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ON THE COMPOSITION OF IRREDUCIBLE MORPHISMS

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ABSTRACT. We study the relationship between compositions of irreducible morphisms and the powers of the radical of their module category, taking into account the degrees of such morphisms.

1. INTRODUCTION

These are the notes of the lecture given by the author at the XIII "Dr. Antonio Monteiro" Congress in Bahía Blanca, Argentina, concerning compositions of irreducible morphisms and the powers of the radical of their module categories.

We consider A to be a finite dimensional k-algebra over an algebraically closed field k, and $\mathrm{mod}A$ the category of finitely generated left A-modules.

The Jacobian radical of $\mathrm{mod}A$, denoted by $\Re(\mathrm{mod}A)$, is the ideal in $\mathrm{mod}A$ generated by all non-isomorphisms between indecomposable A-modules. The powers of the radical are defined inductively. By $\Re^\infty(\mathrm{mod}A)$ we denote the intersection of all powers $\Re^i(\mathrm{mod}A)$, with $i \geq 1$, of $\Re(\mathrm{mod}A)$.

Irreducible morphisms (subsection 2.2) play an important role in the representation theory of artin algebras. In [3], R. Bautista showed an important relation between an irreducible morphism and the powers of the radical of their module category. More precisely, the author proved that if $f\colon X \to Y$ is a morphism between indecomposable modules, then f is irreducible if and only if $f \in \Re(X,Y)\backslash\Re^2(X,Y)$. Hence, the composition of n irreducible morphisms between indecomposable A-modules belongs to $\Re^n(\mathrm{mod}A)$.

We are interested in the problem of deciding when the composition of n irreducible morphisms between indecomposable A-modules belongs to $\Re^{n+1}(\mathrm{mod}A)$. In order to give an answer to this problem, S. Liu defined the concept of left and right degree of an irreducible morphism (Definition 2.1). Lately, this concept has shown to be a very useful tool to solve many problems in the representation theory of artin algebras. In particular, for a finite dimensional algebra A over an algebraically closed field, it allows to determine if an algebra is of finite representation type knowing the degrees of some particular irreducible morphisms, see Theorem 2.9. In addition, if A is of finite representation type then this notion also allows to find the minimal vanishing power of the radical of a module category, see Theorem 2.11. Many other problems have been solved using degrees, see for example [7, 5, 9, 16, 17].

In [13], the authors gave an answer to the problem of when the non-zero composition of n irreducible morphisms $f_i\colon X_i \to X_{i+1}$ with $X_i \in \mathrm{ind}A$ for $i = 1,\dots,n$ belongs to $\Re^{n+1}(X_1,X_{n+1})$. More precisely, for such a solution they assume that $\dim_k \mathrm{Irr}(X_i,X_{i+1}) = 1$ for $i = 1,\dots,n$. To achieve the result they used covering techniques, see [13, Proposition 5.1]. Recently, in [14] the same authors proved that such mentioned result still holds whenever we consider A to be a finite dimensional k-algebra over a perfect field, without any assumptions on the dimension of the set of irreducible morphisms, see Theorem 2.6.

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The aim of these notes is to present a new approach to the problem of deciding when the composition of n irreducible morphisms between indecomposable A-modules belongs to $\Re^{n+1}(\mathrm{mod}A)$, taking into account the right and left degrees of such irreducible morphisms and without any assumptions on the compositions. These results and their proofs can be found in [10].

2. Notations and preliminary results

Throughout these notes, by an algebra we mean a finite dimensional basic k-algebra over an algebraically closed field k.

2.1. **Quivers and algebras.** A *quiver* Q is given by a set of vertices Q_0 and a set of arrows Q_1, together with two maps $s, e\colon Q_1 \to Q_0$, where for an arrow $\alpha \in Q_1$, $s(\alpha)$ is the starting vertex of α and $e(\alpha)$ is the ending vertex of α.

P. Gabriel proved that if A is an algebra then there exists a quiver Q_A, called the *ordinary quiver of* A, such that A is the quotient of the path algebra kQ_A by an admissible ideal.

Let A be an algebra. We denote by $\mathrm{mod}A$ the category of finitely generated left A-modules and by $\mathrm{ind}A$ the full subcategory of $\mathrm{mod}A$ which consists of one representative of each isomorphism class of indecomposable A-modules.

We say that an algebra A is of finite representation type or that A is a representation-finite algebra if there exists a finite number of isomorphism classes of indecomposable A-modules. Otherwise, we say that A is an algebra of infinite representation type.

For basic background on the representation theory of artin algebras we refer the reader to [1, 2].

2.2. **Irreducible morphisms and the radical of a module category.** A morphism $f\colon X \to Y$ with $X, Y \in \mathrm{mod}A$ is called *irreducible* provided it does not split and whenever $f = gh$, then either h is a split monomorphism or g is a split epimorphism.

Given $M \in \mathrm{ind}A$ then $\mathrm{End}(M)$ is a local ring. Moreover, it is known that the radical of the endomorphisms of M, which we denote by $\Re(\mathrm{End}(M))$, is of the set of non-isomorphisms.

For X and Y in $\mathrm{mod}A$ the above concept can be generalized as follows: $\Re(X,Y)$ is the set of all the morphisms $f\colon X \to Y$ such that, for each $M \in \mathrm{ind}A$, each $h\colon M \to X$ and each $h'\colon Y \to M$ the composition $h'fh \in \Re(\mathrm{End}(M))$.

The powers of $\Re(X,Y)$ are defined inductively. More precisely, $f \in \Re^n(X,Y)$ if and only if there exists $M_i \in \mathrm{mod}A$ such that $f = \Sigma_{i=1}^r h_i g_i$ with $g_i \in \Re(X, M_i)$ and $h_i \in \Re^{n-1}(M_i, Y)$. By $\Re^\infty(X,Y)$ we denote the intersection of all powers $\Re^i(X,Y)$ of $\Re(X,Y)$, with $i \geq 1$.

In particular, when X and Y are indecomposable A-modules, $\Re(X,Y)$ is the set of non-isomorphisms from X to Y.

It is not hard to see that if $f\colon X \to Y$ is an irreducible morphism with X or Y indecomposable A-modules then $f \in \Re(X,Y) \setminus \Re^2(X,Y)$. Moreover, $\Re^\infty(X,Y)$ is an ideal of the module category $\mathrm{mod}A$.

For X, Y indecomposable A-modules, we denote by $\mathrm{Irr}(X,Y)$ the quotient group $\Re(X,Y)/\Re^2(X,Y)$ and by k_X the division ring $\mathrm{End}(X)/\Re(X,X)$. It is known that $\mathrm{Irr}(X,Y)$ is a $k_Y - k_X^{op}$-bimodule. Moreover, if k is an algebraically closed field then $\mathrm{End}(X)/\Re(X,X) \simeq k$.

Finally, we observe that for $X, Y \in \mathrm{mod}A$ the descending chain

$$\mathrm{Hom}(X,Y) \supset \Re(X,Y) \supset \Re^2(X,Y) \supset \Re^2(X,Y) \supset \cdots \supset \Re^n(X,Y)$$

becomes stable. Hence, there exists a positive integer m such that $\Re^m(X,Y) = \Re^\infty(X,Y)$.

2.3. **The Auslander–Reiten quiver.** For an algebra A, the Auslander–Reiten quiver of $\mathrm{mod}A$ is a valued oriented graph denoted by Γ_A and defined as follows.

(a) For each indecomposable module M we associate a vertex $[M]$, and two vertices $[M]$ and $[M']$ are the same if and only if $M \simeq M'$.
(b) There is an arrow between the vertices $[M]$ and $[N]$ if there is an irreducible morphism $\mathrm{mod}A$ from M to N. The arrow $[M] \to [N]$ has valuation (a,b) if there is a right minimal almost split morphism $aM \oplus X \to N$, where M is not isomorphic to a summand of X, and a left minimal almost split morphism $M \to bN \oplus Y$, where N is not isomorphic to a summand of Y.

We observe that if A is a finite dimensional k-algebra over an algebraically closed field then the valuations of the arrows are (a,b) with $a = b$. Moreover, if in addition A is a representation-finite algebra then $a = b = 1$.

The vertices corresponding to the projective A-modules are called projective vertices and the ones corresponding to the injective A-modules are called injective vertices.

Recall that if X is a module and $P_1 \xrightarrow{f} P_0 \to X \to 0$ its minimal projective presentation in $\mathrm{mod}A$, then the transpose $\mathrm{Tr}X$ of X is the cokernel of $\mathrm{Hom}(f,A)$. This allows to define $\tau = D\mathrm{Tr}$, called *the Auslander–Reiten translation* of Γ_A, where D is the duality functor. The functor $D\mathrm{Tr}$ induces a correspondence between the non-projective vertices and the non-injective ones. We denote by τ^{-1} its inverse.

We do not distinguish between an indecomposable module in $\mathrm{mod}A$ and the corresponding vertex $[X]$ in Γ_A.

The Auslander–Reiten quiver is the union of their connected components.

A component Γ of Γ_A is said to be generalized standard if $\Re^\infty(X,Y) = 0$ for all $X,Y \in \Gamma$. Generalized standard Auslander–Reiten components have been defined by Skowroński in [24].

For a detailed account of this theory, we refer the reader to [1, 2, 23, 24].

A sequence of non-zero morphisms $X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_n \xrightarrow{f_n} X_{n+1}$ with $X_i \in \mathrm{ind}A$ for $i = 1,\dots,n$ is said to be a path in $\mathrm{mod}A$ if all the morphisms f_i are not isomorphisms and it is called a path in Γ if all the morphisms f_i are irreducible. A path in Γ is said to be of length n if the sequence has exactly n irreducible morphisms.

We distinguish now two important paths in Γ_A. The first one was defined by R. Bautista in [3] and the second one by S. Liu in [21].

- A path $Y_0 \to Y_1 \to \cdots \to Y_n$ in Γ_A is sectional provided $\tau^{-1}Y_i \not\simeq Y_{i+2}$, for $i = 0,\dots,n-2$.
- A path $Y_0 \to Y_1 \to \cdots \to Y_n$ in Γ_A is said to be pre-sectional if for each $i, 1 \le i \le n-1$, such that $Y_{i-1} \simeq \tau Y_{i+1}$, there is an irreducible morphism $Y_{i-1} \oplus \tau Y_{i+1} \to Y_i$; or, equivalently, if $\tau^{-1}Y_{i-1} \simeq Y_{i+1}$, there is an irreducible morphism $Y_i \to \tau^{-1}Y_{i-1} \oplus Y_{i+1}$. Every sectional path is a pre-sectional path.

A partial solution to the problem of when the composition of n irreducible morphisms belongs to the $(n+1)$-power of the radical of their module category was given by K. Igusa and G. Todorov in [20, Theorem 13.3]. They proved that if

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n$$

is a sectional path of length n then their composition $f_n \dots f_2 f_1 \in \Re^n(X_0,X_n) \setminus \Re^{n+1}(X_0,X_n)$.

Later in 1992, S. Liu generalized such a result for pre-sectional paths proving that if

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n$$

is a pre-sectional path of length n then there exist irreducible morphisms $h_i\colon X_i \to X_{i+1}$ for $i = 0,\dots,n-1$ such that their composition $h_n \dots h_2 h_1 \in \mathfrak{R}^n(X_0,X_n) \setminus \mathfrak{R}^{n+1}(X_0,X_n)$.

2.4. **Degrees of irreducible morphisms.** In [21] S. Liu introduced the useful notion of left and right degree of an irreducible morphism. Next, we state the definition of left degree.

Definition 2.1. *Let $f\colon X \to Y$ be an irreducible morphism in* mod$A$*, with X or Y indecomposable. The left degree $d_l(f)$ of f is infinite, if for each integer $n \geq 1$, each module $Z \in$* ind$A$ *and each morphism $g\colon Z \to X$ with $g \in \mathfrak{R}^n(Z,X) \backslash \mathfrak{R}^{n+1}(Z,X)$ we have that $fg \notin \mathfrak{R}^{n+2}(Z,Y)$. Otherwise, the left degree of f is the least natural m such that there is an A-module Z and a morphism $g\colon Z \to X$ with $g \in \mathfrak{R}^n(Z,X) \backslash \mathfrak{R}^{n+1}(Z,X)$ such that $fg \in \mathfrak{R}^{m+2}(Z,Y)$.*

The *right degree $d_r(f)$* of an irreducible morphism f is dually defined.

In [13, Theorem A], the authors characterized this notion whenever we deal with a finite dimensional k-algebra over an algebraically closed field. They reduced the study of the degree of an irreducible morphism to the study of the degree in a suitable covering that they called *the generic covering*. This characterization allows us to compute degrees in a more handy way than using Liu's definition. Next, we state these results.

Theorem 2.2 (C., Le Meur, Trepode)**.** *If $f\colon X \to Y$ is an irreducible morphism with X or Y indecomposable then $d_l(f)$ is finite if and only if there exist $n \geq 1$, $Z \in$* ind$A$ *and a morphism $g \in \mathfrak{R}^n(Z,X) \setminus \mathfrak{R}^{n+1}(Z,X)$ such that $fg = 0$.*

Proposition 2.3 (C., Le Meur, Trepode)**.** *Let A be a finite dimensional k-algebra over an algebraically closed field k. Let $f\colon X \to Y$ be an irreducible morphism in* mod$A$ *with X indecomposable. Let $n \geq 1$ be an integer. The following conditions are equivalent:*

(a) *$d_l(f) = n$;*
(b) *the morphism $i\colon \mathrm{Ker}(f) \hookrightarrow X$ lies in $\mathfrak{R}^n(\mathrm{Ker}(f),X) \backslash \mathfrak{R}^{n+1}(\mathrm{Ker}(f),X)$, where i is the inclusion morphism.*

Dual results hold for the right degree.

The above results were first proven in [18] for generalized standard Auslander–Reiten components with length (when paths in Γ_A having the same starting vertex and the same ending vertex have the same length) over an artin algebra and later in [6] for standard components and standard algebras.

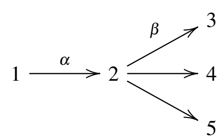
As an immediate consequence of the above results we get the following corollary which shall be useful for our further purposes.

Corollary 2.4 (C., Le Meur, Trepode)**.** *Let A be an algebra and $f\colon X \to Y$ an irreducible morphism with X or Y indecomposable. Then*

(1) *If f is an irreducible epimorphism then $d_r(f) = \infty$.*
(2) *If f is an irreducible monomorphism then $d_l(f) = \infty$.*

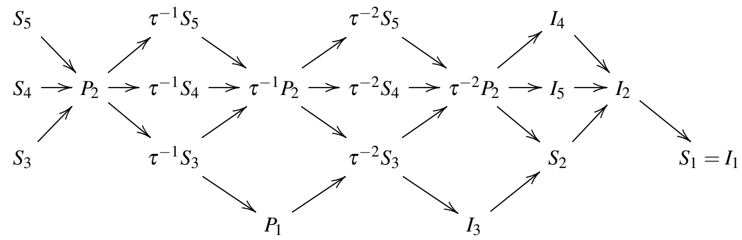
Next, we compute the left and right degrees of some irreducible morphisms.

Example 2.5. Let A be the representation-finite algebra given by the quiver



with $\beta\alpha = 0$.

The Auslander–Reiten quiver Γ_A of A has the following shape:



We compute the left degree of the irreducible epimorphisms $f_i\colon \operatorname{rad} P_i \to P_i$ for $i = 1,2$ and the right degree of the irreducible monomorphisms $g_i\colon I_i \to I_i/\operatorname{Soc} I_i$ for $i = 2,\dots,5$. Since Γ_A is a generalized component with length then to compute the left degree of any irreducible epimorphism $f\colon X \to Y$ it is enough to consider the length of any path from $\operatorname{Ker} f_i$ to X, see [18].

We observe that a dual result holds for the right degree.

Then, $d_r(f_i) = 5$ for $i = 1,2$ and $d_l(g_2) = 1$, $d_l(g_3) = 5$ and $d_l(g_i) = 6$ for $i = 4,5$.

For a detailed account on degree theory we refer the reader to [7, 5, 13, 18, 21, 22].

2.5. **On non-zero compositions of irreducible morphisms.** In 2011, the authors of [13] gave an answer to the problem of when the non zero composition of n irreducible morphisms between indecomposable A-modules belongs to the $(n+1)$-th power of the radical of their module category, in case A is a finite dimensional k-algebra over an algebraically closed field, and in 2014 for a finite dimensional k-algebra over a perfect field (see [14]). In both cases, the characterizations obtained were the same. More precisely, they proved the following theorem.

Theorem 2.6 (C., Le Meur, Trepode)**.** *Let A be a finite dimensional k-algebra over a perfect field. Let Γ be a component of Γ_A, $n \geq 1$, and $X_1,\dots,X_{n+1} \in \Gamma$. The following conditions are equivalent:*

(1) *There exist irreducible morphisms $h_i\colon X_i \to X_{i+1}$ for each i such that $0 \neq h_n \dots h_1 \in \Re^{n+1}(X_1, X_{n+1})$.*
(2) *There exist irreducible morphisms $f_i\colon X_i \to X_{i+1}$ and morphisms $\varepsilon_i\colon X_i \to X_{i+1}$ such that $f_n \dots f_1 = 0$, $\varepsilon_n \dots \varepsilon_1 \neq 0$, and $\varepsilon_i = f_i$ or $\varepsilon_i \in \Re^2(X_i, X_{i+1})$ for each i.*

The aim of these notes is to present a new approach to the problem, considering degrees of irreducible morphisms.

2.6. **On degrees and representation-finite algebras.** The result given in Theorem 2.2 allows to solve many problems such as to determine when an algebra is representation-finite. The next characterization of a representation-finite algebra has been established in [13, Theorem A].

Theorem 2.7 (C., Le Meur, Trepode)**.** *Let A be a finite dimensional k-algebra over an algebraically closed field. The following conditions are equivalent:*

(a) *A is finite representation type.*
(b) *For every non-simple indecomposable injective A-module I, the irreducible morphism $I \to I/\operatorname{soc} I$ has finite left degree.*
(c) *For every non-simple indecomposable projective A-module P, the irreducible morphism $\operatorname{rad} P \to P$ has finite right degree.*
(d) *For every irreducible epimorphism $f\colon X \to Y$ with X or Y indecomposable, the left degree of f is finite.*

(e) *For every irreducible monomorphism $f\colon X \to Y$ with X or Y indecomposable, the right degree of f is finite.*

An interesting question is to know if for a representation-finite algebra we can determine the irreducible morphism which has the greatest left (right) degree.

In [8], the author gave the following answer to such a problem that we state below.

Proposition 2.8. *Let A be a finite dimensional k-algebra over an algebraically closed field of finite representation type. Then*

(1) *There exist a simple module S and an irreducible epimorphism $\theta_s\colon I_s \to I_s/S$ such that for any other irreducible epimorphism $f\colon X \to Y$ with X or Y indecomposable we have that $d_l(f) \leq d_l(\theta_s)$, where I_s is the injective envelope of S.*

(2) *There exists a simple module S' and an irreducible monomorphism $\iota_{s'}\colon \operatorname{rad} P_{s'} \to P_{s'}$ such that for any other irreducible monomorphism $f\colon X \to Y$ with X or Y indecomposable we have that $d_r(f) \leq d_r(\iota_{s'})$, where $P_{s'}$ is the projective cover of S'.*

Note that the greatest left degree and the greatest right degree may not coincide, see Example 2.5.

As an immediate consequence of Proposition 2.8, we get the next characterization.

Theorem 2.9. *Let A be a finite dimensional k-algebra over an algebraically closed field. The following conditions are equivalent:*

(a) *A is finite representation type.*

(b) *There exists an irreducible epimorphism $\theta\colon I \to I/\operatorname{soc}(\mathrm{I})$ of finite left degree with I a non-simple indecomposable injective, such that for any other irreducible epimorphism $f\colon X \to Y$ with X or Y indecomposable we have that $d_l(f) \leq d_l(\theta)$.*

(c) *There exists an irreducible monomorphism $\iota\colon \operatorname{rad} P \to P$ of finite right degree with P a non-simple indecomposable projective, such that for any other irreducible monomorphism $f\colon X \to Y$ with X or Y indecomposable we have that $d_r(f) \leq d_r(\iota)$.*

2.7. **On the minimal vanishing power of the radical.** We start this subsection recalling this important characterization due to Auslander.

Theorem 2.10 (Auslander)**.** *Let A be an artin algebra. Then A is of finite representation type if and only if there is a positive integer n such that $\mathfrak{R}^n(X,Y) = 0$ for all $X, Y \in \operatorname{mod} A$.*

Our next question is the following: *Is it possible to find a minimal lower bound $m \geq 1$, which does not depend on the maximal length of all indecomposable modules, such that the m-th power of the radical of* $\operatorname{mod} A$ *vanishes?*

In [4] we gave a positive answer to the above question whenever A is a finite dimensional algebra over an algebraically closed field of finite representation type. We found a bound depending on the left and right degrees of certain irreducible morphisms. Furthermore, in [15] the authors got an answer for any representation-finite artin algebra.

Now, we explain briefly how we find such a bound for a $A \simeq kQ/I$ a finite dimensional k-algebra over an algebraically closed field of finite representation type.

Let $a \in Q_0$. If either $P_a = S_a$ or $I_a = S_a$ then we write $n_a = 0$ and $m_a = 0$, respectively. Otherwise, we consider the irreducible morphisms $\iota_a\colon \operatorname{rad}(P_a) \hookrightarrow P_a$ and $g_a\colon I_a \to I_a/\operatorname{soc}(I_a)$ and we write $n_a = d_r(\iota_a)$ and $m_a = d_l(g_a)$.

Theorem 2.11. *Let $A \simeq kQ_A/I_A$ be a finite dimensional algebra over an algebraically closed field and assume that A is of finite representation type. Consider $m = \max\{n_a + m_a\}_{a \in Q_0}$. Then $\mathfrak{R}^m(\operatorname{mod} A) \neq 0$ and $\mathfrak{R}^{m+1}(\operatorname{mod} A) = 0$.*

We end this section with the following example.

Example 2.12. Consider the algebra A given in Example 2.5. Applying the above result we get that $\Re^7(\mathrm{mod}A) = 0$.

3. ON DEGREES AND CHAINS OF IRREDUCIBLE MORPHISMS

We are interested in studying the relationship between compositions of irreducible morphisms between indecomposable modules and the powers of the radical of their module category depending on the degrees of such irreducible morphisms. All these results were obtained in [10]. We refer the reader to that article for a detailed account of their proofs.

By definition of degree of an irreducible morphism it is immediate that if all morphisms $f_1,\dots,f_n$ have infinite left and right degree then $f_n\dots f_1 \in \Re^n\backslash\Re^{n+1}$. Then, we are interested in studying chains where at least one irreducible morphism has finite left or right degree.

First, we recall the following useful result from [21, p. 41].

Lemma 3.1 (Liu). *If $f_n\dots f_1 \in \Re^{n+1}$ then there is an irreducible morphism f_i and an irreducible morphism f_j, with $i,j = 1,\dots,n$, such that $d_l(f_i) < \infty$ and $d_r(f_j) < \infty$.*

As an immediate consequence we get the following result.

Lemma 3.2. *Let A be an algebra and $f_i\colon M_{i-1} \to M_i$ be irreducible morphisms between indecomposable A-modules, for $i = 1,\dots,s$. Then the following conditions hold:*

(1) *If f_i are all irreducible epimorphisms (n monomorphisms) for $i = 1,\dots,s$ then $f_s\cdots f_1 \notin \Re^{s+1}(M_0,M_s)$.*
(2) *If $f_s\cdots f_1 \in \Re^{s+1}(M_0,M_s)$ and f_1 is an irreducible epimorphism (f_s is an irreducible monomorphism) then $f_s\cdots f_2 \in \Re^s(M_1,M_s)$ ($f_{s-1}\cdots f_1 \in \Re^s(M_0,M_{s-1})$, respectively).*

Proof. (i). It is an immediate consequence of Lemma 3.1 and Corollary 2.4.

(ii). Assume that $f_s\cdots f_2 \in \Re^{s-1}(M_1,M_s)\backslash\Re^s(M_1,M_s)$. By Corollary 2.4 (1), since $d_r(f_1) = \infty$ then $f_s\cdots f_1 \in \Re^s(M_0,M_s)\backslash\Re^{s+1}(M_0,M_s)$, getting a contradiction to our assumption. □

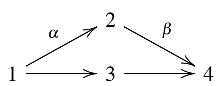
By the above lemmas, to decide if a composition of n irreducible morphisms belongs to $\Re^{n+1}(\mathrm{mod}A)$ it is enough to consider chains of irreducible morphisms starting in a monomorphism, ending at an epimorphism, and such that at least there is an irreducible monomorphism and an irreducible epimorphism with finite right and left degree, respectively.

Proposition 3.3. *Let A be an algebra. Let $M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to \cdots \to M_{s-1} \xrightarrow{f_s} M_s$ be a chain of irreducible morphisms with $M_i \in \mathrm{ind}A$, for $i = 0,\dots,s$, where f_1 is a monomorphism and f_s is an epimorphism. Then the following conditions hold:*

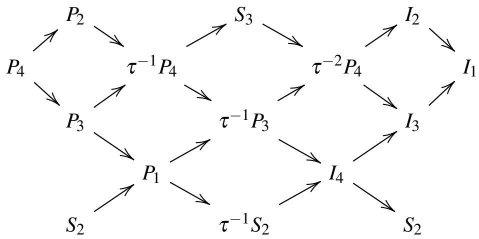
(a) *If for each i such that $f_i\colon M_{i-1} \to M_i$ is an epimorphism we have that $d_l(f_i) > i-1$, then $f_s\cdots f_1 \in \Re^s(M_0,M_s)\backslash\Re^{s+1}(M_0,M_s)$.*
(b) *If for each i such that $f_i\colon M_{i-1} \to M_i$ is a monomorphism we have that $d_r(f_i) > s-i$, then $f_s\cdots f_1 \in \Re^s(M_0,M_s)\backslash\Re^{s+1}(M_0,M_s)$.*

The following example illustrates the above proposition.

Example 3.4. Let A be the representation-finite algebra given by the quiver



with $\beta\alpha = 0$. The Auslander–Reiten quiver Γ_A of A is



where the two copies of S_2 are identified.

Consider the chain of irreducible morphisms:

$$P_1 \xrightarrow{f_1} \tau^{-1}P_3 \xrightarrow{f_2} \tau^{-2}P_4 \xrightarrow{f_3} I_3.$$

Observe that f_1 is a monomorphism and f_2, f_3 are epimorphisms. By Proposition 3.3 (a), we claim that $f_3 f_2 f_1 \in \Re^3(P_1,I_3)\backslash\Re^4(P_1,I_3)$. In fact, $d_l(f_2) = 2$ and $d_l(f_3) = 3$. Hence we have the result.

We also observe that Proposition 3.3 (b) does not give information to decide about this composition since $d_r(f_1) = 2$.

Now, we consider the cases where in a given chain there are irreducible epimorphisms (monomorphisms) $f_i \colon M_{i-1} \to M_i$ such that $d_l(f_i) \leq i-1$ ($d_r(f_i) \leq s-i$, respectively), for $i = 1,\dots,s$.

First, we recall [12, Lemma 3.4], that will be useful for our considerations.

Proposition 3.5 (C., Coelho, Trepode). *Assume that* $\dim_k \operatorname{Irr}(X_i,X_{i+1}) = 1$ *for* $i = 1,\dots,n$ *with* $X_i \in \operatorname{ind} A$ *for* $i = 1,\dots,n+1$. *Then the following conditions are equivalent:*

(1) *There are irreducible morphisms* $f_i \colon X_i \to X_{i+1}$ *in* $\operatorname{mod} A$ *for* $i = 1,\dots,n$ *such that* $f_n \dots f_1 \in \Re^n(X_1,X_{n+1})\backslash\Re^{n+1}(X_1,X_{n+1})$.

(2) *Given irreducible morphisms* $h_i \colon X_i \to X_{i+1}$ *in* $\operatorname{mod} A$ *for* $i = 1,\dots,n$, *then one has that* $h_n \dots h_1 \in \Re^n(X_1,X_{n+1})\backslash\Re^{n+1}(X_1,X_{n+1})$.

By a path we mean a sequence of irreducible morphisms in $\operatorname{mod} A$ between indecomposable A-modules and by a path from X to Y of length zero we mean that the path is not in $\Re(\operatorname{mod} A)$. As a consequence we have that $X \simeq Y$.

The next result holds for algebras such that the irreducible morphisms $f \colon X \to Y$ in the chain are such that $\dim_k(\operatorname{Irr}(X,Y)) = 1$.

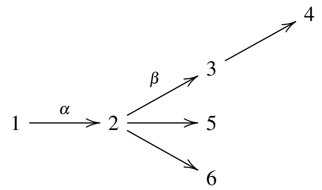
Proposition 3.6. *Let* A *be an algebra and* $M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to \cdots \to M_{s-1} \xrightarrow{f_s} M_s$ *be a chain of irreducible morphisms between indecomposable* A*-modules* M_i, *for* $i = 0,\dots,s$, *where* f_1 *is a monomorphism and* f_s *is an epimorphism. Assume that* $\dim_k(\operatorname{Irr}(M_i,M_{i+1})) = 1$ *for* $i = 0,\dots,s-1$. *Then the following conditions hold:*

(a) *If for each* i *such that* $f_i \colon M_{i-1} \to M_i$ *is an epimorphism with* $d_l(f_i) = m_i \leq i-1$ *and if either there is not a path from* M_0 *to* $\operatorname{Ker} f_i$ *of length* $(i-1)-m_i$ *or all paths from* M_0 *to* $\operatorname{Ker} f_i$ *of length* $(i-1)-m_i$ *are zero, then* $f_s \cdots f_1 \in \Re^s(M_0,M_s)\backslash\Re^{s+1}(M_0,M_s)$.

(b) *If for each* i *such that* $f_i \colon M_{i-1} \to M_i$ *is a monomorphism with* $d_r(f_i) = n_i \leq s-i$ *and if either there is not a path from* $\operatorname{Coker} f_i$ *to* M_s *of length* $(s-i)-n_i$ *or all paths from* $\operatorname{Coker} f_i$ *to* M_s *of length* $(s-i)-n_i$ *are zero, then* $f_s \cdots f_1 \in \Re^s(M_0,M_s)\backslash\Re^{s+1}(M_0,M_s)$.

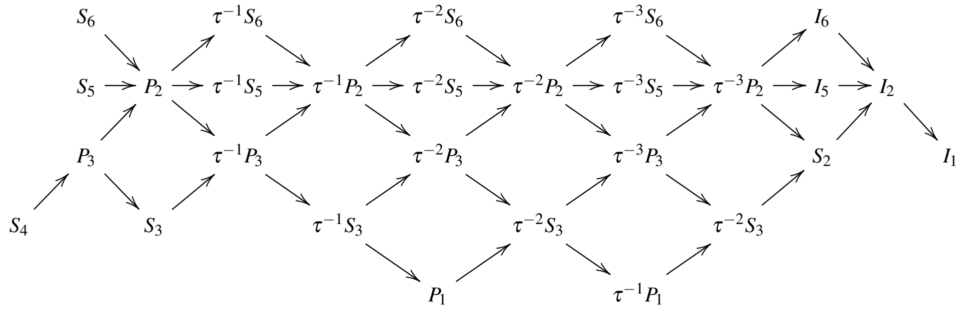
Next, we present an example having a chain of irreducible morphisms where we can apply the above result and also to give a solution when we consider the monomorphisms in Example 3.4.

Example 3.7. (a). Let A be the representation-finite algebra given by the quiver



with $\beta\alpha = 0$.

The Auslander–Reiten quiver Γ_A of A is



Consider the following chain of irreducible morphisms:

$$S_5 \stackrel{f_1}{\to} P_2 \stackrel{f_2}{\to} \tau^{-1}P_3 \stackrel{f_3}{\to} \tau^{-1}P_2 \stackrel{f_4}{\to} \tau^{-2}P_3 \stackrel{f_5}{\to} \tau^{-2}P_2.$$

The irreducible epimorphisms f_2, f_4 and f_5 are such that $d_l(f_2) = 2$, $d_l(f_4) = 2$ and $d_l(f_5) = 5$. By Proposition 3.3 (a), we only need to analyze f_4. Since $\operatorname{Ker} f_4 \simeq S_3$ and there is not an irreducible morphism from S_5 to S_3 then $f_5 f_4 f_3 f_2 f_1 \in \Re^5(S_5, \tau^{-2}P_2) \backslash \Re^6(S_5, \tau^{-2}P_2)$.

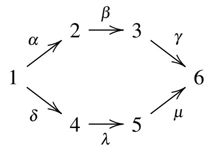
(b). By Proposition 3.6 we can give an answer to the fact that $f_3 f_2 f_1 \in \Re^3 \backslash \Re^4$ in Example 3.4, taking into account the irreducible monomorphism f_1 with $d_r(f_1) = 2$.

Finally, it is left to consider the following case.

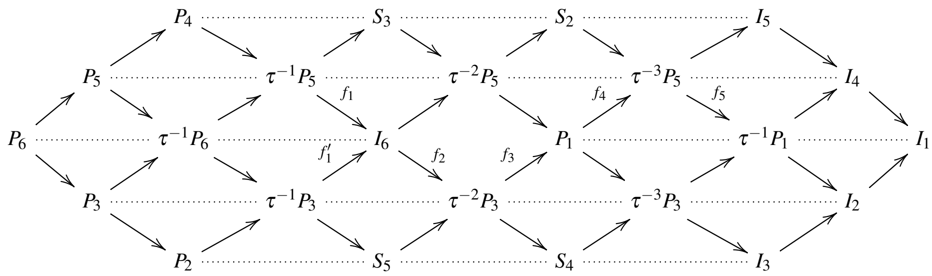
Proposition 3.8. *Let A be an algebra and $M_0 \stackrel{f_1}{\to} M_1 \stackrel{f_2}{\to} M_2 \to \cdots \to M_{s-1} \stackrel{f_s}{\to} M_s$ be a chain of irreducible morphisms between indecomposable A-modules M_i, for $i = 0, \dots, s$, where f_1 is a monomorphism and f_s is an epimorphism. Then the following conditions hold:*

- (i) *Assume there is an irreducible epimorphism $f_i\colon M_{i-1} \to M_i$ with $d_l(f_i) = m_i \leq i-1$ and a non-zero path of irreducible morphisms from M_0 to $\operatorname{Ker} f_i$ in $\operatorname{mod} A$ of length $i-1-m_i$. Let $i \in \{1, \dots, s\}$ be the maximal integer satisfying the mentioned conditions.*
  - (a) *If all paths of irreducible morphisms $M_0 \to M_1 \to \cdots \to M_{i-1} \to M_i$ are non-zero then $f_s \cdots f_1 \in \Re^s(M_0, M_s) \backslash \Re^{s+1}(M_0, M_s)$.*
  - (b) *If there is a zero path of irreducible morphisms $M_0 \to M_1 \to \cdots \to M_{i-1} \to M_i$ and $\dim_k(\operatorname{Irr}(M_i, M_{i+1})) = 1$ for $i = 0, \dots, s-1$ then $f_s \cdots f_1 \in \Re^{s+1}(M_0, M_s)$.*
- (ii) *Assume there is an irreducible monomorphism $f_i\colon M_{i-1} \to M_i$ with $d_r(f_i) = n_i \leq s-i$ and a non-zero path of irreducible morphisms from $\operatorname{Coker} f_i$ to M_s in $\operatorname{mod} A$ of length $s-i-n_i$. Let $i \in \{1, \dots, s\}$ be the minimal integer satisfying the mentioned conditions.*
  - (a) *If all paths of irreducible morphisms $M_i \to M_{i+1} \to \cdots \to M_{s-1} \to M_s$ are non-zero then $f_s \cdots f_1 \in \Re^s(M_0, M_s) \backslash \Re^{s+1}(M_0, M_s)$.*
  - (b) *If there is a zero path of irreducible morphisms $M_i \to M_{i+1} \to \cdots \to M_{s-1} \to M_s$ and $\dim_k(\operatorname{Irr}(M_j, M_{j+1})) = 1$ for $j = i, \dots, s-1$ then $f_s \cdots f_1 \in \Re^{s+1}(M_0, M_s)$.*

Example 3.9. Consider the algebra given by the quiver



with $\gamma\beta\alpha = 0 = \mu\lambda\delta$.
The Auslander–Reiten quiver Γ_A of A is



Consider the chain of irreducible morphisms

$$\tau^{-1}P_5 \xrightarrow{f_1} I_6 \xrightarrow{f_2} \tau^{-2}P_3 \xrightarrow{f_3} P_1 \xrightarrow{f_4} \tau^{-3}P_5 \xrightarrow{f_5} \tau^{-1}P_1.$$

Note that $d_l(f_4) = 2$ and there is an irreducible morphism from $\tau^{-1}P_5$ to $\mathrm{Ker}(f_4) \simeq S_3$. Since all irreducible paths of the form $\tau^{-1}P_5 \to I_6 \to \tau^{-2}P_3 \to P_1 \to \tau^{-3}P_5$ are not zero, then $d_p(f_5 f_4 f_3 f_2 f_1) = 5$.

Summarizing all the above information we state the following theorem.

Theorem 3.10. *Let A be an algebra and $M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to \cdots \to M_{s-1} \xrightarrow{f_s} M_s$ be a chain of irreducible morphisms between indecomposable A-modules M_i with* $\dim_k(\mathrm{Irr}(M_i, M_{i+1})) = 1$ *for $i = 0, \ldots, s-1$. Then,*

1. *$f_s \cdots f_1 \in \Re^{s+1}(M_0, M_s)$ if and only if there is an irreducible morphism $f_i \colon M_{i-1} \to M_i$ for some $i = 0, \ldots, s$ with $d_l(f_i) = m_i \leq i-1$, a non-zero path of irreducible morphisms from M_0 to* $\mathrm{Ker} f_i$ *in* $\mathrm{mod} A$ *of length $i-1-m_i$, and a zero path of irreducible morphisms $M_0 \to M_1 \to \cdots \to M_{i-1} \to M_i$.*
2. *$f_s \cdots f_1 \in \Re^{s+1}(M_0, M_s)$ if and only if there is an irreducible morphism $f_i \colon M_{i-1} \to M_i$ for some $i = 0, \ldots, s$ with $d_r(f_i) = n_i \leq s-i$, a non-zero path of irreducible morphisms from* $\mathrm{Coker} f_i$ *to M_s in* $\mathrm{mod} A$ *of length $s-i-n_i$ and a zero path of irreducible morphisms $M_i \to M_{i+1} \to \cdots \to M_{s-1} \to M_s$.*

We end up this section showing some particular cases where it is easy to decide if the composition of n irreducible morphisms is in $\Re^{n+1}(\mathrm{mod} A)$, that is, the cases when in a given chain we have irreducible morphisms of left or right degree one or two. Moreover, we do not consider any assumptions on the dimension of the irreducible morphisms.

We observe that these results are a consequence of [11, Theorem 2.2] and [7, Proposition 5.1] where the authors studied when the composition of two and three irreducible morphisms are in a greater power of the radical, greater than 3 and 4, respectively.

Proposition 3.11. *Let A be an algebra and*

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to \cdots \to M_{s-1} \xrightarrow{f_s} M_s$$

be a chain of irreducible morphisms between indecomposable A-modules M_i, for $i = 0, \dots, s$, where f_1 is a monomorphism and f_s is an epimorphism. Then the following conditions hold:

(a) *If there is an irreducible epimorphism $f_i \colon M_{i-1} \to M_i$ with $d_l(f_i) = 1$ and* $\operatorname{Ker} f_i \simeq M_{i-2}$ *or $d_l(f_i) = 2$ and* $\operatorname{Ker} f_i \simeq M_{i-3}$ *then $f_s \cdots f_1 \in \Re^{s+1}(M_0, M_s)$.*

(b) *If there is an irreducible monomorphism $f_i \colon M_{i-1} \to M_i$ with $d_r(f_i) = 1$ and* $\operatorname{Coker} f_i \simeq M_{i+1}$ *or $d_r(f_i) = 2$ and* $\operatorname{Coker} f_i \simeq M_{i+2}$ *then $f_s \cdots f_1 \in \Re^{s+1}(M_0, M_s)$.*

We recall that a component Γ of Γ_A is said to satisfy the condition $\alpha(\Gamma) \leq 2$ if $\alpha(X) \leq 2$ for every X in Γ. By $\alpha(X)$ we mean the number of indecomposable direct summands of the middle term of an almost split sequence ending at a non-projective module X.

In addition, if the modules belong to an Auslander–Reiten component Γ with $\alpha(\Gamma) \leq 2$, we get the following result which is a consequence of Theorem 3.10 and [7, Proposition 6.3].

Proposition 3.12. *Let A be an algebra and Γ a component of Γ_A with $\alpha(\Gamma) \leq 2$. Let*

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to \cdots \to M_{s-1} \xrightarrow{f_s} M_s$$

be a chain of irreducible morphisms between indecomposable A-modules M_i, for $i = 0, \dots, s$, where f_1 is a monomorphism and f_s is an epimorphism. Then the following conditions hold:

(a) *If there is an irreducible epimorphism $f_i \colon M_{i-1} \to M_i$ with $d_l(f_i) = m_i \leq i-1$ such that* $\operatorname{Ker} f_i \simeq M_{i-1-m_i}$ *then $f_s \cdots f_1 \in \Re^{s+1}(M_0, M_s)$.*

(b) *If there is an irreducible monomorphism $f_i \colon M_{i-1} \to M_i$ with $d_r(f_i) = n_i \leq s-i$ such that* $\operatorname{Coker} f_i \simeq M_{s-i+n_i}$ *then $f_s \cdots f_1 \in \Re^{s+1}(M_0, M_s)$.*

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