we have, by definition,

$$G_{ik}^* = G_{ik} + 2\varepsilon_{ik} .$$

Hence, we can also estimate that

$$G_{ik}^* = G_{ik} + \frac{2}{\mu} t_{ik} - 2\left(\frac{1-2\mu}{2\mu}\right)\sum_j t_j^j \delta_k^i + F\left(t, \eta\right)\left(t^2 + \eta\, t\right).$$

For the tensor with components c_{kr}^i measuring the change in the Levi-Civita connections, from the "unstrained" natural state to the deformed "strained" state, we estimate

$$c_{kr}^i = F\left(\eta, t\right)\left(t' + \eta' t\right).$$

Then, we also obtain the two following estimates

$$\sum_m t_{im;m} = F\left(\eta, t\right)\left(tt' + \eta' t + \eta t'\right),$$

$$\sum_r t_{hk;rr} + 2\mu \sum_r t_{rr;hk} = F\left(\eta, t\right)\left(\eta\, t'' + \left(t'\right)^2 + t\eta' t' + \eta' t' + \eta'' t + \left(\eta'\right)^2 t + \left(\eta'\right)^2 t^2 + tt''\right)$$

since, in the present case, the three-dimensional compatibility equations are given in terms of the comparison between the Riemannian curvature tensors of the affine shell, when passing from the natural to the deformed state. Observe that, comparing with the expression obtained by F. John [7], such equation in Euclidean geometry, obtained from the corresponding compatibility condition, in that context, is equal to zero for both states of the shell, as expressed in equation (7) of the cited article, i.e.,

$$0 = R_{acdb}^* = \varepsilon_{ab;cd} + \varepsilon_{cd;ab} - \varepsilon_{ad;cb} - \varepsilon_{bc;ad} + \sum_{l,s} G_{ls}^* (c_{ab}^l c_{cd}^s - c_{ad}^l c_{bc}^s),$$

while in the present context of affine geometry we have

$$R^*_{acdb} = \varepsilon_{ab;cd} + \varepsilon_{cd;ab} - \varepsilon_{ad;cb} - \varepsilon_{bc;ad} - \frac{1}{2}\left(\sum_m G^*_{am} R^m_{cbd} + \sum_m G^*_{cm} R^m_{adb}\right) + \sum_{l,s} G^*_{ls}(c^l_{ab}c^s_{cd} - c^l_{ad}c^s_{bc})$$

equation obtained by direct application of Lemma 2 in our previous work [4].

Then, if we denote by $\varepsilon_{ab;cd}$ the second covariant derivatives with respect to the Levi-Civita connection associated to the pseudometric G, we further obtain from the latter equation

$$\varepsilon_{ab;cd} + \varepsilon_{cd;ab} - \varepsilon_{ad;cb} - \varepsilon_{bc;ad} = -\sum_{l,s} G^*_{ls}(c^l_{ab}c^s_{cd} - c^l_{ad}c^s_{bc}) + R^*_{acdb} - \frac{1}{2}\left(\sum_m G^*_{am} R^m_{cbd} + \sum_m G^*_{cm} R^m_{adb}\right),$$

and it is easy to get the following estimates for those tensors

$$R = F(\eta)\left(\eta'' + (\eta)^2 + (\eta')^2\right),$$

$$R^* = F(\eta, t)\left((t')^2 + (\eta')^2 + t'' + \eta'' t + \eta' t' + \eta t\right).$$

In what follows we shall also denote by ε an upper bound for the absolute values of the principal strains at all points of the shell. Let P_0 be a point on the undeformed middle surface M_0 and D the closest affine distance from P_0 to the lateral surface of the shell. Also, $2h$ represents the thickness of the shell and R is the typical length associated with the middle surface, all these quantities having been previously introduced above and in our article [6].

Then, we introduce the quantity

$$\theta = \max\left(\frac{h}{D}, \sqrt{\frac{h}{R}}, \sqrt{\varepsilon}\right)$$

and assume, besides, that the circumstances are such that

$$\theta < \theta_0$$

where θ_0 is a constant which depends only on the choice of the strain energy density W.

We assume that all of the calculations shall be done for a system of normal affine coordinates, as indicated previously, and also fully described in [4]. The middle surface of the shell is represented then in the form of Monge's, i.e. as a graph function, where the origin of coordinates is located precisely at $P_0 \in M_0$, with the axis of coordinates chosen to lie at the tangent plane to M_0 at that point, and with the third axis in the affine normal direction at the same point. The estimates to be computed for the partial derivatives of the function representing M_0, in the system chosen, shall be used immediately to make the corresponding estimates for the successive covariant derivatives $t_{ik;rs...}$, so that the latter estimates shall be independent of the system originally used.

Thus, we define

$$\lambda = \frac{\theta_0 h}{\theta} = \theta_0 \min\left(D, \sqrt{Rh}, \frac{h}{\sqrt{\varepsilon}} \right).$$

and obtain the inequalities

$$h < \lambda < \sqrt{Rh}\,,\ \ h < \frac{R}{4},\ \ \lambda < \frac{D}{2}$$

which are easily seen to be satisfied if we assume, for example, that

$$\theta_0 < \frac{1}{2}$$

and from this we obtain that

$$\frac{1}{R} \le (\theta_0)^2 \frac{h}{\lambda^2} \le \theta_0 \theta \frac{1}{\lambda} \le \frac{1}{2}\frac{\theta}{\lambda}, \qquad\qquad \varepsilon < (\theta_0)^2 \frac{h^2}{\lambda^2} < \frac{h^2}{4\lambda^2}.$$

It is to be further assumed next that θ_0 is chosen so small that for the given strain energy function W all of the above formulae are valid in the region defined by

$$M = \left\{ \left(u^1, u^2, u^3\right) : \sum \left(u^\alpha\right)^2 < \lambda^2, \left|u^3\right| < h \right\}.$$

Also, from now on we shall use the same symbols of approximation as described by F. John [6] , represented by $"O"$ and $"o"$, i.e., the first symbol is used in the conventional, classical way except that dependence on W is allowed. Thus, the relation

$$A = O(B)$$

where $B \ge 0$, means that for a given strain energy function W there exists a positive number K such that

$$|A| \le KB$$

The second one shall be used in an unconventional sense and only in combination with the first. The relation

$$A = O(B) + \mathrm{o}(C)$$

where $B \ge 0$ and $C \ge 0$, shall mean that for a given strain energy function W there exists a function $K(k)$, defined for all positive k such that

$$|A| \le K(k)B + kC$$

for all $k > 0$.

Besides, we shall assume that the strain energy function $W(s_1, s_2, s_3)$ is defined for all values of $|s_i|$ enough small and is as differentiable as needed. Here s_i are the traces of the successive powers of the strain operator. By definition, the "length" of such "strain operator", (1,1)-tensor with components ε_i^m, is $|\varepsilon| := \sqrt{\sum_{i,m} \varepsilon_i^m \varepsilon_i^m}$. For the metric tensor G sufficiently close to the unit matrix, i.e., for $|\eta|$ sufficiently small, we can estimate $|\varepsilon|$ in terms of the eigenvalues of the matrix (ε_i^m), i.e., in terms of the so-called principal strains.

Then, there exists a positive ε_0 only depending on the choice of the strain energy function W such that the strain-stress relations hold for $|\varepsilon| < \varepsilon_0$, and is also follows that, for such values, $t_i^m = O(|\varepsilon|)$.

Hence, for a given function W we can also find bounds t_0, η_0 such that for $|t| < t_0$ and $|\eta| < \eta_0$ all of the previously stated estimates are valid and, besides, $|\varepsilon| < \varepsilon_0$.

6. ESTIMATES FOR THE L_2-NORMS OF SECOND ORDER DERIVATIVES

In what follows, we shall use the following expression of the norm $\|w\|$ for any vector $w = w(u^1, u^2, u^3)$ defined in the working region M specified above

$$\|w\| = \sqrt{\iiint_M |w| du^1 du^2 du^3} .$$

The symbol w' shall denote the gradient of w, i.e., the vector whose components are the first derivatives of the components of w with respect to u^1, u^2, u^3. We shall denote, besides, with $w^{\bullet}$ the "surface" coordinates gradient of w, i.e., the vector of first derivatives with respect to u^1, u^2 only. It is well-known that the components of the stress tensor t_{ik} satisfy the symmetry condition $t_{ik} = t_{ki}$. We can represent the estimates obtained from the ***equations of equilibrium for the Euclidean case*** (see [5, 7] for full details) by

$$\sum_m t_{im;m} = P_i = F(\eta, t)(tt' + \eta' t + \eta t')$$

and the estimates resulting from the ***compatibility conditions*** [4] by

$$\sum_r t_{hk;rr} + 2\mu \sum_r t_{rr;hk} = Q_{hk}$$
$$= F(\eta, t)\left(\eta t'' + (t')^2 + t\eta' t' + \eta' t' + \eta'' t + (\eta')^2 t + (\eta')^2 t^2 + tt''\right)$$

We obtain correspondingly for the ***Affine Theory of Shells***:

$$\sum_r t_{hk;rr} + 2\mu \sum_r t_{rr;hk} = F(\eta,t)\begin{pmatrix} (t')^2 + \eta' t t' + (\eta' t)^2 + (\eta')^2 + t'' + \eta'' t + \\ + \eta' t' + \eta t + \eta'' + (\eta)^2 + \eta'' t^2 + (\eta t)^2 + \\ + (\eta' t)^2 + \eta\eta'' t + (\eta)^3 t + \eta t (\eta')^2 \end{pmatrix}.$$

In fact, by using the previous estimates one may write:

$$\varepsilon_{ab;cd} + \varepsilon_{cd;ab} - \varepsilon_{ad;cb} - \varepsilon_{bc;ad} = \ldots \text{ estimated terms},$$

and, consequently,

$$t_{hk;rr} + \frac{1}{1+\nu} t_{rr;hk} = \frac{\nu}{1+\nu}\left(t_{1,rr}\delta_h^k - t_{1,rk}\delta_r^h - t_{1,kr}\delta_k^r\right) + t_{hr;kk} + t_{rk;hr} + .. + \text{higher order terms},$$

$$\begin{aligned} t_{ij;kl} &= t_{ij,kl} - (\Gamma^h_{ik})_{,l} t_{hj} - \Gamma^h_{ik} t_{hj,l} - (\Gamma^m_{jk})_{,l} t_{im} - \Gamma^m_{jk} t_{im,l} - \Gamma^h_{il}(t_{hk,j} - \Gamma^r_{hj} t_{rk} - \Gamma^s_{kj} t_{sh}) - \\ &\quad - \Gamma^m_{jl}(t_{mi,k} - \Gamma^r_{mk} t_{ri} - \Gamma^s_{ik} t_{sm}) - \Gamma^q_{kl}(t_{qi,j} - \Gamma^r_{qi} t_{rj} - \Gamma^s_{ij} t_{sq}) \end{aligned}.$$

Finally, by using all of the above expressions one may write, for the case of Affine Shells, estimates which resemble the ones obtained for the Euclidean case.

7. References

[1] S. Gigena *Constant Affine Mean Curvature Hypersurfaces of Decomposable Type*, Proc of Symp. in Pure Math, American Math Society, Vol. 54, (1993), Part 3, pp.289-316..

[2] Gigena, S. *Hypersurface Geometry and Related Invariants in a Real Vector Space*, Editorial Ingreso, Córdoba, Argentina, (1996).

[3] Gigena, S. *Ordinary Differential Equations in Affine Geometry*, Le Matematiche,Vol. LI, (1996), Fasc.I, pp.119-151.

[4] Gigena, S. - Binia, M. - Abud, D. *Condiciones de Compatibilidad para Cáscaras Afines*, Mecánica Computacional Vol. XXI, (2002), pp.1862-1881.

[5] Gigena, S.; Binia, M.; Abud, D.; *Ecuaciones de equilibrio en Cáscaras Afines*, Mecánica Computacional, Vol. XXII, (2003), pp.1953-1963.

[6] Gigena, S.; Abud, D.; Binia, M. *Teoría de Cáscaras Afines: Desigualdades Básicas*, Mecánica Computacional, Vol. XXIII, (2004), pp.639-652.

[7] John, F. *Estimates for the Derivatives of the Stresses in a Thin Shell and Interior Shell Equations*, Comm. Pure Appl. Math. Nº 18, (1965), pp.235-267.

[8] John, F. *Refined Interior Equations for Thin Elastic Shells*, Comm. Pure Appl. Math Nº 24, (1971), pp.583-615.

[9] Koiter, W.T. *On the mathematical foundation of shell theory*, ´Proc. Int. Congr. On Mathematics, Nice, (1970), Vol. 3, Paris, 1971, pp.123-130.

[10] Love, A.E.H. *A Treatise on The Mathematical Theory of Elasticity*, 4thed, Dover, (1944).

[11] Millman, R. - Parker, G. *Elements of Differential Geometry*, Prentice-Hall, N. Jersey, (1977).

[12] Möllmann, H. *Introduction to the Theory of Thin Shells*, J. Wiley Sons, (1981).

[13] Nomizu, K. - Sasaki, T. *Affine Differential Geometry*, Cambridge U. Press, (1994).