ACTAS DEL XIII CONGRESO
DR. ANTONIO A. R. MONTEIRO (2015)
2016, Páginas 39–41

ANALYTIC COHOMOLOGY IN CHARACTERISTIC $p > 0$: PROGRESS REPORT

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1. Introduction

This is a report on ongoing joint work with Joachim Cuntz and Ralf Meyer. Let k be a field of characteristic $p > 0$, $V = W(k)$ the ring of Witt vectors. Thus V is a Noetherian, local domain with principal maximal ideal $\mathfrak{m} = \pi V$ and residue field $V/\pi V = k$, complete in the $\mathfrak{m}$-adic topology. We write K for the field of fractions of V. Our goal is to construct a functor

$$H^{an} : k\text{-algebras} \to ((\mathbb{Z}/2\text{-graded complexes of } K\text{-modules}))$$

with the following properties:

(1) Polynomial homotopy invariance: $H^{an}(A) \to H^{an}(A[t])$ is a quasi-isomorphism.
(2) Excision: applying H^{an} to any extension of k-algebras

$$0 \to A \to B \to C \to 0$$

gives rise to a homotopy fibration sequence of $\mathbb{Z}/2$-graded complexes

$$H^{an}(A) \to H^{an}(B) \to H^{an}(C).$$

(3) Matrix invariance: $H^{an}(A) \to H^{an}(M_\infty A)$ is a quasi-isomorphism.
(4) Agreement with Bertherlot's rigid cohomology [3] in the commutative case: if $k \to A$ is a commutative unital algebra of finite type, then

$$H^{an}_n(A) = \prod_j H^{2j-n}_{rig}(A, K).$$

To explain why having such a theory could be useful, assume for a moment that a functor with the properties above exists, and write

$$H^{an}_*(A, B) = H_*(\mathrm{HOM}(H^{an}(A), H^{an}(B))).$$

By the universal property of algebraic bivariant K-theory [1] and the assumed properties of H^{an}, it follows that there is a Chern character

$$kk_*(A, B) \to H^{an}_*(A, B),$$

compatible with composition. In particular, setting $A = k$, we get a Chern character

$$KH_*(B) = kk_*(k, B) \to H^{an}_*(B) := H^{an}_*(k, B)$$

from Weibel's homotopy algebraic K-theory [7] (and thus also from Quillen's K-theory, using the natural transformation $K \to KH$). Specializing to B commutative of finite type, yields maps $ch_{j,n} : KH_n(B) \to H^{2j-n}_{rig}(B)$.

Supported by CONICET, and partially supported by grants MTM2012-36917-C03-02, UBACyT 20020100100386 and PIP 11220110100800.

Recall from [4] that a *bornology* on a K-vector space M is an exhaustive filtration by normed subspaces, and that the completion of M with respect to a given bornology is the colimit of the diagram of Banach K-spaces that results from completing each of the normed subspaces in the filtration. The basic idea for constructing H^{an} is to write $A = TL/I$ as a quotient of the tensor algebra of a free V-module L, use the ideal I to produce a bornology on $TL_K := TL \otimes_V K$ and then take the periodic cyclic complex of the completion $H^{an}(A) = HP(\widehat{TL_K})$. The highly non-trivial technical point is what bornology to take.

2. Rigid cohomology (smooth affine case)

Let $V \to R = V[x_1, \ldots, x_m]$ be a commutative algebra of finite type. The *weak completion* $R^\dagger$ [5] is the following V-subalgebra of the π-adic completion

$$\lim_n R/\pi^n R \supset R^\dagger = \left\{ \sum_n \pi^n f_n(x_1, \ldots, x_m) : (\exists N)\ \deg(f_n) \leq N(n+1) \right\}.$$

Remark 2.1. *Assume $V \to R$ is flat. Let $\mathscr{F}_n R \subset R = V[x_1, \ldots, x_m]/I$ be the submodule generated by the images of the monomials x^α with $|\alpha| \leq n$. Then $R^\dagger \otimes_V K$ is the bornological completion of $R \otimes_V K$ with respect to the bornology defined by the V-submodules*

$$S_{N,C} = \sum_n \pi^{-C+[n]} \mathscr{F}_{N(n+1)} R.$$

Theorem 2.2 (Elkik, [2])**.** *Let $i : k \to A$ be a smooth homomorphism of commutative k-algebras. Then there is a smooth homomorphism $\hat{i} : V \to R$ such that $i = \hat{i} \otimes_V k$.*

If $k \to A$ is smooth and R is as in Elkik's theorem, the rigid cohomology of A is defined to be (see [6])

$$H^*_{rig}(A, K) = H^*((\Omega_{R/V} \otimes_R R^\dagger) \otimes_V K).$$

The first result we have relates the rigid cohomology of A to the periodic cyclic homology of the bornological algebra $R^\dagger \otimes_V K$.

Theorem 2.3. *Let $k \to A$ be smooth and let R be as in Elkik's theorem. Then*

$$HP_n(R^\dagger \otimes_V K) = \prod_j H^{2j-n}_{rig}(A, K).$$

3. Tentative definition of H^{an}

Let $k \to A$ be a (not necessarily commutative) unital algebra. Assume A is of finite type, i.e., assume it is a quotient

$$A = k\{x_1, \ldots, x_m\}/I \tag{3.1}$$

of a polynomial ring in finitely many non-commuting variables. Filter A by the image $\{\mathscr{F}_n(A)\}$ of the degree filtration of $k\{x_1, \ldots, x_m\}$. Let L be the free V-module on the set A/k equipped with the filtration $\{\mathscr{F}_n(F_V(A))\}$ induced by the chosen filtration of A. Write $I = \ker(F_V(A) \to A)$ for the kernel of the canonical surjection, and set $\mathscr{F}_n I = I \cap \mathscr{F}_n(F_V(A))$. On $F_K(A) := F_V(A) \otimes_V K$ consider the bornology defined by the V-submodules

$$S_{\alpha,N,C} = \sum_n \pi^{-C-[\alpha n]} \mathscr{F}_{N(n+1)} I^n, \quad 0 < \alpha < 1,\ N, C \in \mathbb{Z}_{\geq 1}. \tag{3.2}$$

Let $\widehat{F_V(A)}$ be the bornological completion. Define

$$H^{an}(A) = HP(\widehat{F_V(A)}).$$

The following theorem subsumes the properties we have been able to prove so far for this tentative model of H^{an}.

Theorem 3.1.

i) $H^{an}(A)$ *is independent of the choice of presentation* (3.1).

ii) H^{an} *is polynomially homotopy invariant.*

iii) *Let* $A = L/\pi L$ *be a presentation as a quotient of a unital filtered* V*-algebra* L*, such that* L *is a free* V*-module. Equip* $L_K := L \otimes_V K$ *with the bornology* (3.2)*, and let* $\widehat{L_K}$ *be the bornological completion. Assume that* $\mathscr{F}_0 L = V$*, that* $\mathscr{F}_{n+1}L/\mathscr{F}_n L$ *is a free* V*-module* ($n \geq 0$)*, and that there exists* $n > 0$ *such that* $\Omega^n_V L$ *is a projective bimodule. Further assume that* $p > 2$*, and let* $A = L/\pi L$*. Then* $H^{an}_*(A) = HP_*(\widehat{L_K})$.

Corollary 3.2. *Let* $k \to A$ *be smooth commutative and let* $V \to R$ *as in Elkik's theorem. Assume that there exists* L *as in the theorem above such that* $\widehat{L_K} \cong R^\dagger \otimes_V K$*. Then*

$$H^{an}_n(A) = \prod_j H^{2j-n}_{rig}(A, K).$$

Example 3.3. *Let* $A = k[x_1, \dots, x_m]/f$ *be a smooth hypersurface and let* $\partial_i f$ *be the partial derivative. Choose* $g_i \in A$ *such that* $\sum g_i \partial_i f = 1$ *and let* $\hat{f}, \hat{g}_i \in V[x_1, \dots, x_m]$ *be any lifts of* f *and* g_i *whose coefficients are invertible in* V*. Then* $L = V[x_1, \dots, x_m]/\hat{f}$ *and* $R = L[1/(1 - \sum_i \hat{g}_i \hat{f}_i)]$ *satisfy the hypothesis of the corollary above.*

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