A P.D.E. CHARACTERIZATION OF AFFINE HYPERSURFACES WITH PARALLEL DIFFERENCE TENSOR

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Abstract: The use of P.D.E. (Partial Differential Equations) in Differential Geometry has a long standing history. Very particularly this has happened within the topic of Classificatory Problems in Affine Differential Geometry, where it has played an essential role helping to obtain a better understanding of every one of those problems and where, besides, it has provided essential tools for arriving at rigorous solutions which, otherwise, would be very difficult, if not impossible, to obtain. One of those problems is to try to achieve the classification of those affine hypersurfaces with parallel difference tensor, initially treated by F. Dillen and L. Vrancken, who obtained partial results in lower dimensions, without appealing to P.D.E.
It is the purpose of this article to consider the problem and obtain, as the title indicates, a characterization of that class of hypersurfaces by means of Partial Differential Equations, particularly of Monge-Ampère type.

Key Words: *Affine Hypersurfaces, Affine Metric, Difference Tensor, Affine Normal, Monge-Ampère Equations, Normal Connection.*

1. Introduction

The problem of classifying hypersurfaces with affine normal parallel Difference Tensor was started by F. Dillen and L. Vrancken in their interesting paper [1]. The first, very important result they proved reads as follows:

Theorem (A). *Let M^n be an affine hypersurface in $\mathbb{R}^{n+1}$ with parallel difference tensor, $\nabla K = 0$. If K is not equal to zero at some point, then M^n is an improper affine hypersphere whose affine metric is flat. Further, there exists a number $m: 2 \le m \le n$ such that K^{m-1} is different from zero and $K^m = 0$. Besides, M^n is given as the graph of a polynomial of degree $m+1$ with constant Hessian.*

There are other, related results which are proven in the paper and, by using all this, they study diverse instances of the integer values for n and m, to obtain a partial classification of some cases, particularly in lower dimensions $n = 2, 3, 4$. Nevertheless, it should be said, first of all, that the geometrical and analytical properties described in the above theorem are not characterizing. In fact, consider the hypersurface described by the graph immersion

$$f(t_1, t_2, t_3) = t_1 t_2 + t_1^2 t_3 + t_3^2$$

Then, it is easy to see that this satisfies $\nabla K \neq 0$, $K^2 \neq 0$, and $K^3 = 0$.
Furthermore, and as a consequence of this fact, the classification obtained in the mentioned paper is not complete, at least for dimensions 3 and 4.
The object of the present paper is to characterize, by means of Partial Differential Equations of Monge-Ampère type, the geometrical and analytical properties of the class of Hypersurfaces, in reference to the problem considered. We conjecture that this characterization is the appropriate tool to be used with the purpose of trying to complete the classification, initially for lower dimensions and then, hopefully, also for the higher ones. In this sense, let us state here that the method of work to be used has already been introduced by one of us in previous papers, where the classification of affine hypersurfaces with parallel Second Fundamental (Cubic) was studied, see [2, 3, 4], obtaining the full classification of the cited class of

hypersurfaces. The comparison with the results exposed in those articles is clear because, by using the methods introduced in this paper, it can be seen that, in fact, the classification shall depend not only on the integer value m, as described above, but also on two other integer values that we shall label as k and r with $1 \le k \le n/2$, $1 \le r \le n-1$, with the same meaning as in the cited articles.

The present article is organized as follows: in Section 2, we introduce the necessary tools pertaining to the unimodular affine geometry of hypersurfaces, for dimensions greater or equal than two, that are essential for the better understanding of the problem considered. We shall use the so-called structural language for connections, used in [5], although it should be said that other authors refer to this as the language of Koszul for connections or, more directly, Koszul Connections. In the final Section 3 we obtain the *characterizing geometrical and analytical properties* of the class of hypersurfaces under consideration.

2. AFFINE HYPERSURFACE GEOMETRY

Let M^n be an $n-$dimensional manifold of class C^∞ and $F : M \to \mathbb{R}^{n+1}$ an immersion enough smooth, for example we can take directly of class C^∞ in order to avoid further discussions on the matter. We assume that the affine space $\mathbb{R}^{n+1}$ is provided with its usual flat affine connection D and a fixed parallel volume element ω.

A differentiable vector field η it said to be transversal to $F(M)$ if at each point p in M and for any referential $(X_1,\dots,X_n)$, the vectors $(F_*)_p(X_1),\dots,(F_*)_p(X_n),\eta_p$ form a basis of $T_{F(p)}(\mathbb{R}^{n+1}) \simeq \mathbb{R}^{n+1}$. Obviously, this condition is equivalent to requiring that $\omega(X_1,\dots,X_n,\eta) \ne 0$. For the sake of simplicity, we shall identify $F_*(X)$ with X for each $X \in \mathfrak{X}(M)$.

For an arbitrary transversal vector field η we have the following structures:

A non trivial volumes form θ

$$\theta(X_1,\dots,X_n) = \omega(X_1,\dots,X_n,\eta) \tag{2.1}$$

A tensor S of type $(1,1)$ and a form τ by means of the Weingarten's structural equation

$$D_X\eta = -SX + \tau(X)\eta \tag{2.2}$$

A bilinear form h and a $\text{torsion}-\text{free}$ connection ∇ satisfying the formula of Gauss

$$D_X Y = \nabla_X Y + h(X,Y)\eta \tag{2.3}$$

The symmetric bilinear form h is called the affine fundamental form relative to the transversal vector field η.

We are interested in verifying if the couple (∇,θ) defines an affine unimodular structure, that is, if $\nabla\theta = 0$. Since $\nabla\theta = \theta \otimes \tau$, the condition $\nabla\theta = 0$ is equivalent to $\tau = 0$. [5].

If the affine fundamental form h is nondegenerate, we have a volume form ω_h defined by

$$\omega_h(X_1,\dots,X_n) = \left|\det\left[h(X_i,X_j)\right]\right|^{1/2}$$

If we choose an arbitrary transversal vector field η, then we obtain on M the affine fundamental form h, the induced connection ∇ and the induced volume element θ. We want to achieve, by means of an appropriate choice of θ, the following two goals:

(I): $\nabla\theta = 0$

(II): $\omega_h = \theta$.

For each point p in M , there is a transversal vector field ξ defined in a neighborhood of p satisfying the conditions (I) and (II) above [5]. Such a transversal vector field is unique up to sign.
This transversal vector field is called the affine normal field and the induced connection ∇ , the affine fundamental form h , and the affine shape operator S make up the so called Blaschke structure (∇, h, S) on the hypersurface M . The induced connection ∇ is independent of the choice of the sign of ξ and is called the Blaschke connection.
For an immersion of this type we have the following identities:

(2.4)	$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z)$	Codazzi equation for h
(2.5)	$(\nabla_X S)Y = (\nabla_Y S)X$	Codazzi equation for S
(2.6)	$h(SX,Y) = H(X,SY)$	Ricci equation
(2.7)	$\nabla\theta = 0$	Equiaffine condition
(2.8)	$\omega_h = \theta$	Volume condition
(2.9)	$\nabla\omega_h = 0$	Apolarity condition

If the connection ∇ is torsion free, for the curvature tensor field R hold the first and the second Bianchi identities:

(2.10) $$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

(2.11) $$(\nabla_X R)(Y,Z,W) + (\nabla_Y R)(Z,W,X) + (\nabla_Z R)(W,X,Y) + (\nabla_W R)(X,Y,Z) = 0$$

Definition 1. The Ricci tensor field, which is a geometrical object of type $(0,2)$ is given by

(2.12) $$\mathrm{Ric}(Y,Z) = \mathrm{trace}\{X \to R(X,Y)Z\}.$$

The tensor field R satisfies the fundamental equation

(2.13) $$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY$$

and from the latter we obtain

(2.14) $$\mathrm{Ric}(Y,Z) = \mathrm{Tr}(S)h(Y,Z) - h(SY,Z)$$

From the Codazzi equation (2.4) it is seen that the Second Fundamental, Cubic form C given by

$$C(X,Y,Z) = (\nabla h)(X,Y,Z)$$

is symmetric on X and Z .

If we denote by $\widetilde{\nabla}$ the Levi-Civita connection of h , we can consider the tensor field K given by the difference between the connections ∇ and $\widetilde{\nabla}$. This tensor field, which is of type $(1,2)$, called the

Difference Tensor is defined by

$$K(X,Y)=\nabla_X Y-\tilde{\nabla}_X Y$$

Since ∇ and $\tilde{\nabla}$ are free of torsion, we have $K(X,Y)=K(Y,X)$. Moreover, it can be seen that $C(X,Y,Z)=-2h(K(X,Y),Z)$. Therefore, the cubic form C is symmetric in all three arguments.

Proposition 2. *Let $F:M\to\mathbb{R}^{n+1}$ be a nondegenerate immersion, then we have:*

a) $S=0$ *if and only if* $R=0$.

b) $Ric=0$ *if and only if* $S=0$.

Proof**.** See [5].

We have, moreover, the Mainardi-Codazzi equation

$$\begin{aligned}(\tilde{\nabla}C)(Y,Z,W,X)-(\tilde{\nabla}C)(X,Z,W,Y)&=h(X,Z)h(SY,W)-h(Y,Z)h(SX,W)\\&\quad+h(X,W)h(SY,Z)-h(Y,W)h(SX,Z)\end{aligned}\tag{2.15}$$

and, the Gauss condition

$$\begin{aligned}\tilde{R}(X,Y)Z&=\frac{1}{2}\left(h(Y,Z)SX-h(X,Z)SY\right)\\&\quad+\frac{1}{2}\left(h(SY,Z)X-h(SX,Z)Y\right)\\&\quad-[K_X,K_Y]Z\end{aligned}\tag{2.16}$$

where $\tilde{R}$ is the curvature tensor corresponding to the affine metric.

Definition 3. A Blaschke hypersurface M is called an *improper affine hypersphere* if $S=0$. If $S=\lambda I$, where λ is a nonzero constant, then M is called a *proper affine hypersphere.*

Proposition 4. *Let $F:M\to\mathbb{R}^{n+1}$ be a nondegenerate immersion and assume that M is a connected manifold, then the following statements are equivalent:*

a) $F(M)$ *is an affine hypersphere.*

b) ∇C *is totally symmetric.*

c) $\tilde{\nabla}C$ *is totally symmetric*.

Proof**.** By using $\nabla_X Y=\tilde{\nabla}_X Y+K_X Y$ it is easy to see that

$$A=(\tilde{\nabla}_X C)(Y,Z,W)-(\tilde{\nabla}_Y C)(X,Z,W)$$

can be written as

$$A=(\nabla_X C)(Y,Z,W)-(\nabla_Y C)(X,Z,W),\tag{2.17}$$

which implies that ∇C is totally symmetric if and only if $\tilde{\nabla}C$ is totally symmetric.

If $\tilde{\nabla}C$ (or ∇C) is totally symmetric, by (2.15) we have

(2.18)
$$\begin{aligned}0 &= h(X,Z)h(SY,W)-h(Y,Z)h(SX,W)\\ &+h(X,W)h(SY,Z)-h(Y,W)h(SX,Z)\end{aligned}$$

From this equality we obtain, by taking the trace of the linear map, $SY=\frac{1}{n}\mathrm{Tr}(S)Y$, that is, M is an affine hypersphere.
On the other hand, the equality $S=\lambda I$ implies, by (2.15), that $\tilde{\nabla}C$ is totally symmetric. □

Lemma 5. *The following formula is a consequence of the definitions of* K, C *and* $C(X,Y,Z)=-2h(K(X,Y),Z)$.

(2.19)
$$h((\nabla_X K)(Y,Z),W)=-\frac{1}{2}(\nabla_X C)(Y,Z,W)+2h(K_X K_Y Z,W).$$

Proof**.** If we write $A=h((\nabla_X K)(Y,Z),W)$, we can further obtain that

(2.20)
$$\begin{aligned}A &= h(\nabla_X(K(Y,Z))-K(\nabla_X Y,Z)-K(Y,\nabla_X Z),W)\\ &= h(\nabla_X K_Y Z,W)-h(K_Z\nabla_X Y,W)-h(K_Y\nabla_X Z,W)\end{aligned}$$

Now, by using the equality $C(X,Y,Z)=-2h(K(X,Y),Z)$, we see that the quantity H given by $H=-h(K_Z\nabla_X Y,W)-h(K_Y\nabla_X Z,W)$, can be written as

$$\begin{aligned}H &= -\frac{1}{2}(\nabla_X C)(Y,Z,W)+h(K_X K_Y Z,W)\\ &-h(\nabla_X K_Y Z,W)+h(K_Y Z,K_X W)\end{aligned}$$

Replacing in (2.20) we obtain

(2.21)
$$\begin{aligned}h((\nabla_X K)(Y,Z),W) &= h(\nabla_X K_Y Z,W)-\frac{1}{2}(\nabla_X C)(Y,Z,W)\\ &= h(K_X K_Y Z,W)-h(\nabla_X K_Y Z,W)+h(K_Y Z,K_X W)\\ &= -\frac{1}{2}(\nabla_X C)(Y,Z,W)+h(K_X K_Y Z,W)+h(K_Y Z,K_X W)\end{aligned}$$

and the last term of the right hand side can be written as

$$\begin{aligned}h(K_Y Z,K_X W) &= h(K_X W,K_Y Z)=-\frac{1}{2}C(X,W,K_Y Z)\\ &= -\frac{1}{2}C(X,K_Y Z,W)=h(K_X K_Y Z,W)\end{aligned}$$

Then finally

(2.22) $$h(K_Y Z, K_X W) = -\frac{1}{2}(\nabla_X C)(Y,Z,W) + 2h(K_X K_Y Z, W)$$

and the lemma is proved. □

Lemma 6. ∇K *is totally symmetric if and only if* $S = \lambda I$ *and* $[K_X, K_Y] = 0$ *for each* X *and* Y.

Proof**.** Interchanging X and Y in (2.20) and subtracting we obtain that $A = h((\nabla_X K)(Y,Z),W) - h((\nabla_Y K)(X,Z),W)$ can be written

$$A = -\frac{1}{2}((\nabla_X C)(Y,Z,W) - (\nabla_Y C)(X,Z,W)) + 2h([K_X, K_Y]Z, W)$$

By using (2.15), (2.17) and (2.13) the first term T in the right hand can be written

(2.23)
$$\begin{aligned} T &= -\frac{1}{2}(h(X,Z)h(SY,W) - h(Y,Z)h(SX,W)) \\ &\quad -\frac{1}{2}(h(X,W)h(SY,Z) - h(Y,W)h(SX,Z)) \\ &= -\frac{1}{2}(-h(R(X,Y)Z,W) - h(R(X,Y)W,Z)) \\ &= -\frac{1}{2}(R(X,Y)h)(Z,W) \end{aligned}$$

which is symmetric in Z and W, and the second is clearly skew-symmetric in Z and W.
Now, ∇K is totally symmetric if and only if

$$-\frac{1}{2}(R(X,Y)h)(Z,W) + 2h([K_X, K_Y]Z, W) = 0$$

which is equivalent to $(R(X,Y)h)(Z,W) = 0$ and $h([K_X, K_Y]Z, W) = 0$, but, by (2.23), $(R(X,Y)h)(Z,W) = 0$ if, and only if, ∇C is totally symmetric. Finally, by Proposition 4, $F(M)$ is an affine hypersphere, that is, $S = \lambda I$. □

We remark that the condition $[K_X, K_Y] = 0$ implies that $J = 0$. It follows then that, if $K \neq 0$, the affine metric is indefinite. (J is the Pick invariant classically defined by $J =: \frac{1}{n(n-1)} h(K,K)$).

Lemma 7. If $\nabla K = 0$ and $K \neq 0$, then M is an improper affine hypersphere.
Proof**.** See [1].

Lemma 8. If $[K_X, K_Y]Z = 0$ for all Y and Z, then K_X is nilpotent for each X.
Proof**.** See [1].

3. CHARACTERIZING GEOMETRICAL AND ANALYTICAL PROPERTIES

We are interested in studying nondegenerate hypersurfaces with affine normal parallel difference tensor which are not hyperquadrics, i.e. which satisfy the conditions $\nabla K = 0,\ K \neq 0$. Thus, the first result to be presented next is the one that expresses, from the theory developed so far, that the affine immersion fulfilling those two conditions, say $F(M)$, must be an improper affine hypersphere, that it can be expressed in the form of Monge's, i.e. as a graph immersion, and that the graph function f satisfies certain particular conditions.

Theorem 9. *Let M^n be a nondegenerate affine hypersurface in $\mathbb{R}^{n+1}$ with $\nabla K = 0$ and $K \neq 0$, then the following properties hold:*

a) *M^n is an improper affine hypersphere.*

b) *M^n is expressible in the form of Monge's, i.e., a graph immersion, and with respect to a suitable affine system of coordinates the graph function f satisfies a Monge-Ampère type equation $\det(\partial_i \partial_j f) = \pm 1$. Moreover, there exists a number $m \in \{2,3,...,n\}$ such that f is a polynomial of degree $m+1$.*

Proof. M^n is an improper affine hypersphere by Lemma 7 above.
Since M is an improper affine hypersphere, we have $S = 0$, then ∇ is flat and the affine normal ξ is constant. Let us choose a coordinate system so that $\xi = (0,0,...,0,1)$, then M is given by $F(t_1,...,t_n) = (t_1,...,t_n, f(t_1,...,t_n))$. Since $(t_1,...,t_n)$ are $\nabla-\text{flat}$ coordinates on M we have

$$h\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}\right) = \frac{\partial^2 f}{\partial t_i \partial t_j} = \partial_i \partial_j f \tag{3.1}$$

and the Hessian of f satisfies $\det(\partial_i \partial_j f) = \pm 1$.
From (3.1) we have that

$$(\nabla^{m-2} h)\left(\frac{\partial}{\partial t_{i_1}}, \cdots, \frac{\partial}{\partial t_{i_m}}\right) = \frac{\partial^m f}{\partial t_{i_1} \partial t_{i_2} \cdots \partial t_{i_m}} = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_m} f\ . \tag{3.2}$$

By Lemma 8 above and Lemma (3.4) of [1] (Page 49) there exists $m \in \{2,3,...,n\}$ such that $\nabla^m h = 0$. Hence , from (3.2) f is a polynomial of degree $m+1$. □

Our next goal is to present geometrical and analytical conditions which, when satisfied by a given graph function, are characterizing.

Theorem 10. *Let M^n be a nondegenerate affine hypersurface in $\mathbb{R}^{n+1}$ which is expressible in the form of Monge, i.e. as a graph immersion with respect to some affine system of coordinates in the ambient space $(t_1,...,t_n,t_{n+1})$ as $t_{n+1} = f(t_1,...,t_n)$, such that:*

a) *The graph function f is a polynomial function of degree $m+1$.*

b) The graph function f is a solution to the Monge-Ampère type equation $\det\left(\partial_i \partial_j f\right) = \pm 1$.

c) The quantities $A^i_{jk} = \sum_r h^{ir}\left(\partial_r \partial_j \partial_k f\right)$ *in those coordinates satisfy the conditions* $A^i_{jk} = \text{constant}$*, with at least one of them different from zero.*

Then, M^n is an affine hypersurface with parallel difference tensor which is not a hyperquadric, i.e. it satisfies $\nabla K = 0$ and $K \neq 0$. Besides, in that case we further have that K is a nilpotent operator with (the degree of the above polynomial function) $m \in \{2, 3, \ldots, n\}$, $K^{m-1} \neq 0$, $K^m = 0$.

Proof**.** In the above mentioned coordinate system, we have that, $\Gamma^i_{jk} = 0$, and $h_{ij} = \partial_i \partial_j f$, [2], so that we have the local coordinates C_{ijk} for the cubic form C are given by $C_{ijk} = \partial_i \partial_j \partial_k f$, then $K^i_{jk} = -\frac{1}{2}\sum_r h^{ir} C_{rjk} = -\frac{1}{2}\sum_r h^{ir}\left(\partial_r \partial_j \partial_k f\right) = -\frac{1}{2} A^i_{jk}$, therefore, clearly $K \neq 0$ and $\nabla K = 0$. Hence M^n is an improper affine hypersphere by the previous Theorem 9 and $\det\left(\partial_i \partial_j f\right) = \pm 1$.

Finally, the last result follows from lemma 8 above. □

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