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GAUGED COTANGENT BUNDLE REDUCTION WITH SINGULARITIES

MIGUEL RODRÍGUEZ-OLMOS

Dedicated to Hernán Cendra

ABSTRACT. This article surveys in a non-technical way the current status of the theory of symplectic and Poisson reduction of cotangent lifted Lie group actions exhibiting singularities. We explain how the involved singular spaces obtained through several reduction processes can be globally realized as stratified fiber bundles, where each of the strata is a smooth bundle that can be identified with the reduced space obtained via some classical reduction scheme for a cotangent lifted action where the singularities have been essentially removed.

1. INTRODUCTION

The well known Marsden–Weinstein reduction scheme is a method to obtain a new symplectic manifold from an old one equipped with a Hamiltonian action of a Lie group. If (M,ω) is a symplectic manifold we say that a smooth Lie group action $G\times M\to M$ is Hamiltonian if G acts by symplectomorphisms and there exists a map $\mathbf{J}:M\to\mathfrak{g}^*$ satisfying

(i) $\omega(\xi_M,\cdot)=\mathbf{d}\langle\mathbf{J},\xi\rangle$ for every Lie algebra element $\xi\in\mathfrak{g}$, and
(ii) $\mathbf{J}(g\cdot z)=\mathrm{Ad}^*_{g^{-1}}(\mathbf{J}(z))$ for every $g\in G$, $z\in M$.

Condition (ii) is just a statement on the G-equivariance of $\mathbf{J}$, while the vector field ξ_M in (i) is the fundamental vector field for the G-action corresponding to the Lie algebra element ξ, i.e. $\xi_M(z)=\frac{d}{dt}\big|_{t=0}e^{t\xi}\cdot z$.

In a simple setup, if we are in the previous situation and the group action is free and proper then the Marsden–Weinstein reduction result [13] states that for every $\mu\in\mathfrak{g}^*$, the level set $\mathbf{J}^{-1}(\mu)$ is a smooth manifold on which the G-action restricts to an action of the coadjoint isotropy group, defined as

$$G_\mu=\{g\in G:\mathrm{Ad}^*_g\mu=\mu\}.$$

Moreover, the quotient space $\mathbf{J}^{-1}(\mu)/G_\mu$ is a smooth symplectic manifold equipped with a natural symplectic form ω_μ uniquely defined by

$$\pi^*\omega_\mu=\iota^*\omega,$$

where $\iota:\mathbf{J}^{-1}(\mu)\to M$ and $\pi:\mathbf{J}^{-1}(\mu)\to\mathbf{J}^{-1}(\mu)/G_\mu$ are the canonical inclusion and the group projection respectively.

A second kind of reduction scheme that one can perform on a symplectic manifold equipped with a Hamiltonian action is Poisson reduction [12]. In the present setup, and under the same freeness and properness assumptions on the action $G\times M\to M$, we can form the quotient manifold M/G, and this space is equipped with a natural reduced Poisson bracket defined as

$$\{f_1,f_2\}_{\mathrm{red}}\circ\pi=\omega(X_{f_1\circ\pi},X_{f_2\circ\pi}),$$

Palabras clave. Marsden–Weinstein reduction, symplectic and Poisson manifolds, momentum maps, stratified spaces.

where $f_1, f_2 \in C^\infty(M/G)$, $\pi : M \to M/G$ is the group projection and $X_{f_i \circ \pi}$ is the Hamiltonian vector field on M defined implicitly as

$$\omega(X_{f_i\circ\pi}, \cdot) = \mathbf{d}(f_i \circ \pi).$$

There exists an intimate relationship between both reduction schemes which is the following: the symplectic leaves of the reduced Poisson manifold M/G correspond exactly to the connected components of the Marsden–Weinstein reduced spaces $\mathbf{J}^{-1}(\mu)/G_\mu$ when μ ranges over $\mathfrak{g}^*$.

An important particular case of Hamiltonian actions on symplectic manifolds is given by cotangent lifted actions on cotangent bundles. Let Q be a smooth manifold and T^*Q its cotangent bundle. T^*Q is a symplectic manifold in a natural way equipped with a symplectic form ω_Q expressed locally as

$$\omega_Q = \mathbf{d}x_i \wedge \mathbf{d}y_i,$$

where $\{x_i\}$ are local coordinates on the base and $\{y_i\}$ linear coordinates in the fibers. If a Lie group G acts smoothly on Q (without any particular further conditions) then the cotangent lifted action $G \times T^*Q \to T^*Q$ is automatically Hamiltonian with respect to ω_Q (it is also free and proper if the original action on Q is so) and admits a momentum map defined by the relation

$$\langle \mathbf{J}(\alpha_x), \xi \rangle = \langle \alpha_x, \xi_Q(x) \rangle \quad \text{for any } x \in Q,\ \alpha_x \in T_x^*Q,\ \xi \in \mathfrak{g}. \tag{1}$$

Note that the cotangent bundle projection $\tau : T^*Q \to Q$ is equivariant with respect to the original action on Q and its cotangent lift to T^*Q. Now, since T^*Q is equipped with two different geometric structures, a symplectic form and a fibration, and both are left invariant by the group action, it is reasonable to ask if under the processes of Marsden–Weinstein or Poisson reduction, the various reduced spaces obtained are also going to exhibit a fibered structure, besides the symplectic or Poisson structure like in the general case. This is the starting point of the theory of cotangent bundle reduction, which seeks to give fibered realizations of the reduced spaces $\mathbf{J}^{-1}(\mu)/G_\mu$ and $(T^*Q)/G$ when reduction is applied to a cotangent lifted action on a cotangent bundle T^*Q. The main results of the theory can be summarized in Figure 1, which is a convenient way to think about the so-called "gauge picture" of Mechanics.

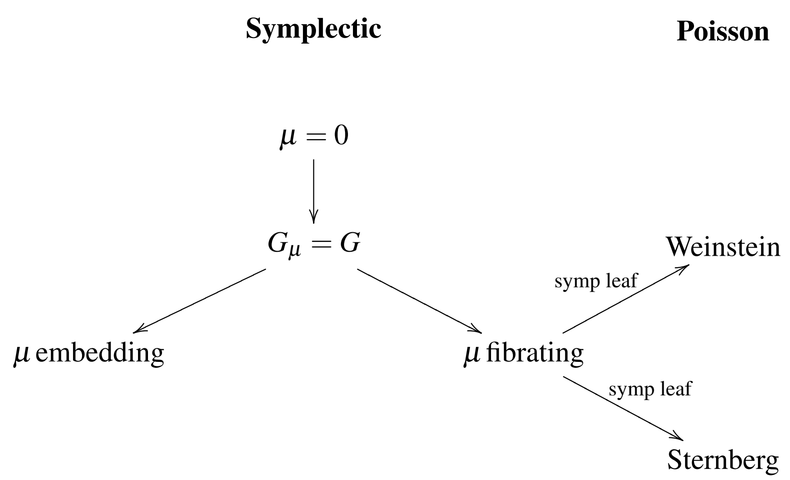

FIGURE 1. Cotangent Bundle Reduction

The cases $\mu = 0$, $G_\mu = G$, μ fibrating and μ embedding all correspond to the Marsden–Weinstein reduction process and are different ways of realizing the reduced symplectic spaces $\mathbf{J}^{-1}(\mu)/G_\mu$ as fiber bundles, which vary according to different properties of the momentum value μ. On the other hand, the two cases denoted "Weinstein" and "Sternberg" correspond to Poisson reduction and are two different fibered realizations of the same Poisson reduced space $(T^*Q)/G$. We shall now briefly review these cases.

$\mu = 0$. This is the first result of the theory of cotangent bundle reduction, obtained originally by Satzer in [22]. It shows that if $\mu = 0$ there exists a canonical symplectic diffeomorphism

$$\varphi_0 : (\mathbf{J}^{-1}(0)/G, \omega_0) \longrightarrow_0 (T^*(Q/G), \omega_{Q/G}).$$

That is, for the case of a cotangent lifted Hamiltonian action, the Marsden–Weinstein reduced space at $\mu = 0$ is precisely (up to a symplectic isomorphism) the cotangent bundle of the quotient space Q/G of the base manifold Q. In this sense the content of this result is that one can, starting with a Lie group action $G \times Q \to Q$, either construct the symplectic cotangent bundle of Q and then reduce via the Marsden–Weinstein scheme at zero momentum, or just quotient Q by the action of G and then construct its cotangent bundle equipped with the corresponding canonical symplectic form, and both processes give the same outcome, up to a symplectic diffeomorphism. Also it is interesting to note that using this result one can *define* the cotangent bundle of a quotient space Q/G as the Marsden–Weinstein reduced space of T^*Q via the cotangent lifted action of G to the total space.

$G_\mu = G$ (totally isotropic momentum). This is historically the second result of the theory of cotangent lifted actions. In the particular case when the group G is Abelian, this case was already studied in [22]. A more general approach which does not require this hypothesis was taken in [8] and [1]. This is also the first case in which twisting terms appear in the fibered realizations of the Marsden–Weinstein reduced spaces for cotangent lifted actions. These twisting terms modify the canonical symplectic form of the cotangent bundle involved in the fibered model of the reduced space. They are sometimes also called magnetic or Coriolis terms, due to their role in Mechanics. The most standard way of defining such a twisting term is the following. Let $\mu \in \mathfrak{g}^*$ and choose a principal connection $\mathscr{A}$ in the bundle $Q \to Q/G$. The one-form α_μ on Q defined by

$$\alpha_\mu = \langle \mathscr{A}, \mu \rangle \tag{2}$$

can be shown to be G_μ invariant. Let $\pi : Q \to Q/G_\mu$ be the orbit map for the restriction of the G-action to G_μ. Then, there is a unique two-form B_μ on Q/G_μ satisfying

$$\pi^* B_\mu = \mathbf{d}\alpha_\mu. \tag{3}$$

Notice that in the case of a totally isotropic element μ one has $G_\mu = G$ and therefore $Q/G_\mu = Q/G$. The fibered realization for the Marsden–Weinstein space at such a momentum value μ then consists in a symplectic diffeomorphism

$$\varphi_\mu : (\mathbf{J}^{-1}(\mu)/G, \omega_\mu) \longrightarrow (T^*(Q/G), \omega_{Q/G} - \tau^* B_\mu).$$

Obviously the map φ_μ will depend on the choice of the connection $\mathscr{A}$, and in particular it would be possible to choose as the target the "untwisted" or canonical cotangent bundle $(T^*(Q/G), \omega_{Q/G})$ if it is possible to find a flat connection. In the classical theory of particles and fields the twisting term B_μ corresponds to the strength of a gauge field $\mathscr{A}$ coupled to classical particles that have as configuration space the physical spacetime Q/G and so the symplectic form of their phase space $T^*(Q/G)$ (and therefore their classical dynamics) are modified by this term if the gauge field has non-vanishing strength. A typical example of this situation would be if $G = S^1$ and Q is the trivial bundle $S^1 \times \mathbb{R}^4$, in which case since S^1

is Abelian every momentum value (identified with the electric charge of a classical particle) is totally isotropic and the term B_μ is precisely the strength of an electromagnetic field.
The embedding and fibrating pictures. These are two different ways of obtaining fibered realizations of a Marsden–Weinstein reduced space for cotangent lifted actions and general momentum values. If μ is an arbitrary element of $\mathfrak{g}^*$ the embedding picture (see [8], [1]) realizes the reduced space as a subbundle of a twisted cotangent bundle via a symplectic embedding

$$\varphi_\mu : (\mathbf{J}^{-1}(\mu), \omega_\mu) \to (T^*(Q/G_\mu), \omega_\mu - \tau^* B_\mu),$$

which depends on the choice of a connection $\mathscr{A}$ on $Q \to Q/G$ producing the twisting term B_μ.

The fibrating picture on the other hand (see [11]) realizes the reduced space via a symplectic diffeomorphism

$$\varphi_\mu : (\mathbf{J}^{-1}(\mu), \omega_\mu) \to (V, \omega_{Q/G, \mathscr{O}_\mu, \mathscr{A}}),$$

where

$$\mathscr{O}_\mu \to V \to T^*(Q/G)$$

is a bundle over $T^*(Q/G)$ having as standard fibre the coadjoint orbit $\mathscr{O}_\mu \subset \mathfrak{g}^*$ passing through the chosen momentum value μ. The total space is equipped with a symplectic form of the form

$$\omega^\mu_{Q/G} = \omega_{Q/G} + \mathrm{KKS}_{\mathscr{O}_\mu} + \mathscr{B}_\mu,$$

where there is a contribution coming from the canonical symplectic form $\omega_{Q/G}$ on $T^*(Q/G)$, the base of the bundle V, another term denoted $\mathrm{KKS}_{\mathscr{O}_\mu}$, which is related to the Konstant–Kirillov–Souriau symplectic form on the fiber $\mathscr{O}_\mu$, and a third term $\mathscr{B}_\mu$ which is related to the twisting term B_μ of the principal connection $\mathscr{A}$, on which the map φ_μ depends.

Note that in the particular case when $G_\mu = G$ is totally isotropic this realization coincides with the previous totally isotropic momentum case since the coadjoint orbit $\mathscr{O}_\mu$ is trivial. Analogously, in that case the symplectic embedding φ_μ in the embedding picture becomes injective and also the embedding picture coincides with the totally isotropic momentum case, which in turn contains the $\mu = 0$ as a particular case, since from (2) and (3) we see that in the zero momentum case the twisting term must be trivial. Therefore, as far as the symplectic left part of Figure 1 goes, each arrow represents a step forward in generality since the realization of the target has as particular case the one on the source.
The Weinstein and Sternberg pictures. These are two very closely related fibered realizations of the Poisson reduced space T^*Q/G. They were independently obtained in the same year in [25] and [26]. In both cases (which only differ in the order that some fiber bundle constructions like pull-back and association are performed) there is a Poisson diffeomorphism (depending on the choice of a connection $\mathscr{A}$ on $Q \to Q/G$)

$$\varphi_\mu : (T^*Q/G, \{\cdot,\cdot\}_{\mathrm{red}}) \longrightarrow (S, \{\cdot,\cdot\}_{Q/G,\mathscr{A}})$$

where $\mathfrak{g}^* \to S \to T^*(Q/G)$ is a bundle over $T^*(Q/G)$ having as standard fiber the dual Lie algebra $\mathfrak{g}^*$. This space is equipped with the Poisson structure

$$\{\cdot,\cdot\}_{Q/G,\mathscr{A}} = \{\cdot,\cdot\}_{Q/G} + \mathrm{LiePoiss} + \mathscr{B}$$

where $\{\cdot,\cdot\}_{Q/G}$ is a contribution related to the Poisson structure on $T^*(Q/G)$ corresponding to the canonical symplectic form $\omega_{Q/G}$ on the base $T^*(Q/G)$, LiePoiss is a term constructed from the Lie–Poisson structure on the fiber $\mathfrak{g}^*$ and $\mathscr{B}$ is a term involving the curvature of $\mathscr{A}$.

Finally, the fibrating pictures make explicit the fact stated earlier that the Marsden–Weinstein reduced spaces correspond to the symplectic leaves of the Poisson reduced spaces.

In fact, we can think of the fibrating version of the symplectic reduced space $\mathscr{O}_\mu \to V \to T^*(Q/G)$ as a subbundle of the reduced Poisson space $\mathfrak{g}^* \to S \to T^*(Q/G)$ obtained by just taking the restriction on each fiber from $\mathfrak{g}^*$ to the coadjoint orbit $\mathscr{O}_\mu$, which, if equipped with the Konstant–Kirillov–Souriau symplectic form, is precisely a symplectic leaf of $\mathfrak{g}^*$ equipped with the Lie–Poisson structure. This is the meaning of the two transversal arrows in Figure 1.

The program of singular cotangent bundle reduction started in [14] and [5] and continued in several references such as [7, 9, 16, 18, 21, 23] seeks to obtain realizations analog to those of Figure 1 once the freeness assumption in the group action has been dropped. This leads to a number of difficulties mainly related to the fact that the several quotient spaces appearing in the theory are no longer smooth manifolds. Although other approaches do exist, we will treat this issue by working within the framework of stratified spaces.

Singular cotangent bundle reduction is still an unfinished topic. In the remainder of this article we will expose the basic results of the theory for the cases $\mu = 0$ and totally isotropic μ. Partial results about the embedding and fibrating pictures were already obtained in [7, 14, 16] for a particular class of actions with singularities but, up to our knowledge, the general case has not yet been fully addressed. The singular version of Poisson cotangent bundle reduction, encompassing singular analogues of the Weinstein and Sternberg spaces is the content of [17].

2. Stratified spaces and orbit types

We now recall the basic aspects of the theory of stratified spaces and its role in reduction theory. See [19] and [15] for an in-depth exposition. The main example that we will use is the stratification of the orbit space for a proper Lie group action.

Let X be a topological space and $\mathscr{Z}_X = \{X_i : i \in I\}$ a collection of locally closed disjoint subspaces of X which are smooth manifolds in the induced topology. We say that

$$X = \bigsqcup_{i \in I} X_i$$

induces a stratification of X if the above partition of X is locally finite and satisfies

$$X_i \cap \overline{X_j} \neq \emptyset \Longleftrightarrow X_i \subset \overline{X_j} \backslash X_j \quad \text{for } i \neq j.$$

In that case we say that X_i and X_j are in incidence relation. Therefore, a stratification of a singular topological space X describes it as a union of smooth manifolds glued together along their boundaries. The sets X_i are called the strata of the induced stratification (although usually a technical maximality condition that will not be explained here is also required).

The notion of mappings between stratified spaces will also play a notable role in the description of singular reduced cotangent bundles. Let X and Y be topological stratified spaces, being their respective partitions $\mathscr{Z}_X = \{X_i : i \in I\}$ and $\mathscr{Z}_Y = \{Y_r : r \in J\}$. Let $f : X \to Y$ be a continuous map. We will say that f is a stratified morphism if for every $i \in I$, $f(X_i) \subset Y_r$ for some $r \in J$ and the restriction $f|_{X_i} : X_i \to Y_r$ is smooth. If for each $i \in I$ the restriction of f is a diffeomorphism (resp. submersion, embedding, etc.) we will say that f is a stratified diffeomorphism (resp. submersion, embedding, etc.). In particular a stratified fiber bundle will consist in a stratified fibration $f : X \to Y$.

Let $G \times M \to M$ be a proper action of a Lie group G on the smooth manifold M. The orbit space M/G is not a manifold in general, however it has a natural stratification induced by its partition into orbit types. Let $H \subset G$ be a compact subgroup of G. The orbit type set

$M_{(H)}$ is defined as

$$M_{(H)} = \{z \in M : G_z \text{ is conjugate to } H\},$$

where $G_z \subset G$ denotes the stabilizer of $z \in M$ for the G-action.

It can be proved [4] that the connected components of $M_{(H)}$ are embedded submanifolds of M. We will assume from now on that every orbit type submanifold is connected, since otherwise every aspect of the theory persists just by applying it component by component. In addition, by construction, orbit types are G-invariant and satisfy

(i) $M_{(H)}/G$ is either empty or a smooth manifold.
(ii) The partition

$$M/G = \bigsqcup_{H \subset G} M_{(H)}/G \tag{4}$$

induces a stratification of M/G called the orbit type stratification.

Using this framework, Sjamaar and Lerman obtained in their seminal paper [24] the following result, which is an analog of the Marsden–Weinstein symplectic reduction theorem in the case of proper Lie group actions with singularities.

Theorem 1. *Let (M,ω) be a symplectic manifold equipped with a proper and Hamiltonian action of the Lie group G with associated momentum map $\mathbf{J} : M \to \mathfrak{g}^*$. Then the partition*

$$\mathbf{J}^{-1}(0)/G = \bigsqcup_{H \subset G} (\mathbf{J}^{-1}(0) \cap M_{(H)})/G \tag{5}$$

induces a stratification of $\mathbf{J}^{-1}(0)/G$. Moreover, the manifolds $(\mathbf{J}^{-1}(0) \cap M_{(H)})/G$, if not empty, are equipped with natural reduced symplectic forms $\omega_0^{(H)}$ characterized by

$$(\pi^H)^* \omega_0^{(H)} = (\iota^H)^* \omega,$$

where $\iota^H : \mathbf{J}^{-1}(0) \cap M_{(H)} \to M$ and $\pi^H : \mathbf{J}^{-1}(0) \cap M_{(H)} \to (\mathbf{J}^{-1}(0) \cap M_{(H)})/G$ are the natural inclusion and group projection, respectively. This stratification is called the symplectic stratification of $\mathbf{J}^{-1}(0)/G$.

The above theorem realizes the singular quotient $\mathbf{J}^{-1}(0)/G$ as a union of smooth symplectic manifolds for which the symplectic structures are obtained in a Marsden–Weinstein style. Obviously, since a free action is such that all the stabilizers are trivial, in that case the above partition consists in a single element, namely the one corresponding to $H = \{e\}$, which is of course the full smooth quotient $\mathbf{J}^{-1}(0)/G$, and so the Marsden–Weinstein reduction result at momentum $\mu = 0$ is recovered. Theorem 1 was further generalized in [3] to deal with the case of general μ (see also [2] for an alternative approach not based on the theory of stratified spaces). Although the completely general case is slightly more involved, we collect for later convenience the following particular case. If μ satisfies $G_\mu = G$ then the results of Theorem 1 carry over to a description of the reduced space $J^{-1}(\mu)/G$ just substituting $\mathbf{J}^{-1}(0)$ by $\mathbf{J}^{-1}(\mu)$ and $\omega_0^{(H)}$ by $\omega_\mu^{(H)}$.

3. Reduction at momentum zero

Notice that the starting point for Theorem 1 was a symplectic manifold equipped with an action of a Lie group G with singularities that respected the existing geometric structure (the symplectic form). This result then describes the topological symplectic quotient $\mathbf{J}^{-1}(0)/G$ in terms of a stratification that includes this geometric data, since the strata are symplectic manifolds. With this point of view, the approach for studying the particular case of cotangent bundle reduction for singular actions consists in describing the topological symplectic quotient relative to a cotangent lifted action $G \times T^*Q \to T^*Q$ respecting both the symplectic and fibered structures present on T^*Q. Therefore, using the same guiding principle, we wish to obtain a description of $\mathbf{J}^{-1}(0)/G$ as a stratified fiber bundle in which the strata of the total space recover some geometric structure from the original canonical symplectic form of T^*Q. That is, both geometric structures are present in the resulting stratification.

In order to attack this problem, we start by noticing the following: The expression of momentum map for the cotangent lifted action (1) implies that $\mathbf{J}^{-1}(0) \subset T^*Q$ is at each point $z \in Q$ the annihilator of the tangent space to the group orbit $G \cdot x$. Therefore the cotangent bundle projection $\tau : T^*Q \to Q$ induces a singular G-equivariant fibration $\tau : \mathbf{J}^{-1}(0) \to Q$ that descends to a continuous projection

$$\tau^0 : \mathbf{J}^{-1}(0)/G \to Q/G.$$

We wish to realize this natural projection as a stratified fibration, together with geometric structure on the smooth strata of $\mathbf{J}^{-1}(0)/G$ induced from the original symplectic form ω_Q. There are obvious candidates for which should be the stratifications on the source and target spaces. Namely the symplectic stratification (5) on $J^{-1}(0)/G$ and the orbit type stratification (4) on Q/G. The following two properties are straightforward to prove (see [21]).

Proposition 1. *Let the Lie group G act on the manifold Q and on T^*Q by cotangent lifts with associated momentum map given by* (1). *Then, for any compact subgroup $H \subset G$:*

(i) $(\mathbf{J}^{-1}(0) \cap (T^*Q)_{(H)})/G \neq \emptyset$ *if and only if* $Q_{(H)}/G \neq \emptyset$.
(ii) $\tau^0((\mathbf{J}^{-1}(0) \cap (T^*Q)_{(H)})/G) = \overline{Q_{(H)}/G}$.

In particular, it follows from (ii) that if both spaces are equipped with the above mentioned stratifications, τ^0 cannot be a stratified morphism since it does not map strata to strata. The solution to this fact is to refine the symplectic stratification of $\mathbf{J}^{-1}(0)/G$ in the following way: it is showed in [18] that, given two subgroups $H, K \subset G$, the sets

$$\mathbf{J}^{-1}(0) \cap (T^*Q)_{(H)} \cap T^*Q\big|_{Q_{(K)}}$$

are non-empty if and only if $H \subset K$, and in that case they are smooth G-invariant subbundles of $T^*Q\big|_{Q_{(K)}}$. The main result describing the stratified fibered structure of the singular reduced spaces at $\mu = 0$ for cotangent lifted actions is the following.

Theorem 2 ([18])**.** *Let the Lie group G act on the manifold Q and on T^*Q by cotangent lifts with associated momentum map given by* (1). *Then the partition*

$$\mathbf{J}^{-1}(0)/G = \bigsqcup_{H \subset K} S_{K \to H} \tag{6}$$

where

$$S_{K \to H} = \frac{\mathbf{J}^{-1}(0) \cap (T^*Q)_{(H)} \cap T^*Q\big|_{Q_{(K)}}}{G},$$

induces a stratification of the reduced space $\mathbf{J}^{-1}(0)/G$ satisfying:

(i) *If Q/G is equipped with the orbit type stratification* (4) *then the induced continuous projection*
$$\tau^0 : \mathbf{J}^{-1}(0)/G \to Q/G$$
is a stratified fibration, with $\tau^0(S_{K \to H}) = Q_{(K)}/G$, for any pair $H \subset K$.
(ii) *If $H \subset K$, then $S_{K \to H}$ is a coisotropic submanifold of the symplectic stratum $((\mathbf{J}^{-1}(0) \cap (T^*Q)_{(H)})/G, \omega_0^{(H)})$.*

(iii) *In particular, if* $H = K$ *then* $S_{H\to H}$ *is an open and dense symplectic submanifold of* $((\mathbf{J}^{-1}(0)\cap(T^*Q)_{(H)})/G, \omega_0^{(H)})$*, symplectomorphic to* $(T^*(Q_{(H)}/G), \omega_{Q_{(H)}/G})$.

The stratification induced by (6) *is called the coisotropic stratification of* $\mathbf{J}^{-1}(0)/G$*, and the manifolds* $S_{K\to H}$ *are its coisotropic strata.*

The previous theorem shows that by (i) the coisotropic stratification of $\mathbf{J}^{-1}(0)/G$ is compatible with the natural projection τ^0 and the natural orbit type stratification of Q/G, in the sense that τ^0 becomes a stratified fibration. Furthermore, it is compatible with the symplectic stratification of $\mathbf{J}^{-1}(0)/G$ at a topological level, since by (ii) every coisotropic stratum $S_{K\to H}$ is contained in a unique symplectic stratum $(\mathbf{J}^{-1}(0)\cap(T^*Q)_{(H)})/G$. In addition, it is also compatible with the symplectic geometry of the symplectic stratification, since also by (ii) each coisotropic stratum has a definite coisotropic character within its ambient symplectic stratum. Finally, the coisotropic stratification is a natural generalization of Satzer's result on cotangent bundle reduction at $\mu = 0$, since by (iii) each symplectic stratum is almost everywhere the cotangent bundle of a quotient manifold equipped with its canonical symplectic form.

4. Singular connections and reduction at totally isotropic momentum

We will consider from now on totally isotropic momentum values, i.e. elements $\mu\in\mathfrak{g}^*$ which lie in the image of $\mathbf{J}$ and satisfy $G_\mu = G$. There are two main difficulties when trying to generalize the coisotropic stratification from $\mu = 0$ to totally isotropic momentum values in the case of cotangent lifted Lie group actions with singularities. The first one is that unlike in the zero momentum case, the restriction $\tau : \mathbf{J}^{-1}(\mu)\to Q$ is no longer surjective and therefore the singular reduced space $\mathbf{J}^{-1}(\mu)/G$ will not be realizable as a stratified fiber bundle over Q/G. The second problem is more delicate, and is related with the appearance of twisting terms in the free theory. Since in the presence of singularities the action $G\times Q\to Q$ does not define a principal fiber bundle, it is not clear how to define a connection in order to obtain the twisting term present in the reduced symplectic form.

The first problem was addressed in [21], which studies the topological and fibered structure of the reduced space $\mathbf{J}^{-1}(\mu)/G$. The main properties that will be relevant for the present exposition are collected in the following result.

Proposition 2 ([21])**.** *Let the Lie group* G *act properly on the manifold* Q *and on* T^*Q *by cotangent lifts with associated momentum map given by* (1)*. Let* $\mu\in\mathfrak{g}^*$ *be a value of* $\mathbf{J}$ *satisfying* $G_\mu = G$*. Then*

(i) $\tau(\mathbf{J}^{-1}(\mu)) = Q^\mu$*, where*

$$Q^\mu = \{z\in Q : \mu\in(\mathfrak{g}_z)^\circ\}.$$

Here $(\mathfrak{g}_z)^\circ\subset$ *denotes the annihilator of* $\mathfrak{g}_z$*, the Lie algebra of the stabilizer of* z.

(ii) *The partition*

$$Q^\mu/G = \bigsqcup_{H\subset G,\,\mu\in\mathfrak{h}^\circ} Q_{(H)}/G, \tag{7}$$

with $\mathfrak{h} = \mathrm{Lie}(H)$*, induces a stratification, called the* μ*-orbit type stratification of* Q^μ/G.

Notice that it follows from (ii) that Q^μ/G with the μ-orbit type stratification is then a stratified subset of Q/G endowed with the orbit type stratification defined in (4), in the sense that the canonical inclusion $Q^\mu/G\to Q/G$ is a stratified morphism.

For the second problem we will develop a basic exposition of the theory of singular connections introduced in [16]. Let the Lie group G act properly on Q and consider the singular vector bundle $p : \nu \to Q$ defined by

$$\nu = \bigcup_{z \in M} \mathfrak{g}/\mathfrak{g}_z.$$

We will denote the elements of ν as classes $(z,[\xi])$, where $\xi \in \mathfrak{g}$ and $p(z,[\xi]) = z$. There is a natural action of $G \times \nu \to \nu$ covering the G-action on Q, and given by

$$g \cdot (z,[\xi]) = (g \cdot z, [\mathrm{Ad}_g \xi]) \in p^{-1}(g \cdot x).$$

Notice that this action is well defined since for every $\eta \in \mathfrak{g}_z$ we have $\mathrm{Ad}_g \eta \in \mathfrak{g}_{g \cdot z}$.

Next, notice that given a compact subgroup $H \subset G$, for every $z \in Q_{(H)}$, the stabilizer G_z is conjugate to H in G, and therefore $\mathfrak{g}_z$ belongs to the same adjoint orbit as $\mathfrak{h}$ in $\mathfrak{g}$. This fact, together with a straightforward application of Palais' Tube Theorem allows us to conclude that the restrictions $\nu\big|_{Q_{(H)}}$ are smooth subbundles of ν.

Definition 1. *Let the Lie group G act properly on Q. A singular connection for this action is a continuous surjective bundle map*

$$\mathscr{A} : TQ \to \nu$$

covering the identity, and satisfying

(i) *For each compact subgroup $H \subset G$ the restriction $\mathscr{A}^H : TQ\big|_{Q_{(H)}} \to \nu\big|_{(H)}$ is a surjective submersion.*

(ii) *$\mathscr{A}$ is G-equivariant: $\mathscr{A}(g \cdot v) = g \cdot \mathscr{A}(z)$, for every $v \in TM$, $g \in G$.*

(iii) *For all $\xi \in \mathfrak{g}$, $\mathscr{A}(\xi_M(z)) = (z,[\xi])$.*

It is possible to show that singular connections always exist for proper Lie group actions as a consequence of the existence of G-invariant partitions of unity. Once chosen a singular connection on Q and fixing an element $\mu \in \mathfrak{g}^*$ satisfying $G_\mu = G$, for each $H \subset G$ satisfying $\mu \in \mathfrak{h}^\circ$ we can define a one-form $\alpha_\mu^{(H)}$ on $Q_{(H)}$ in the following way: if $v \in T_z Q_{(H)} \subset T_z Q$ and $\mathscr{A}(v) = (z,[\xi])$, then

$$\alpha_\mu^{(H)}(v) = \langle \mu, \xi \rangle.$$

Notice that α_μ is well defined precisely when $\mu \in \mathfrak{h}^\circ$ and $G_\mu = G$, since if $(x,[\xi'])$ is a different representative of $\mathscr{A}(v)$, then $\xi' = \xi + \mathrm{Ad}_g \eta$ for some $\eta \in \mathfrak{h}$. Therefore

$$\langle \mu, \xi' \rangle - \langle \mu \xi \rangle = \langle \mu, \mathrm{Ad}_g \eta \rangle = \langle \mathrm{Ad}^*_{g^{-1}} \mu, \eta \rangle = \langle \mu, \eta \rangle = 0.$$

Similar arguments show that this form is G-invariant and that it induces a two-form $B_\mu^{(H)}$ on each μ-orbit type stratum of Q^μ in the following way. Let $\pi_{(H)} : Q_{(H)} \to Q_{(H)}/G$ be the group projection. Both $Q_{(H)}/G$ and $\pi_{(H)}$ are smooth manifolds by the exposition in Section 2. Then define $B_\mu^{(H)}$ as the unique two-form on $Q_{(H)}/G$ satisfying

$$\mathbf{d}\alpha_\mu^{(H)} = \pi_{(H)}^* B_\mu^{(H)}.$$

We are now in a position to generalize Theorem 2 to the case of totally isotropic momentum.

Theorem 3 ([20])**.** *Let the Lie group G act properly on the manifold Q and on T^*Q by cotangent lifts with associated momentum map given by (1), and let $\mu \in \mathfrak{g}^*$ be a value of* $\mathbf{J}$ *satisfying $G_\mu = G$. Then the partition*

$$\mathbf{J}^{-1}(\mu)/G = \bigsqcup_{H \subset K,\, \mu \in \mathfrak{k}^\circ} S^\mu_{K \to H} \tag{8}$$

where

$$S^{\mu}_{K\to H} = \frac{\mathbf{J}^{-1}(\mu)\cap(T^*Q)_{(H)}\cap T^*Q\big|_{Q_{(K)}}}{G},$$

induces a stratification of the reduced space $\mathbf{J}^{-1}(\mu)/G$ *satisfying:*

(i) *If* Q^{μ}/G *is equipped with the* μ*-orbit type stratification* (7) *then the naturally induced continuous projection*

$$\tau^{\mu}: \mathbf{J}^{-1}(\mu)/G \to Q^{\mu}/G$$

is a stratified fibration, satisfying $\tau^{\mu}(S^{\mu}_{K\to H}) = Q_{(K)}/G$*, for any pair* $H\subset K$ *with* $\mu\in\mathfrak{k}^{\circ}$.

(ii) *If* $H\subset K$ *and* $\mu\in\mathfrak{k}^{\circ}$*, then* $S^{\mu}_{K\to H}$ *is a coisotropic submanifold of* $((\mathbf{J}^{-1}(\mu)\cap(T^*Q)_{(H)})/G,\omega^{(H)}_{\mu})$.

(iii) *In particular, if* $H=K$ *and* $\mu\in\mathfrak{h}^{\circ}$ *then* $S^{\mu}_{H\to H}$ *is an open and dense symplectic submanifold of* $((\mathbf{J}^{-1}(\mu)\cap(T^*Q)_{(H)})/G,\omega^{(H)}_{\mu})$*, symplectomorphic to* $(T^*(Q_{(H)}/G),\ \omega_{Q_{(H)}/G}-\tau^*_{(H)}B^{(H)}_{\mu})$*, where* $\tau_{(H)}: T^*(Q_{(H)}/G)\to Q_{(H)}/G$ *is the cotangent bundle projection.*

The stratification induced by (6) *is called the coisotropic stratification of* $\mathbf{J}^{-1}(\mu)/G$*, and the manifolds* $S^{\mu}_{K\to H}$ *the coisotropic strata.*

Similar comments to those following Theorem 2 about the adequacy of the coisotropic stratification and its compatibility with all the geometric structures present in the problem also apply to the case of non-zero totally isotropic momentum.

5. Final remarks

We conclude this article with some thoughts about the current status of the program of singular cotangent bundle reduction. For general proper actions $G\times Q\to Q$, the only results existing so far correspond to the cases $\mu=0$ ([18]), $G_{\mu}=G$ ([20]) and the Sternberg picture of the singular Poisson reduced space ([17]). For the cases of the embedding picture at general μ, and the Weinstein and Sternberg realizations of the singular Poisson reduced space, at this moment there are only partial results corresponding to particular cases of Lie group actions on Q presenting only one orbit type. These results were obtained respectively in [14], [7] and [16]. One of the main objectives of the program of singular cotangent bundle reduction consists precisely in obtaining descriptions of the various reduced spaces involved in these cases for general proper actions.

Another important topic not discussed here is the study of the local properties of the resulting stratified reduced spaces. For the case of proper Hamiltonian Lie group actions on general symplectic manifolds, it is proved in [24] that the reduced space $\mathbf{J}^{-1}(0)/G$, equipped with the symplectic stratification is an example of a locally trivial conical Whitney space. This has been later generalized to arbitrary momentum values in [3] and [15]. The main tool employed to this end is the local model for the neighborhood of a group orbit of a Hamiltonian action given by the Hamiltonian tube theorem of Marle, Guillemin and Sternberg ([6] and [10]). This local part of the theory has not been so far successfully applied to singular cotangent bundle reduction, and to determine if the new coisotropic stratifications enjoy the same local properties as the symplectic one, is still an open problem. The main reason for this is that the construction of the Hamiltonian tube is not fine enough to take into account the fibered structure present in the case of a cotangent lifted Hamiltonian action. In [23] a Hamiltonian tube specially adapted to cotangent lifted actions has been obtained for some particular types of orbits, but the general case is still unsolved. This is the main obstacle preventing the study of the local properties of the coisotropic stratifications of Theorems 2 and 3.

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UNIVERSIDAD POLITÉCNICA DE CATALUÑA
E-mail: `miguel.rodriguez.olmos@upc.edu`