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THE HOMOTOPY ANALYSIS METHOD IN THE SEARCH FOR PERIODIC ORBITS

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ABSTRACT. The Homotopy Analysis Method (HAM) is a recently developed technique designed to solve differential equations and of other types. In the first part of these notes a review of some classical results on periodic orbits is given. Then the implementation of HAM to solve these problems is shown. Finally, some results for delay differential equations are shown.

1. INTRODUCTION

These notes contain the material of the lecture given by the author at the XII Congress Dr. Antonio Monteiro. The talk is about the application of the Homotopy Analysis Method (HAM) for finding periodic orbits in dynamical systems modeled by ordinary differential equations and differential equations with delay.

The search and study of the periodic orbits in dynamical systems is of continuing importance since its inception. This occurs both from the theoretical and practical standpoint. Just mention the index theory, the theory of Poincaré–Bendixon or the famous sixteenth Hilbert problem to check the enormous impact of this issue in the field of mathematics and its applications ([24, 15, 12]).

The need to find explicit expressions of the orbits has motivated the development of several methods, including the HAM initially developed by Shijun Liao [18, 19, 18]. The HAM has been used to find periodic orbits in various situations. For instance, to approximate the limit cycle in the van der Pol equation [9], to find solutions for the mKdV equation [27] or cycles around a center [7].

In the search for periodic orbits with the Homotopy Analysis Method a series solution is constructed in the spirit of the Poincaré–Lindstedt method [22, 14, 26]. The successive terms of the solution are found by imposing conditions that ensure the cancellation of the secular terms in the solutions of a properly chosen linear operator. This procedure is summarized in [7].

The organization is as follows. Some classical elementary results on periodic orbits are shown in section 2. In section 3 the Poincaré–Lindstedt method, perhaps the closest antecedent of HAM, is described. Then the HAM, as used in the search for periodic orbits, is shown. Finally, two examples are given: one is the simple pendulum and the other is a version of the van der Pol equation to which a linear term with delay is added.

2. PERIODIC ORBITS

A dynamical system is essentially a state space where time evolution occurs. It can be seen as the action of a uniparametric group (or semigroup) on a space, usually a differentiable manifold. In these notes we consider dynamical systems modeled by ordinary

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differential equations of the type

$$x' = f(x,t), \qquad \text{with} \quad x(t) \in D \subset \mathbb{R}^n, \quad f\colon D \times \mathbb{R} \to \mathbb{R}^n.$$

Here the evolution is given by the flow of the vector field f.

A periodic solution is a solution of the above equation for which there is a $T > 0$ such that $x(t+T) = x(t)$ for all t. The smallest T with this property is the period.

Various criteria have been established (especially in the bi-dimensional case, see [12, 15]) to determine the existence of periodic orbits. Here are some of them.

Gradient fields. If the field is a gradient $x' = -\operatorname{grad} V(x)$, then it is easy to see that a periodic orbit could not exist.

Dulac criterion. Given by the following theorem.

Theorem 1. *Let R be a simply connected region in* $\mathbb{R}^2$ *and consider the following system in R*

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y), \end{aligned}$$

where f *and* g *are* C^1 *functions. Suppose that there is a* C^1 *function,* $h(x,y)$ *such that* $\operatorname{div}(hf, hg)$ *has a definite sign in R. Then the system has no periodic orbits in R.*

In the particular case where $h = 1$ it is known as Bendixon criterion.

Lyapunov function criterion. If there is a monotonically decreasing function along the orbits, then the system has no periodic orbits.

Index Theory. We can say informally that the index of a curve in a vector field is the number of times the field is rotated counterclockwise along the curve. The index of an equilibrium is the index of a curve arbitrarily close to the equilibrium that enclose no other equilibrium. Writing I_C or I_x for the index of a curve C or an equilibrium x we have the following lemma

Lemma 1. *If a closed curve C encloses n fixed points* $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$*, then*

$$I_C = I_{\bar{x}_1} + \cdots + I_{\bar{x}_n}.$$

From this lemma we draw some conclusions, for example:

- Any closed orbit in the plane must enclose equilibria whose indices add up $+1$.
- In particular, if the field has no equilibria, then there cannot exist periodic orbits.
- If a periodic orbit contains a unique equilibrium, then it can not be hyperbolic.

Poincaré–Bendixon theorem. Another important result is the theorem of Poincaré–Bendixon which essentially describes the periodic attractors in the plane. It can be stated as follows

Theorem 2. *Let R be a closed and bounded region of the plane. Consider the system* $x' = f(x)$*, where* f *is at least* C^1*. Suppose that R contains no equilibria of* f*. Assume further that there is an orbit,* γ*, of* f *that remains in R for all* t*. Then* γ *is either a closed orbit or it asymptotically approaches a closed orbit, that is a limit cycle exists in R.*

A periodic orbit may belong to various dynamical scenarios. For example it may be isolated, then it is called a limit cycle. Figure 1 shows the limit cycle of the van der Pol equation, perhaps the best known example [25, 11]. Other examples of limit cycles are emerging cycles from a a Hopf bifurcation [12, 15].

Another usual type of periodic orbits are located at centers. Here there is a continuum of concentric cycles around a non hyperbolic equilibrium. This name is usually reserved for the case of conservative systems or bi-dimensional systems. In Figure 2 the phase portrait

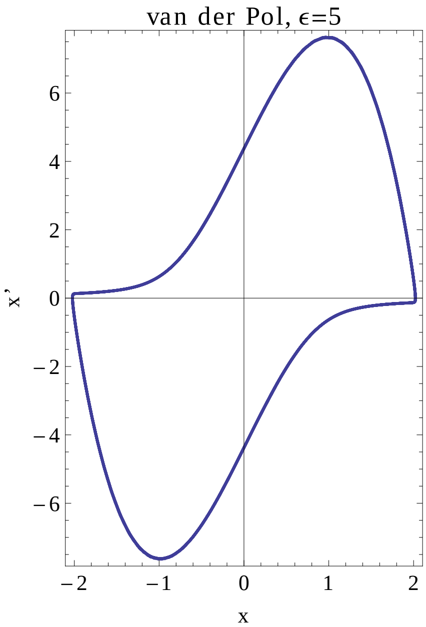

FIGURE 1. Limit cycle of the van der Pol equation with $\varepsilon = 5$.

of a simple pendulum is shown. To the left is the usual representation in the plane θ-θ' and on the right a 3-dimensional representation on the cylinder, which is the manifold where the dynamics actually occurs.

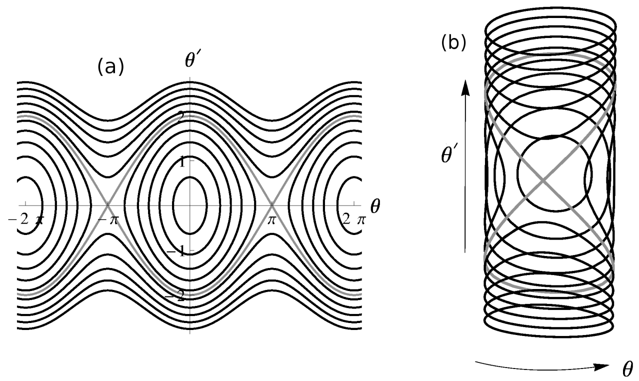

FIGURE 2. Phase portrait of a simple pendulum. (a): θ-θ', plane, (b) in the cylinder.

A more complex scenario is shown in the case of a chaotic attractor. For example, the most famous of all, the Lorenz attractor represented in phase space in Figure 3 left. In a neighborhood of the attractor there is a dense distribution of unstable periodic orbits; one of them is shown on the right in the same figure. These orbits are interpreted as knots. The article [10] highlights the remarkable relationship between the periodic orbits of the Lorenz attractor and the periodic orbits of the modular flow in the space of lattices (as appearing in number theory). In the article are also displayed multiple animations of these wonderful mathematical objects.

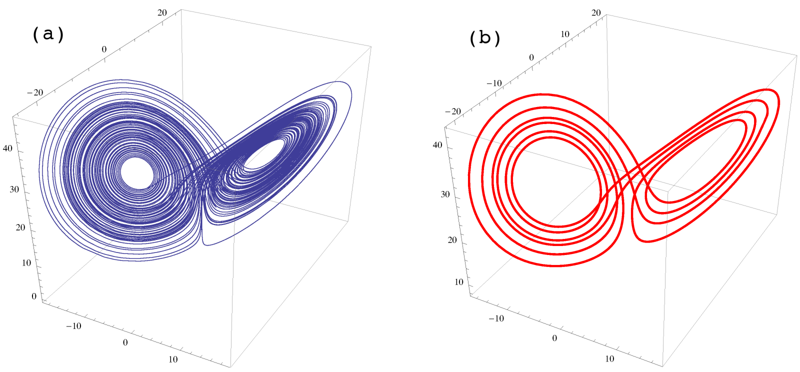

FIGURE 3. (a): Lorenz attractor, (b) periodic orbit in the vicinity of the attractor

3. POINCARÉ–LINDSTEDT METHOD

This section describes the method of Poincaré–Lindstedt as presented in [26] (see also reference [14]). The method is based on the Poincaré expansion theorem which is stated later ([22, 26]). We consider the initial value problem

$$x' = f(t,x,\varepsilon), \quad x(t_0) = \eta,$$

where it is assumed that $f(t,x,\varepsilon)$ can be expanded in a convergent Taylor series around ε in a certain domain. The unperturbed problem is

$$x_0' = f(t,x_0,0).$$

This problem has a periodic solution, $x_0(t)$, with initial condition $x_0(t_0)$. We assume that the solution for $\varepsilon \neq 0$ has initial condition

$$x(t_0) = x_0(t_0) + \mu,$$

with constant μ. Setting $x(t) = y(t) + x_0(t)$ we obtain

$$y' = F(t,y,\varepsilon), \quad y(t_0) = \mu,$$

where $F(t,y,\varepsilon) = f(t,y+x_0(t),\varepsilon) - f(t,x_0(t),0)$. The next theorem shows that there exist solutions in series around $\varepsilon = 0$.

Theorem 3 (Poincaré expansion theorem)**.** *We consider the initial value problem* $y' = F(t,y,\varepsilon)$, $y(t_0) = \mu$, *with* $|t - t_0| \leq d$, $y \in D \subset \mathbb{R}^n$, $0 \leq \varepsilon \leq \varepsilon_0$, $0 \leq \mu \leq \mu_0$. $F(t,y,\varepsilon)$ *continuous in* t, *and* ε. *It can also be expanded in convergent power series with respect to* y *and* ε *for* $||y|| \leq \rho$, $0 \leq \varepsilon \leq \varepsilon_0$, *then* $y(t)$ *can be expanded in convergent power series with respect to* ε *and* μ *in a neighbourhood of* $\varepsilon = \mu = 0$, *convergent on the time-scale* 1.

The time-scale 1 means that the solution is valid for small time, independent of ε. Below the conditions under which the solutions for $\varepsilon \neq 0$ are periodic are shown.

3.1. **Periodicity conditions.** Consider the equation

$$x'' + x = \varepsilon f(x,x',\varepsilon), \tag{1}$$

where $\varepsilon > 0$ and $(x,x') \in D \subset \mathbb{R}^2$. If $\varepsilon = 0$ then the solutions are periodic with period 2π. We assume that there are periodic solutions for small ε. Suppose that in D and $0 \leq \varepsilon \leq \varepsilon_0$

the requirements of the expansion theorem of Poincaré are satisfied. Setting $T = T(\varepsilon)$, $x(0) = a(\varepsilon)$ and $x'(0) = 0$ then the expansion theorem of Poincaré gives

$$\lim_{\varepsilon\to 0} x(t,\varepsilon) = a(0)\cos t.$$

on time-scale 1.

Calling $\omega t = \theta$, and $\omega^{-2} = 1-\varepsilon\eta(\varepsilon)$ the equation (1) is written as

$$\begin{aligned} x''+x &= \varepsilon\left(\eta x+(1-\varepsilon\eta)f(x,(1-\varepsilon\eta)^{-1}x',\varepsilon)\right)\\ &= \varepsilon g(x,x',\varepsilon,\eta), \end{aligned}$$

with initial conditions $x(0) = a(\varepsilon)$ and $x'(0) = 0$. The solution can be obtained with the following formula

$$x(\theta) = a\cos(\theta)+\varepsilon\int_0^{\theta}\operatorname{sen}(\theta-\tau)g(x(\tau),x'(\tau),\varepsilon,\eta)\,d\tau.$$

If this solution is periodic then it must verify $x(\theta) = x(\theta+2\pi)$, then

$$\int_{\theta}^{\theta+2\pi}\operatorname{sen}(\theta-\tau)g(x(\tau),x'(\tau),\varepsilon,\eta)\,d\tau = 0.$$

Equivalently

$$\begin{aligned} \int_0^{2\pi}\operatorname{sen}(\tau)g(x(\tau),x'(\tau),\varepsilon,\eta)\,d\tau &= 0\\ \int_0^{2\pi}\cos(\tau)g(x(\tau),x'(\tau),\varepsilon,\eta)\,d\tau &= 0. \end{aligned}$$

In particular, for $\varepsilon = 0$

$$\begin{aligned} \int_0^{2\pi}\operatorname{sen}(\tau)f(a(0)\cos(\tau),-a(0)\operatorname{sen}(\tau),0)\,d\tau &= 0\\ \pi\eta(0)a(0)+\int_0^{2\pi}\cos(\tau)f(a(0)\cos(\tau),-a(0)\operatorname{sen}(\tau),0)\,d\tau &= 0. \end{aligned}$$

This system of nonlinear equations gives the values $a(0)$ and $\eta(0)$ that generate the possible periodic solutions. The solutions of these equations correspond to the cancellation of the so-called secular terms in the solution of the original equations, as will be seen in the example below. In particular, the condition for unique solution is

$$\frac{\partial(F_1,F_2)}{\partial(a,\eta)}\neq 0$$

giving the condition

$$a(0)\int_0^{2\pi}\left(\frac{1}{2}\operatorname{sen}(2\tau)\frac{\partial f}{\partial x}(a(0)\cos(\tau),-a(0)\operatorname{sen}(\tau),0)\,d\tau \right. \\ \left. -\operatorname{sen}^2(\tau)\frac{\partial f}{\partial y}(a(0)\cos(\tau),-a(0)\operatorname{sen}(\tau),0)\right)\,d\tau\neq 0.$$

3.2. **Example: van der Pol.** Hereinafter the application of the Poincaré–Lindstedt method to the van der Pol equation is shown. We consider the equation

$$x'' + \varepsilon(x^2 - 1)x' + x = 0.$$

A new variable $\theta = \omega t$ is introduced so that the new period is 2π. The equation is written as

$$\omega^2 x'' + \varepsilon\omega(x^2 - 1)x' + x = 0.$$

Then we substitute the following expansions

$$x(\theta) = x_0(\theta) + \varepsilon x_1(\theta) + \cdots$$
$$\omega = \omega_0 + \varepsilon\omega_1 + \cdots.$$

Note that $\omega_0 = 1$. Linear differential equations are obtained for each n by considering powers of ε. The first three are

$$x_0'' + x_0 = 0$$
$$x_1'' + x_1 = -2\omega_1 x_0'' - (x_0^2 - 1)x_0'$$
$$x_2'' + x_2 = -(\omega_1^2 + 2\omega_2)x_0'' - 2\omega_1 x_1'' - (x_0^2 - 1)(x_1' + \omega_1 x_0') - 2x_0 x_1 x_0'.$$

The initial conditions are $x_0(0) = a$, $x_1(0) = x_2(0) = 0$, and $x_0'(0) = x_1'(0) = x_2'(0) = 0$.

These equations are solved using the freedom to choose the coefficients ω_i and a to eliminate the resonant (also called secular) terms. These are the terms corresponding to the first harmonic (frequency 1) and thus give rise to non-periodic solutions of the type $\theta\sin\theta$ or $\theta\cos\theta$.

Substituting $x_0(\theta) = a\cos\theta$ into the equation for x_1 gives

$$x_1''(\theta) + x_1(\theta) = 2a\omega_1\cos\theta - a\left(1 - \frac{a^2}{4}\right)\operatorname{sen}\theta + \frac{a^3}{4}\operatorname{sen}3\theta.$$

Setting $\omega_1 = 0$ and $a = 2$ gives the equation

$$x_1''(\theta) + x_1(\theta) = 2\operatorname{sen}3\theta.$$

The solution is $x_1(\theta) = (3\sin\theta - \sin 3\theta)/4$.

Replacing x_0 and x_1 into the equation for x_2 gives

$$x_2''(\theta) + x_2(\theta) = \left(4\omega_2 + \frac{1}{4}\right)\cos\theta - 6\cos 3\theta + 5\cos 5\theta.$$

Then we choose $\omega_2 = -1/16$. The result up to order 2 is

$$x(\theta) = \frac{1}{96}\big(192\cos\omega\theta + \varepsilon(72\sin\omega\theta - 24\sin 3\omega\theta) - \varepsilon^2(13\cos\omega\theta + 18\cos 3\omega\theta - 5\cos 5\omega\theta)\big),$$

where $\omega = 1 - \varepsilon^2/16$. In the Figure 4 the solution of order 2 is shown as a function of time. A numerical solution is also shown. It should be noted the good match for a value of $\varepsilon = 0.9$.

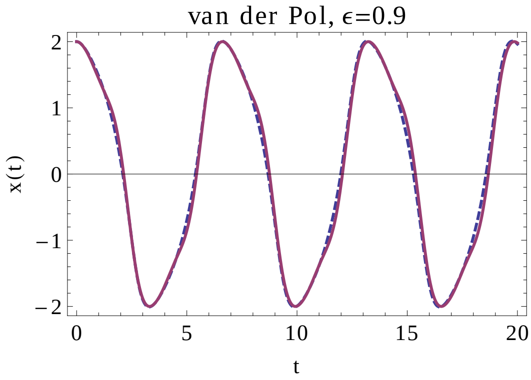

FIGURE 4. solid line: Poincaré–Lindstedt, order 2; dashed line: numerical solution.

4. THE HOMOTOPY ANALYSIS METHOD

The HAM was proposed in Liao's thesis in 1992. It is an analytical technique applied to solving nonlinear ordinary and partial differential equations (and of other types). It basically consists in a continuous deformation of the solution of a known linear problem to obtain the solution of the nonlinear problem. The solution is expressed as a series of functions in a given base.

Among the methods created to solve nonlinear differential equations we mention perturbative methods ([14, 21]), which depend on the existence of small or large parameters such as the Poincaré–Lindstedt method. Other methods are the Lyapunov small artificial parameter method, the Adomian decomposition method, the δ-expansion method or the HAM. The last mentioned method may be considered, to some extent, a generalization of the above mentioned, especially the Poincaré–Linsdstedt method, with which it has many features in common. It has experienced a major development today (see for example [1, 5, 7, 16, 19, 17, 18, 20]).

4.1. **Description of the method.** We consider the differential equation

$$y' = f(y,s), \quad \text{with } y(s) \in \mathbb{R}^n, \;\; f : D \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n.$$

Suppose that the system has a periodic orbit of frequency ω and amplitude a. After making the replacements $t = \omega s$ and $y = ax$, the normalized equation becomes

$$\omega a x' = f(ax, t/\omega), \tag{2}$$

In the new variables the equation has a solution of amplitude and frequency 1. We write (2) as $N[x,\omega,a] = 0$, with initial conditions $x(0) = a$, $x'(0) = 0$. In the general case we write

$$N[x, g_1, g_2, \dots, g_m] = 0,$$

where $x(t) \in \mathbb{R}^n$ and $g_i \in \mathbb{R}$, $i = 1,\dots,m$, are constants to be determined. The frequency is always among them. Besides we have the initial conditions.

To find the periodic solution $x_P(t)$, a family of operators, dependent on the deformation parameter $q \in [0,1]$, is constructed. The family is written as

$$\mathscr{H}_q[\phi] = (1-q)\, \mathscr{L}[\phi - x_0] - q\, h\, \mathscr{N}_q[\phi],$$

where $\phi(t,q)$ is the homotopy, $h \neq 0$ is a real parameter, $x_0(t)$ is an initial approximation that verify initial conditions and $\mathscr{L}$ is a suitably chosen linear operator. Finally $\mathscr{N}_q$ is the operator

$$\mathscr{N}_q[\phi] = N[\phi(t,q), \gamma_1(q), \gamma_2(q), \ldots, \gamma_m(q)].$$

We search for analytical functions $\phi(t,q), \gamma_1(q), \ldots, \gamma_m(q)$, such that

i. $\mathscr{H}_q[\phi] = 0$ for all $q \in [0,1]$,
ii. $\phi(t,q)$ verifies the initial conditions for $q \in [0,1]$.

If these functions exist then taking $q = 0$ and $q = 1$ we have

$$\mathscr{H}_0[\phi] = \mathscr{L}[\phi(t,0) - x_0(t)] = 0 \quad \text{and} \quad \mathscr{H}_1[\phi] = -h\mathscr{N}_1[\phi(t,1)] = 0.$$

Then $x_P(t) = \phi(t,1)$, $g_1 = \gamma_1(1), \ldots, g_m = \gamma_m(1)$ is the solution.

For finding the functions $\phi(t,q), \gamma_1(q), \ldots, \gamma_m(q)$ we consider its series expansions

$$\phi(t,q) = \sum_{k=0}^{+\infty} x_k(t) q^k, \quad \gamma_1(q) = \sum_{k=0}^{+\infty} g_{1k} q^k, \quad \ldots \quad , \gamma_m(q) = \sum_{k=0}^{+\infty} g_{mk} q^k.$$

It is explicitly assumed that the successive $x_i(t)$ are generated by certain base of functions $B = \{\beta_1, \beta_2, \ldots\}$. For example trigonometric functions in the search for periodic solutions. Replacing the above expressions in $\mathscr{H}_q[\phi] = 0$, and taking the k-th derivative with respect to q at $q = 0$ we obtain for $k = 1, 2, \ldots$

$$\mathscr{L}[x_k(t) - (1 - \delta_{1k}) x_{k-1}(t)] = \frac{h}{(k-1)!} \frac{\partial^{k-1} \mathscr{N}_q[\phi]}{\partial q^{k-1}} \bigg|_{q=0}.$$

Considering that $\phi(t,0) = x_0(t)$ satisfies the initial conditions, then it should be imposed $x_k(0) = x_k'(0) = 0$ for $k \geq 1$.

The terms $x_k(t)$ are calculated by solving the equations with given initial conditions. We impose that each term be periodic. Depending on the linear operator, certain conditions must be verified to prevent the k-th term contains non-periodic functions (of the form $t \cos t$ or $t \operatorname{sen} t$). These conditions allow us to calculate the terms g_{ik}, $i = 1, \ldots, m$. For $k = 1$ we obtain a system of nonlinear equations with unknowns $g_{10}, \ldots, g_{m0}$, while for $k \geq 2$ the system is linear.

It remains to determine the value of h. The solutions thus obtained for g_i, $i = 1, \ldots, m$ and $x_P(t)$ are functions of h. When the order goes to infinity such functions converge to a value independent of h, for values of this parameter for which the series is convergent. Figure 5 (a) shows the so called h-curves where this behavior can be seen.

4.2. **Meaning of** h. A concise solution with the HAM can only be obtained in a few cases. Consider as an example the first order equation with initial condition (showed in Liao's book [18])

$$x' + x^2 = 1, \quad x(0) = 0.$$

This equation has no periodic solutions, however it is useful to clarify the role of h in the HAM. The solution can be obtained by direct integration. It is $x(t) = \tanh t$. The perturbative solution (for small t) is easily obtained

$$x(t) = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \cdots = \sum_{n=0}^{\infty} \alpha_n t^{2n+1}.$$

The radius of convergence of this series is $\pi/2$.

To obtain the solution by the HAM we choose the following linear operator $\mathscr{L} = \partial/\partial t$, the basis functions $B = \{t, t^3, t^5, \ldots\}$ and the initial solution $x_0(t) = t$. Thus, we obtain the following solution to order m

$$x(t) = \sum_{n=0}^{m} \mu_{m,n}(h)\, \alpha_n\, t^{2n+1},$$

where

$$\mu_{m,n}(h) = (-h)^n \sum_{j=0}^{m-n} \binom{n-1+j}{j} (1+h)^j.$$

This expression is called generalized Taylor expansion by Liao. The functions $\mu_{m,n}(h)$ have the property

$$\mu_{m,n}(-1) = 1 \quad \text{if} \quad n \le m \qquad \text{and} \qquad \lim_{m\to\infty} \mu_{m,n}(h) = \begin{cases} 1, & \text{if} \quad |1+h| < 1, \\ \infty & \text{if} \quad |1+h| > 1. \end{cases}$$

In [2] and [23] it was noted that the so-called generalized Taylor expansion seems to correspond to a shift in the point around which the development is done. As follows

$$\begin{aligned} f(t) &= \lim_{m\to\infty} \sum_{n=0}^{m} \frac{f^{(n)}(t_1)}{n!}(t-t_1)^n \\ &= \lim_{m\to\infty} \sum_{n=0}^{m} \frac{f^{(n)}(t_1)}{n!} \sum_{k=0}^{n} \binom{n}{k} (t-t_1)^k (t_0-t_1)^{n-k} \\ &= \lim_{m\to\infty} \sum_{n=0}^{m} \mu_{m,n}(f,t_0,t_1) \frac{f^{(n)}(t_0)}{n!}(t-t_0)^n, \end{aligned}$$

where

$$\mu_{m,n}(f,t_0,t_1) = \left(\frac{f^{(n)}(t_0)}{n!}\right)^{-1} \sum_{k=n}^{m} \frac{f^{(k)}(t_1)}{k!} \binom{k}{n} (t_0-t_1)^{k-n}.$$

For example taking $f(t) = 1/(1+t)$ we obtain

$$f(t) = \lim_{m\to\infty} \sum_{n=0}^{m} \mu_{m,n}(h)(-1)^n t^n.$$

This expression is the Taylor expansion around $t_0 = -1/h - 1$. With $-2 < h < 0$ we have a convergence region $1 < t < -1 + 2/|h|$. Thus, with proper choice of h, it is possible to increase the region of convergence of the solution.

The following result (which is shown in [2]) shows that by varying h it is expected to exist segments in which the Taylor series converges to the function.

Theorem 4. *Let $g\colon [a,b] \to \mathbb{R}$ continuous and $f\colon [a,b] \to \mathbb{R}$. Suppose that all the derivatives of f exist and are uniformly bounded, ie there is an $M \in \mathbb{R}$ such that*

$$\max_{t\in[a,b]} |f^{(k)}(t)| \le M \quad \textit{for all} \quad k.$$

Let $G_n(t,\alpha)$ be the Taylor polynomial of degree n for $f(t)$ around $\alpha \in (a,b)$. Suppose that $\alpha = g(h)$, then for every $\varepsilon > 0$ and $\gamma \in (a,b)$ there exists $n \in \mathbb{N}$ and an interval (c,d) such that for all $h \in (c,d)$ and $n \ge \mathbb{N}$

$$|f(\gamma) - G_n(\gamma, g(h))| < \varepsilon.$$

Corollary 1. *Suppose that $f(t)$ is sufficiently differentiable in $[a,b]$, $g(h)$ is continuous in $[a,b]$ and $G_n(t,g(h))$ is the Taylor polynomial of degree n around $g(h)$. Then for all $\gamma \in (a,b)$ the function $G_n(\gamma, g(h))$ shows an horizontal region when n goes to infinity.*

5. EXAMPLES

This section shows some examples, which were presented in [6, 7]. The first is the simple pendulum. Despite its simplicity it has features that make it interesting. First, the non-linearity is not of polynomial type. Then, in this case it is necessary to use the formula of Jacobi-Anger [3] to write the equations. Furthermore the phase space is a cylinder and the usual angular coordinate is not suitable for analyzing rotating solutions. We found a suitable coordinate change to implement HAM in the search of these solutions.

The other example is a differential equation with delay ([13, 4, 8]). It is an infinite-dimensional dynamical system, however, the HAM can be applied to this case. It is particularly suitable for analysis of bifurcations as we briefly show in several cases.

5.1. **The simple pendulum.**

5.1.1. *Vibrations.* The equation of a simple pendulum is $\theta'' + \operatorname{sen}\theta = 0$. The application of the HAM to this system is studied in [7]. First we proceed to change variables, yielding

$$\omega^2 a\theta''(t) + \operatorname{sen}(a\theta(t)) = 0.$$

Now, the periodic solution $\theta_P(t)$ has unit amplitude and frequency. The initial conditions are $\theta_P(0) = 1$ and $\theta_P'(0) = 0$. We define

$$\begin{aligned}\mathcal{N}_q[\phi] &= N[\phi(t,q),\Omega(q),A(q)] \\ &= \Omega(q)^2 A(q)\frac{\partial^2\phi(t,q)}{\partial t^2} + \operatorname{sen}(A(q)\phi(t,q)).\end{aligned}$$

The linear operator is

$$\mathscr{L}[\phi] = \frac{\partial^2\phi}{\partial t^2} + \phi.$$

We take $\theta_0(t) = \cos t$. The equation for $k = 1$ is

$$\theta_1''(t) + \theta_1(t) = h(-\omega_0^2 a_0 \cos t + \operatorname{sen}(a_0 \cos t)),$$

with initial condition $\theta_1(0) = \theta_1'(0) = 0$. Then we obtain

$$\theta_1(t) = \frac{h}{a_0}\cos t(\cos a_0 - \cos(a_0\cos t)) + h\operatorname{sen} t\left(-\frac{1}{2}\omega_0^2 a_0 t + \int_0^t \cos s \operatorname{sen}(a_0 \cos s)ds\right).$$

To eliminate secular terms we apply the formulas of Jacobi–Anger, which are of the form

$$\cos(a\cos t) = 2\sum_{n=0}^{\infty}(-1)^n \mathrm{J}_{2\mathrm{n}}(a)\cos(2nt).$$

Then we obtain

$$\theta_1''(t) + \theta_1(t) = h\left(-\omega_0^2 a_0\cos t + 2\sum_{n=0}^{+\infty}(-1)^n J_{2n+1}(a_0)\cos((2n+1)t)\right),$$

giving

$$\omega_0 = \sqrt{\frac{2J_1(a_0)}{a_0}}.$$

Setting the value of a_0 the above equation allows us to find ω_0, and calculate $\theta_1(t)$. We obtain

$$\theta_1(t) = 2h\sum_{n=1}^{+\infty}(-1)^n\frac{J_{2n+1}(a_0)}{1-(2n+1)^2}\left(\cos((2n+1)t) - \cos t\right).$$

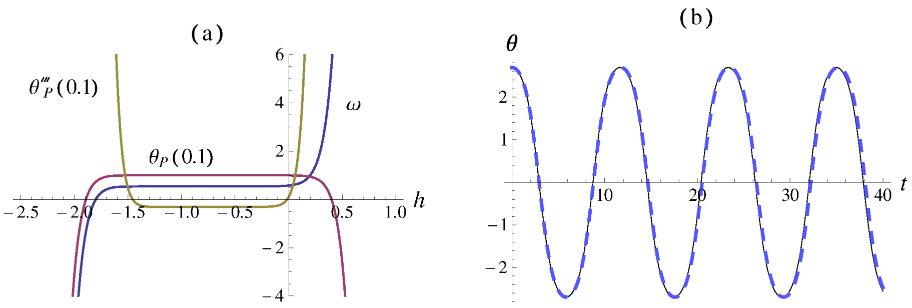

FIGURE 5. (a) h-curves for oscillations of the simple pendulum, (b) trajectory as a function of time: (---) exact, (—) HAM.

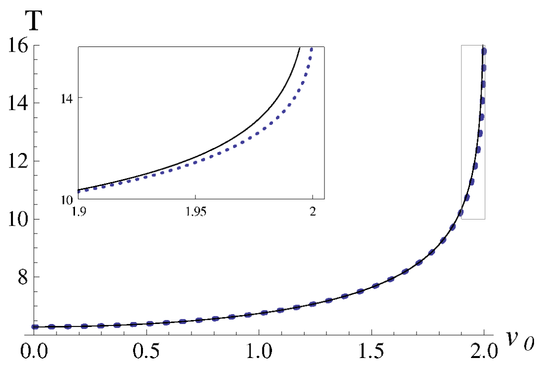

FIGURE 6. Period of the oscillations as a function of maximum velocity. (—) exact, $(\cdots)$ HAM.

The equation for $\theta_2(t)$ is

$$\theta_2''(t)+\theta_2(t)=\theta_1''(t)+\theta_1(t)$$
$$+h\left(a_0\omega_0^2\theta_1''(t)-(\omega_0^2a_1+2\omega_0\omega_1a_0)\cos t+(a_0\theta_1(t)+a_1\cos t)\cos(a_0\cos t)\right).$$

After replacing ω_0 and $\theta_1(t)$ we obtain $\theta_2(t)$. The process can be continued this way to high orders by using symbolic computation programs. For example in Figure 5 several h-curves and the trajectory for an initial velocity $\theta'(0)=1.95$ and order 15 are shown. Also in Figure 6 the obtained period compared with the exact one is shown. The coincidence is remarkable.

5.1.2. *Rotations.* In order to obtain rotational solutions we do the following coordinate transformation $u=e^{\theta'}\cos\theta$ and $v=e^{\theta'}\operatorname{sen}\theta$. In the new coordinates the equations of the pendulum are

$$u'=-uv\left(u^2+v^2\right)^{-1/2}-\frac{1}{2}v\ln(u^2+v^2)$$
$$v'=-v^2(u^2+v^2)^{-1/2}+\frac{1}{2}u\ln(u^2+v^2).$$

Assuming that there is a solution of frequency ω such that $(u(0),v(0)) = (e^{\xi},0)$ then, after a new change of coordinates we obtain

$$\omega u' = -uv\left(u^2+v^2\right)^{-1/2} - v\xi - \frac{1}{2}v\ln(u^2+v^2)$$
$$\omega v' = -v^2(u^2+v^2)^{-1/2} + u\xi + \frac{1}{2}u\ln(u^2+v^2).$$

Here we take the operator $\mathscr{L}$

$$\mathscr{L}[\phi_1,\phi_2] = \begin{pmatrix} \partial/\partial t & 1 \\ -1 & \partial/\partial t \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \partial\phi_1/\partial t + \phi_2 \\ -\phi_1 + \partial\phi_2/\partial t \end{pmatrix},$$

and $\mathscr{N}_q$ given by

$$\mathscr{N}_q[\phi_1,\phi_2] = N[(\phi_1(t,q),\phi_2(t,q)),\Omega(q),\Xi(q)] = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$$
$$= \begin{pmatrix} \Omega\,\partial\phi_1/\partial t + \phi_1\phi_2(\phi_1^2+\phi_2^2)^{-1/2} + \phi_2\Xi + \frac{1}{2}\phi_2\ln(\phi_1^2+\phi_2^2) \\ \Omega\,\partial\phi_2/\partial t + \phi_2^2(\phi_1^2+\phi_2^2)^{-1/2} - \phi_1\Xi - \frac{1}{2}\phi_1\ln(\phi_1^2+\phi_2^2) \end{pmatrix}.$$

The initial conditions we must take are $(u_0(t),v_0(t))^T = (\cos t, \operatorname{sen} t)^T$. The equations for $k=1$ are

$$u_1'(t) + v_1(t) = h(-\omega_0 \operatorname{sen} t + \cos t \operatorname{sen} t + \xi_0 \operatorname{sen} t)$$
$$-u_1(t) + v_1'(t) = h(\omega_0 \cos t + \operatorname{sen}^2 t - \xi_0 \cos t),$$

with initial conditions $u_1(0) = v_1(0) = 0$. The term $(u_1(t),v_1(t))^T$ is periodic if the coefficients of $\cos t$ and $\operatorname{sen} t$ vanish in the following expression

$$\left(\frac{\partial}{\partial t}N_1 - N_2\right)\Bigg|_{q=0} = 2(-\omega_0+\xi_0)\cos(t) + \frac{1}{2}(-1+3\cos(2t)),$$

namely the term is periodic if $\omega_0 = \xi_0$, and similar expressions for higher orders. Thus we obtain

$$u_1(t) = h(\cos(t) + \frac{1}{2}(-1-\cos(2t)))$$
$$v_1(t) = h(\operatorname{sen}(t) - \frac{1}{2}\operatorname{sen}(2t)).$$

The process can be continued to high orders by using symbolic computation programs as in the previous case. Several trajectories in phase space in both mentioned coordinate systems are shown in Figure 7. They correspond to order 10. In Figure 8 the obtained period compared with the exact one is also shown for order 10.

5.2. **Equation of van der Pol with a delayed feedback.** We consider the van der Pol equation with a delayed feedback as discussed in [6]

$$x''(t) + \varepsilon(x^2(t)-1)x'(t) + x(t) = d\varepsilon x(t-\tau).$$

After the change of variables the equation is written as

$$\omega^2 x''(t) + \varepsilon\omega(a^2x^2(t)-1)x'(t) + x(t) = d\varepsilon x(t-\omega\tau),$$

The HAM allows us to know very accurately the periodic orbits in this system. It allows us to detect and analyze bifurcations. The methodology consists of taking a line in the parameter space and finding periodic orbits on its points. Besides, the stability of these orbits can be studied by various methods. In this case we used a continuation method with Chebyshev polynomials.

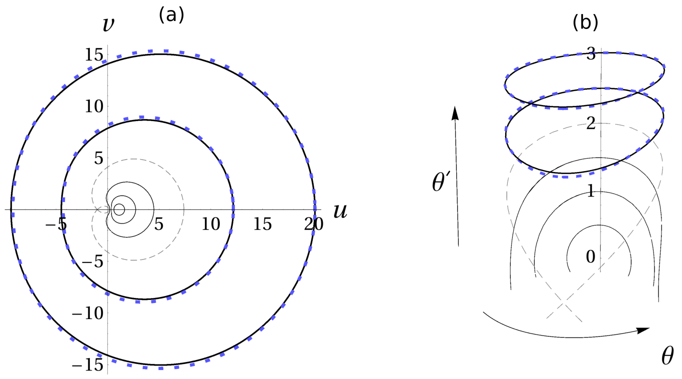

FIGURE 7. Rotations in phase space, (a): coordinates u-v, (b): in the cylinder with coordinates θ-θ'. (—) exact, $(\cdots)$ HAM.

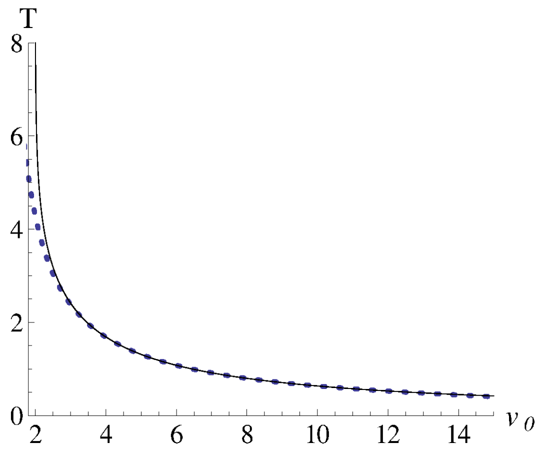

FIGURE 8. Period of the oscillations as a function of maximum velocity. (—) exact, $(\cdots)$ HAM.

As an example of this study we show the following bifurcations:

- 3 : 4 resonant double Hopf. For $\varepsilon = 0.139057$, $d = 2.22971$ and $\tau = 7.90083$. The eigenvalues are $\pm i1.125888$ and $\pm i0.844416$. Analysis of the cycles near the bifurcation allows us to find a Neimark-Sacker bifurcation (NS). This determines the appearance of $2D$ torus T_1 and T_2 shown in Figure 9. Also a 1 : 2 resonance is shown.
- Folds near double Hopf. For $\varepsilon = 0.5$, $\tau = 12.254248$ and $d = 1.511726$. The trivial equilibrium does not change its stability and cycles that appear are unstable. Figure 10 shows the curves of Hopf and folds in the plane of the parameters d-τ. It also compares the amplitudes obtained with the HAM with those calculated with the software PDECONT.
- 1 : 1 resonant double Hopf. For $\varepsilon = 0.254659$, $d = 7.85363$ and $\tau = 0.991860$ with $\omega = 1$. This situation is shown in Figure 11; the notation is the same as that of Figure 9.

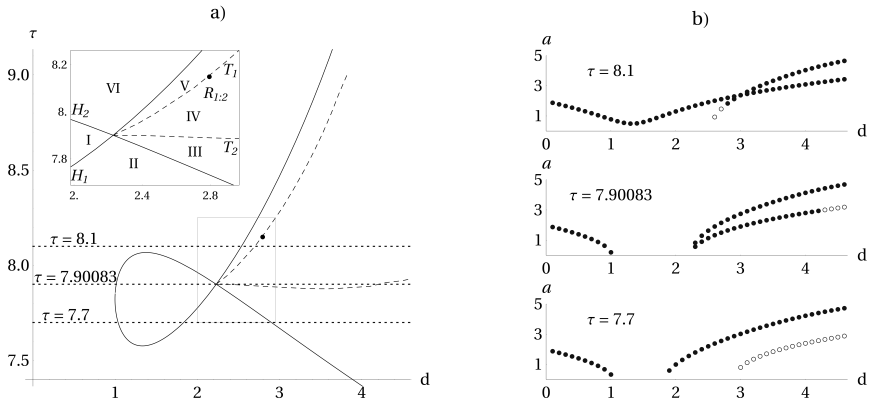

FIGURE 9. Neighborhood of a double Hopf point, $\varepsilon = 0.139057$. (a): Hopf bifurcation curves in the plane of the parameters d-τ and emerging branches from the $3:4$ resonance, (b) amplitudes of the periodic orbits corresponding to the indicated values of τ.

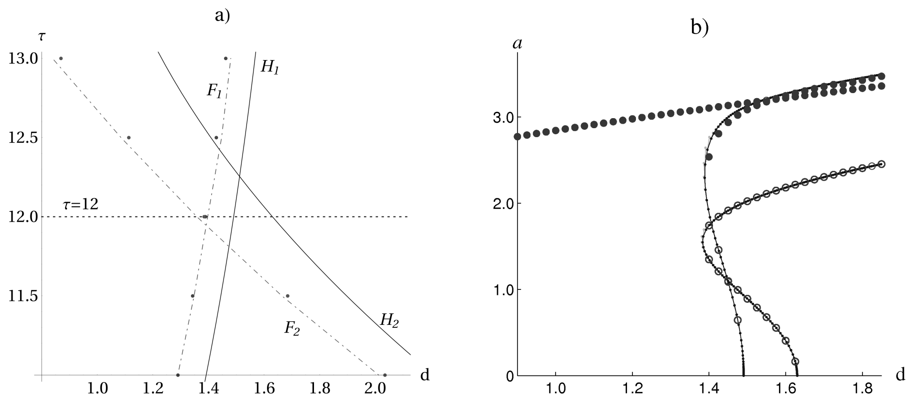

FIGURE 10. Neighborhood of a double Hopf point, $\varepsilon = 0.5$. (a) Hopf bifurcation curves in the plane of the parameters d-τ in the vicinity of the double Hopf point. Also fold curves F_1 y F_2 are shown, (b) amplitude of periodic curves corresponding to $\tau = 12.254248$. Continuous curves: PDECONT, points: HAM (full points: stable, hollow points: unstable).

- $1:4$ resonance on the curves T_1 and T_2, the eigenvalues are $e^{\pm i\pi/2}$. In these points the cycles change its stability. In this case the dynamics is very complex. See Figure 12.

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REFERENCES

[1] S. Abbasbandy. The application of homotopy analysis method to nonlinear equations arising in heat transfer. *Physics Letters A*, 360:109–113, 2006. MR 2288118.

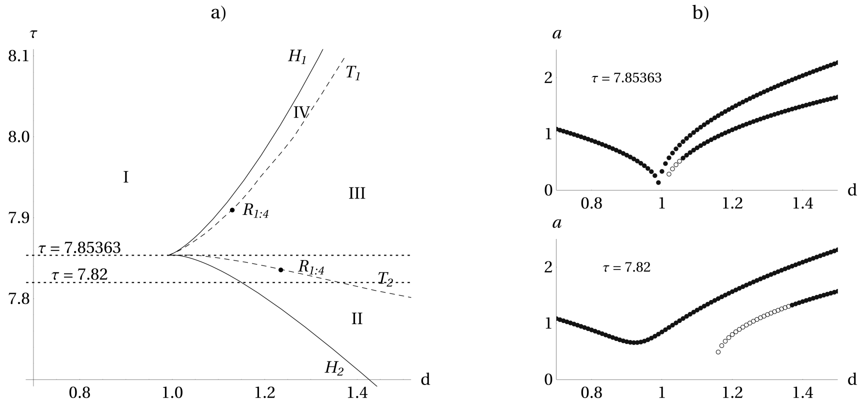

FIGURE 11. Neighborhood of a 1 : 1 resonant double Hopf, $\varepsilon = 0.254659$. (a): Hopf bifurcation curves in the plane of the parameters d-τ and emerging bifurcations from the 1 : 1 resonance, (b) amplitudes of the periodic orbits corresponding to the indicated values of τ.

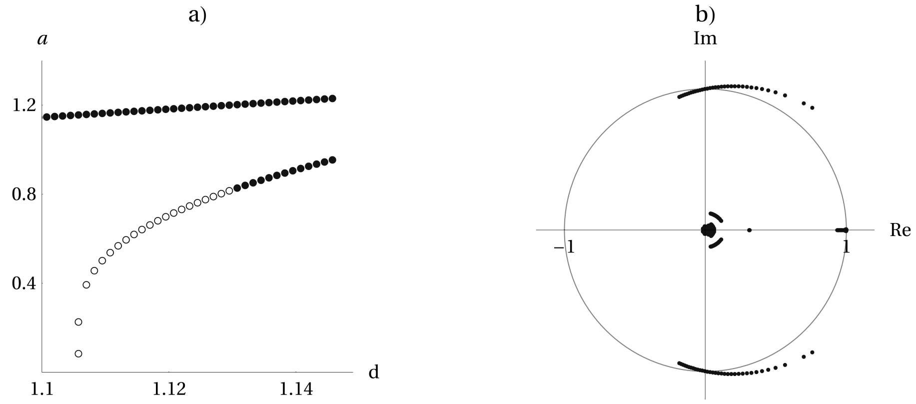

FIGURE 12. 1 : 4 resonance (a) amplitude of the cycles and stability across the resonance. (b) Floquet multipliers obtained by the Chebyshev method. They cross the unit circle when the parameter d increases.

[2] S. Abbasbandy, E. Shivanian, and K. Vajravelu. Mathematical properties of $\hbar$-curve in the frame work of the homotopy analysis method. *Commun Nonlinear Sci Numer Simulat*, 16:4268–4275, 2011. MR 2806738.

[3] Milton Abramowitz and Irene A. Stegun, editors. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, 1965.

[4] Balakumar Balachandram, Tamás Kalmár-Nagy, and David E. Gilsinn, editors. *Delay Differential Equations, Recent Advances and New Directions*. Springer, 2009. MR 2722942.

[5] A. Bel and W. Reartes. Bifurcaciones en la ecuación de van der Pol realimentada con retardo. In M. C. Maciel L. R. Castro and S. M. Castro, editors, *Anales del III Congreso de Matemática Aplicada, Computacional e Industrial, III MACI 2011*, pages 627–630. ASAMACI, mayo 2011.

[6] A. Bel and W. Reartes. The homotopy analysis method in bifurcation analysis of delay differential equations. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 22(8), 2012. MR 2975482.

[7] Andrea Bel, Walter Reartes, and Ana Torresi. Global study of the simple pendulum by the homotopy analysis method. *Eur. J. Phys.*, 33:231–241, 2012.

[8] Ed Bueler. Chebyshev collocation for linear, periodic, ordinary and delay differential equations: A posteriori estimates. *Preprint*, 2004. arXiv:math/0409464.

[9] Y. M. Chen and J. K. Liu. A study of homotopy analysis method for limit cycle of van der Pol equation. *Communications in Nonlinear Science and Numerical Simulation*, 14(5):1816–1821, 2009.

[10] Etienne Ghys and Jos Leys. Lorenz and modular flows: A visual introduction. Feature Column, American Mathematical Society website, 2006. `http://www.ams.org/samplings/feature-column/fcarc-lorenz`.

[11] Jean-Marc Ginoux and Christophe Letellier. Van der Pol and the history of relaxation oscillations: Toward the emergence of a concept. *Chaos*, 22, 2012. `http://dx.doi.org/10.1063/1.3670008`.

[12] John Guckenheimer and Philip Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, volume 42 of *Applied Mathematical Sciences*. Springer, 1983. MR 0709768.

[13] Jack K. Hale and Sjoerd M. Verduyn Lunel. *Introduction to Functional Differential Equations*, volume 99 of *Applied Mathematical Sciences*. Springer-Verlag, 1993. MR 1243878.

[14] J. Kevorkian and J. D. Cole. *Perturbation Methods in Applied Mathematics*. Springer-Verlag, 1968. MR 0608029.

[15] Yuri A. Kuznetsov. *Elements of Applied Bifurcation Theory*. Springer, 1995. MR 1344214.

[16] S. Liao and Y. Tan. A general approach to obtain series solutions of nonlinear differential equations. *Studies in Applied Mathematics*, 119:297–354, 2007. MR 2358643.

[17] Shijun Liao. An analytic approximate approach for free oscillations of self-excited systems. *International Journal of Non-linear Mechanics*, 39:271–280, 2004. MR 2009253.

[18] Shijun Liao. *Beyond Perturbation, Introduction to Homotopy Analysis Method*. Chapman & Hall/CRC, 2004. MR 2058313.

[19] Shijun Liao. On the homotopy analysis method for nonlinear problems. *Applied Mathematics and Computations*, 147:499–513, 2004. MR 2012589.

[20] Shijun Liao. Notes on the homotopy analysis method: some definitions and theorems. *Commun Nonlinear Sci Numer Simulat*, 14:983–997, 2009. MR 2468931.

[21] A. H. Nayfeh. *Introduction to Perturbation Techniques*. John Wiley & Sons, 1985. MR 0597894.

[22] H. Poincaré. *Les Méthodes Nouvelles de la Mécanique Céleste*, tome I. Solutions périodiques. Non-existence des intégrales uniformes. Solutions asymptotiques. Gauthier-Villars et fils, 1892.

[23] Cheng shi Liu. The essence of the homotopy analysis method. *Appl. Math. Comput.*, 216:1299–1303, 2010. MR 2607239.

[24] Steven H. Strogatz. *Nonlinear Dynamics and Chaos With Applications To Physics, Biology, Chemistry, and Engineering*. Westview Press, 1994.

[25] B. van der Pol. On relaxation-oscillations. *The London, Edinburgh and Dublin Phil. Mag. J. Sci.*, 2(7):978–992, 1927.

[26] F. Verhulst. *Nonlinear Differential Equations and Dynamical Systems*. Springer-Verlag, 1992. MR 1422255.

[27] Chun Wang, Yong-yan Wu, and Wan Wu. Solving the nonlinear periodic wave problems with the homotopy analysis method. *Wave Motion*, 41:329–337, 2005. MR 2123919.

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