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# WHAT IS LOGICAL TRUTH?

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*[Die Mathematik] bildet eine Art Philosophie mit positiven Resultate; sie ist aber auch eine Kunst im tiefsten Sinne des Wortes.*

—Paul du Bois-Reymond [1910]

Our targets are formal propositional logic and, for it, definitions of logical truth. Here, formulae of that logic, e.g.

$$(\neg p \wedge q) \to r,$$

are called 'schemes.' Such metavariables as

$$\Theta(p, q, r)$$

range over schemes. Substitutions of formally arithmetic sentences for letters like $p$, $q$, and $r$ in schemes play a pivotal role, as well as universal and existential quantification—over truth-values—binding the $p$, $q$, and $r$ places. Throughout, our method of metamathematical investigation is intuitionistic, although both conventional and intuitionistic mathematicians can grasp and accept (or reject) the proposed definitions.

## 1. Sentential versus valuational

What precisely does it mean either to assert or to deny that a scheme of propositional logic expresses a logical truth? It is a datum that mathematicians of all stripes use the very same words (in English)—'logical truth'—either to assert or to deny that the *tertium non datur* or **TND** scheme

$$p \vee \neg p$$

expresses a logical truth. In (the Dutch, French, and German cognates of) those very same terms, L.E.J. Brouwer and his Amsterdam associates first doubted and later rejected the logical truth of **TND** [3]. Clearly, since the famous *Grundlagenstreit* was not one vast equivocation, what the intuitionist rejects about the logical truth of **TND** means exactly, neither more nor less than, what the conventionalist asserts. If a conventional mathematician and an intuitionist are to debate the point and disagree, then their respective correlate words, including 'logical truth,' must carry the same meanings in their conflicting statements. (By the way, when the intuitionist says, 'Not,' he or she means just what the conventionalist means by 'Not.') Otherwise, their seemingly conflicting statements may not disagree at all!

Yet, there seem to be two reigning explications (or traditions of explication) of the notion 'logical truth,' even within conventional mathematics. On the one hand, there is the tradition endorsed by David Hilbert and later taken up by W.V.O. Quine, on which logical truth is defined in explicitly metalinguistic terms: via truth and all allowable substitutions into a scheme. One finds this endorsement on prominent display in Hilbert and Ackermann's **Grundzüge** [13]:

> It is now the first task for logic *to find those combinations of statements that are always true, regardless whether their component statements represent true or false assertions*. [13, p. 12] [Translation from the German by the author. Italics as in the original.]

Four or so decades later, Quine pledged allegiance to the same idea:

> A *logical truth*, then, is definable as *a sentence from which we get only truths when we substitute sentences for its simple sentences*. [22, p. 50] [Italics as in the original]

This Hilbert–Ackermann–Quine idea of logical truth is explicitly *sentential*. It takes concrete form by means of Gödel numbers $s$ for sentences of an object language, a numerical substitution function *Sub* defined over the numbers, and a predicate *Tr* on Gödel numbers marking out truth for sentences of the object language. Accordingly, the sentential characterization of logical truth is also substitutional and Gödelian. For example, the logical truth of **TND** is rendered in the sentential way as follows:

$$\forall s.\ Tr(Sub(s, \ulcorner p \urcorner, \ulcorner p \vee \neg p \urcorner)).$$

$\forall$ quantifies universally over natural numbers in a metalanguage, $s$ ranges over code numbers of sentences of the object language, corner quotes

$$\ulcorner \ldots \urcorner$$

signify the mapping from linguistic items to their Gödel numbers, and

$$Sub(s, \ulcorner p \urcorner, \ulcorner p \vee \neg p \urcorner)$$

is the encoded result of substituting the object language sentence numbered $s$ into all appearances of the sentence letter numbered $\ulcorner p \urcorner$ within the Gödelized scheme $p \vee \neg p$. Hence, the display

$$\forall s.\ Tr(Sub(s, \ulcorner p \urcorner, \ulcorner p \vee \neg p \urcorner))$$

may be read, "All results of substituting Gödelized sentences of the object language for $p$ in the scheme $(p \vee \neg p)$ yield truths of the object language."

Experienced foundationalists will be wary of the sentential understanding. For one thing, it seems to require fussy metamathematical filigree: encodings of individual symbols, schemes and their Gödel numberings, definable substitution functions, not to mention axiomatized truth predicates or even full-bore truth definitions. Clarity is a friend to correctness, while filigree is too often its bitter enemy. Perhaps more telling is that sentential logical truth is too closely tied to the substitutions on offer in the chosen object language, and hence to the expressive limitations on it. But what is true *logically* should not be circumscribed *contingently*, cooped up by what we just happen to have learnt to enunciate to date.

Given that, the Russell–Tarski valuational conception of logical truth comes as a welcome alternative. Russell wrote this in his **Introduction to Mathematical Philosophy**:

> Not only the principles of deduction, but all the primitive propositions of logic, consist of assertions that certain propositional functions are always true. If this were not the case, they would have to mention particular things or concepts—Socrates, or redness, or east and west, or what not—and clearly it is not the province of logic to make assertions which are true concerning one such thing or concept but not concerning another. It is part of the definition of logic (but not the whole of its definition) that all its propositions are completely general, *i.e.* they all consist of the assertion that some

> propositional function containing no constant terms is always true. [23, p. 159]

Admittedly, Russell was not always consistent in his use of the term 'propositional function.' This time, however, its referent is plain: it denotes abstract functions that are either nullary—and hence truth-values themselves—or $n+1$-ary—hence any functions from entities into truth-values. These functions are not sentences, sentence forms, or other shards of syntax, but *echt* values of 0th or higher order. With propositional functions, it is easy to set out Russell's analysis of 'the **TND** is logically true:'

$$\forall p\ (p \vee \neg p)$$

with variable '$p$' ranging over truth-values, and with $\vee$ and $\neg$ denoting the truth functions familiar to freshmen. (We will not follow Russell into the thicket of his type theory by earmarking symbols with indices for types or orders.)

In his [26], Tarski enunciated a definition of logical consequence, and hence of logical truth, in equally valuational terms, employing quantification over truth-values:

> *The sentence X* follows logically *from the sentences of the class K if and only if every model of the class K is also a model of the sentence X*. [Translation by J.H. Woodger. Italics as in the original.]

A Tarskian model of a propositional scheme is an assignment of truth-values to variables, and, according to the above definition, a sentence $X$ is logically true when it is true in every such model. Therefore, **TND** will be logically true by Tarski's lights, as by Russell's, just in case

$$p \vee \neg p$$

for every truth-value $p$.

Now, unlikely as it may seem, could these two disparate efforts at construing the one notion of logical truth—sentential versus valuational—be co-extensive? Are there essentially true logical relations on truth-values, ones that logicians (and not just logicians of intuitionistic stripe) wish to capture and study, that cannot be captured sententially, in terms of coded sentences of an object language, substitutions of them into schemes, and a truth predicate? If they are co-extensive, what are the precise (intuitionistic) mathematical conditions under which they are provably equivalent extensionally? What is the full mathematical cost incurred—to be paid in mathematical assumptions—when proving that the two coincide, if such proof be possible?

## 2. Truth-values, singletons, and powersets

Truth-values are the members of the powerset of singleton $\{0\}$,

$$\mathscr{P}(\{0\})$$

or $\mathscr{P}$ for short. For $p \in \mathscr{P}$, truth-value $p$ is TRUE just in case $0 \in p$, and FALSE when $0 \notin p$. Either conventionally or intuitionistically, $\mathscr{P}$ is the perfect choice for the official set of truth-values, since it stands in natural bijective correspondence with the quotient, under intuitive equivalence, of the collection of mathematical sentences. Map mathematical sentence $A$ into the set

$$F(A) = \{0 \mid A \text{ is true}\},$$

which is a truth-value. It is easy to see that, for all sentences $A$ and $B$,

- $A$ is true if and only if $0 \in F(A)$,
- $A \leftrightarrow B$ is true if and only if $F(A) = F(B)$,
- $(A \wedge B)$ is true if and only if $0 \in (F(A) \cap F(B))$,

- $(A \vee B)$ is true if and only if $0 \in (F(A) \cup F(B))$, and
- $(A \to B)$ is true if and only if $0 \in (F(A) \Rightarrow F(B))$, where $p \Rightarrow q$ is the truth-value

$$\{0 \mid \text{if } p, \text{ then } q\} \in \mathscr{P}.$$

The function taking any particular truth-value $p \in \mathscr{P}$ into the mathematical sentence

$$0 \in p$$

is inverse to $F$, making $F$ bijective.

In conventional mathematics standardly interpreted, $\mathscr{P}$ contains precisely the elements named

1, $\{0\}$, or TRUE,

and

0, $\emptyset$, or FALSE,

and no others. Intuitionistically, $\mathscr{P}$ surely contains both TRUE and FALSE, but is uncountable in total size. $\mathscr{P}$ is also a complete Heyting algebra, a distributive lattice with all infs $\bigwedge$ and sups $\bigvee$, and such that

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i),$$

for all elements $a$ and $I$-indexed families of elements $b_i$. Moreover, $\mathscr{P}$ is a natural epimorphic Heyting image of every powerset $\mathscr{P}(X)$ with $X \in \mathbb{V}$, $\mathbb{V}$ the universe of all sets, whenever $X$ has at least one member. To see this, let $a \in X$ and map $\mathscr{P}(X)$ onto $\mathscr{P}$ by sending $A \subset X$ into

$$\{0 \mid a \in X\} \in \mathscr{P}.$$

Moreover, $\mathscr{P}$ is characterized by this property: any complete Heyting algebra $\mathscr{A}$ that is an epimorphic image of every powerset of an inhabited set is itself isomorphic to $\mathscr{P}$. For the proof, let $F$ be a Heyting epimorphism from $\mathscr{P}$ onto $\mathscr{A}$, and let $1_{\mathscr{A}}$ be the greatest member of $\mathscr{A}$. Assume that $p, q \in \mathscr{P}$. Recall that, for all $p \in \mathscr{P}$,

$$p = \bigcup_{0 \in p} \{1\}.$$

(The foregoing is a set-theoretical rendering of the statement $0 \in p \leftrightarrow 0 \in p \wedge 1$.) Then, from the assumption that

$$F(p) = F(q),$$

it follows that

$$F(\bigcup_{0 \in p} \{1\}) = F(\bigcup_{0 \in q} \{1\}),$$

and that

$$\bigvee_{0 \in p} \{F(1)\} = \bigvee_{0 \in q} \{F(1)\},$$

since $F$ is a morphism of Heyting algebras. Therefore,

$$\bigvee_{0 \in p} \{1_{\mathscr{A}}\} = \bigvee_{0 \in q} \{1_{\mathscr{A}}\}.$$

Once again, the latter holds because $F$ is a Heyting epimorphism. So,

$$1_{\mathscr{A}} \in \bigvee_{0 \in p} \{1_{\mathscr{A}}\} \text{ if and only if } 1_{\mathscr{A}} \in \bigvee_{0 \in q} \{1_{\mathscr{A}}\}.$$

Therefore,

$$0 \in p \text{ if and only if } 0 \in q,$$

and

$$p = q.$$

In consequence, $f$ is a Heyting isomorphism between $\mathscr{P}$ and $\mathscr{A}$. Informally put, this result shows that $\mathscr{P}$ is the unique (up to isomorphism) complete Heyting algebra in which all setwise universal distinctions are accurately reflected. (No distinction can be registered in the smaller powerset $\mathscr{P}(\emptyset)$ with its single element $\emptyset$.) Every way of demarcating a subset, using either sets or classes, from out of any set-sized domain, can be mapped so faithfully into $\mathscr{P}$ that the elements of $\mathscr{P}$ mirror the demarcation logically.

On this set-theoretic treatment of truth-values, the valuational expression for the logical truth of **TND**,

$$\forall p\ (p \vee \neg p),$$

is now captured in the purely set-theoretic condition

$$0 \in \bigcap_{p \in \mathscr{P}} (p \cup \sim p),$$

where $p$ belongs to $\mathscr{P}$, $\bigcap$ is intersection, $\cup$ pairwise union, and $\sim$ relative complement. The logical truth of the **TND** now boils down to the assertion that, given any $p$ in $\mathscr{P}$, 0 belongs either to $p$ or to $p$'s complement. Henceforth, even though we may write such things as

$$\forall p\ \Theta(p)$$

to convey the valuational logical truth of scheme $\Theta$, we will, behind the scenes, be thinking of it in strictly set-theoretic terms.

At this point, the paired mutual inferences, expressed in set theory, between the two definitions are open to parametric investigation, in other words, without explicit quantification over schemes themselves. First, we ask if (and why) the inference from valuational logical truth to sentential logical truth for a fixed but arbitrary scheme $\Theta(p)$, namely,

$$\text{from } \forall p.\ \Theta(p) \ \text{ to } \ \forall s.\ Tr(Sub(s, \ulcorner x \urcorner, \ulcorner \Theta(x) \urcorner)),$$

(which we call $\mathbf{V} \Rightarrow \mathbf{S}$) holds for each scheme $\Theta(p)$. Second and conversely, we ask if (and why) the converse inference from sentential to valuational,

$$\text{from } \forall s.\ Tr(Sub(s, \ulcorner x \urcorner, \ulcorner \Theta(x) \urcorner)) \text{ to } \forall p.\ \Theta(p),$$

(or $\mathbf{S} \Rightarrow \mathbf{V}$) is true, if it is. In these investigations, both definitions are set out in an interpreted or uninterpreted second-order metalanguage, either classical second-order arithmetic **PAS** or its intuitionistic counterpart **HAS** [28, p. 164ff.]; or perhaps the classical set theory **ZF** or its intuitionistic counterpart **IZF** [29, p. 624ff.]. **HAS** is formulated with basic predicates and requisite axioms governing the primitive recursive number-theoretic relations. We also assume that the object language is that of elementary Peano–Dedekind arithmetic, and that this language is a sublanguage of the metalanguage, modulo some obvious definitions, e.g., of arithmetic notions, in a set-theoretic language. Consequently, any formal semantics attached to the metalanguage applies without further ado to the object language as well. Such a semantics is to be specified within an interpreted, informal metametalanguage, which may be a suitable second-order set theory, conventional or intuitionistic.

## 3. The expressibility assumption

To check that $\mathbf{V} \Rightarrow \mathbf{S}$ is correct mathematically, we work informally within an interpreted metalanguage. For the sake of example only, let $\Theta(p)$ be the **TND** scheme

$$p \vee \neg p.$$

Our results are not restricted to that scheme. Let $s$ be the Gödel number of an arbitrary sentence in the object language. Let $Tr$ define truth for the object language as Tarski would have had it or let $Tr$ be governed by axioms sufficient to guarantee that the $Tr$ predicate commutes with the connectives. Then,

$$Tr(s)$$

is a sentence of the metalanguage. Hence, it makes sense to consider

$$\{0 : Tr(s)\},$$

which is a member of $\mathscr{P}$. The valuational assertion of logical truth, namely,

$$\forall p(p \vee \neg p),$$

implies, when construed within set theory, that

$$0 \in (\{0 : Tr(s)\} \cup \{0 : \neg Tr(s)\}).$$

Therefore,

$$Tr(s) \vee \neg Tr(s).$$

By the individual clauses in the truth definition (or truth axioms), it now follows that

$$Tr(\ulcorner s \vee \neg s \urcorner).$$

Using fundamental properties of Gödelian arithmetization, we see that

$$Tr(Sub(s, \ulcorner x \urcorner, \ulcorner x \vee \neg x \urcorner)).$$

Generalizing, one obtains,

$$\forall s.\ Tr(Sub(s, \ulcorner x \urcorner, \ulcorner x \vee \neg x \urcorner)).$$

As stated above, this argument goes through for any scheme $\Theta(p)$ drawn from the language of standard propositional logic.

A proof of parametric inference $\mathbf{S} \Rightarrow \mathbf{V}$ in the converse direction—from sentential logical truth to valuational—seems to require a further assumption. Specifically, one needs to be assured that, for every truth-value $p$, there is a sentence $s$ of the object language bearing precisely $p$ as truth-value, in symbols,

$$\forall p\ \exists s(Tr(s) \leftrightarrow 0 \in p).$$

We refer to this as the 'Expressibility Assumption' or **EA**. (You must distinguish **EA** from the internal comprehension scheme,

$$\exists p(p \leftrightarrow \Theta(q, r)),$$

expressed in the notation of this essay. It is required that $p$ be distinct from $q$ and $r$. Gabbay treated the comprehension scheme in his [8] as a[n object-language] principle of second-order intuitionistic propositional logic.) Now, assume that **TND** is logically true substitutionally,

$$\forall s.\ Tr(Sub(s, \ulcorner x \urcorner, \ulcorner x \vee \neg x \urcorner)), \qquad (*)$$

and let $p$ be an arbitrary member of $\mathscr{P}$. By **EA**,

$$\exists s(Tr(s) \leftrightarrow 0 \in p).$$

From (*), we obtain

$$Tr(\ulcorner s \vee \neg s \urcorner),$$

where $s$ is a sentence whose existence **EA** guarantees. By the commutativity of $Tr$ with the connectives,

$$Tr(s) \vee \neg Tr(s).$$

Therefore, from **EA** once again, we know that

$$0 \in p \vee 0 \notin p.$$

Because $p$ was an arbitrary member of $\mathscr{P}$, we have that

$$0 \in \bigcap_{p \in P} (p \cup \sim p).$$

In other words,

$$\forall p(p \vee \neg p)$$

holds, and **TND** is logically true in the valuational sense. Please note that the foregoing reasoning is conventionally, as well as intuitionistically, cogent and will go through not merely for **TND** but for any propositional scheme.

If, provably in the metatheory, there exist in $\mathscr{P}$ only the two truth-values TRUE and FALSE or 0 and 1, then **EA** can be demonstrated, provided that the object language is minimally satisfactory. In that kind of language, there will be sentences $s_1$ and $s_2$ such that, in the metalanguage, one can show that $Tr(s_1)$ and that $\neg Tr(s_2)$. Hence, each of the two truth-values is the value of some object-language sentence, that is,

$$\forall p \in \mathscr{P}\ \exists s(Tr(s) \leftrightarrow 0 \in p).$$

Upon such reflections as these, it may seem at first that **EA** is always mathematically nugatory. Notably, that is not the case: **EA** is not provable intuitionistically from the axioms of **ZF** set theory, as we now demonstrate. Let the metatheory be **IZF**, first-order intuitionistic Zermelo–Fraenkel set theory, in a formal language appropriate to that theory. **IZF** includes the **ZF** axioms formulated so as not to imply the **TND**; *vide* [1]. Let the object language be that of elementary, first-order Peano–Dedekind arithmetic. We continue to assume that the latter language is included within the former via translation. Then, take for a model of the metatheory $\mathbb{V}(\mathscr{R})$, first cousin to Scott's original topological interpretation of analysis [24] extended to set theory along the lines of [12]. For the metametatheory, adopt conventional **ZF**. In $\mathbb{V}(\mathscr{R})$, formulae of set theory and arithmetic take on topological values that are open subsets of the real line $\mathscr{R}$. This interpretation satisfies all the axioms of **IZF**. In the metametatheory, one can prove that, for any sentence $\phi$ of elementary arithmetic,

$$\text{either } [\![\phi]\!] = \emptyset \text{ or } [\![\phi]\!] = \mathscr{R}.$$

Consequently, in the metatheory,

$$\mathbb{V}(\mathscr{R}) \models \forall s.\ Tr(Sub(s, \ulcorner x \urcorner, \ulcorner x \vee \neg x \urcorner)).$$

However,

$$\forall p\ (p \vee \neg p)$$

cannot hold in the metatheory, for in Scott's model, **TND** fails strongly:

$$\mathbb{V}(\mathscr{R}) \models \neg \forall p\ (p \vee \neg p).$$

Therefore, over $\mathbb{V}(\mathscr{R})$, the statement that every truth-value is the value of some sentence of the object language, that is **EA**, fails too, and as well as the inference **S** $\Rightarrow$ **V** that it licenses. Therefore, we see that

**Theorem.** In the strictly intuitionistic metamathematics **IZF**, one cannot prove **S** $\Rightarrow$ **V**. Consequently, in the same metatheory, one cannot prove the Expressibility Assumption **EA**. □

The sentential definition of the logical truth of a scheme does not imply the valuational definition in intuitionistic metamathematics, and we have a ready counterexample in the scheme for **TND**. We cannot derive, from the axioms of full, standard, first-order set theory without **TND**, the equivalence between the Quine–Hilbert, sentential characterization and the valuational characterization—in terms of propositional functions—that Russell and Tarski adopted. The inference $\mathbf{S} \Rightarrow \mathbf{V}$ linking the two characterizations and the statement **EA** are both independent of the axioms of intuitionistic set theory. (Incidentally, this independence theorem should not be tossed aside as some further, 'obvious' confirmation of a fatal logical weakness in **IZF**. Recall that **IZF** is more than sufficient unto the needs of Bishop-style analysis. Once again, *vide* [1].)

## 4. $\mathbf{S} \Rightarrow \mathbf{V}$ AND **EA** IN INTUITIONISTIC METAMATHEMATICS

The precise status—within intuitionistic metamathematics—of the inference $\mathbf{S} \Rightarrow \mathbf{V}$ as well as that of **EA** can now be determined. With the intuitionistic second-order arithmetic **HAS** as metatheory, neither the correctness of $\mathbf{S} \Rightarrow \mathbf{V}$ nor that of **EA** implies **TND**. In addition, neither implies the validity of any scheme of propositional logic underivable in Heyting's formal propositional calculus. Therefore, neither breaks currently recognized bounds on intuitionistic logical derivability. In addition, we prove that, over the same metatheory, $\mathbf{S} \Rightarrow \mathbf{V}$ does not itself derive **EA**.

**Theorem.** The Expressibility Assumption **EA** and, hence, the inference $\mathbf{S} \Rightarrow \mathbf{V}$ do not imply **TND** within the intuitionistic second-order arithmetic **HAS**.

*Proof.* Until further notice, the metametatheory will be conventional. Let $\mathbb{S}$ be Sierpinski space with elements $\alpha$ and $\beta$ such that $\{\beta\}$ is open in $\mathbb{S}$, but $\{\alpha\}$ is not. Let $\mathfrak{A}$ be the standard model of **PAS**, conventional second-order arithmetic. Let $\mathfrak{B}$ be a nonstandard or Henkin model of **PAS** together with the formal arithmetic sentence

**PAS** is inconsistent,

as Gödel's Second Incompleteness Theorem allows [10]. The domain $|\mathfrak{A}|$ of $\mathfrak{A}$ is identified with the standard part of $\mathfrak{B}$. Then, forcing for **HAS** is defined over $\mathbb{S}$, interpreting second-order variables so as to range over pairs of subsets $\langle A, B\rangle$ drawn respectively from $\mathfrak{A}$ and $\mathfrak{B}$ that are *allowable*; cf. [27, p. 389ff.] and [28, pp. 166–167].

**Definition.** The pair $\langle A, B\rangle$ is *allowable* whenever $A$ is any subset of the domain $|\mathfrak{A}|$ of $\mathfrak{A}$ while $B$ is a subset of domain $|\mathfrak{B}|$ that exists in the model $\mathfrak{B}$ and is such that $A \subseteq B$.

Of course, both $A$ and $B$ may be empty. Clearly, for every $A \subseteq |\mathfrak{A}|$ there is at least one $B$ such that $\langle A, B\rangle$ is allowable. In the sequel, whenever such expressions as '$\langle A, B\rangle$' or '$\langle C, D\rangle$' appears, it is assumed that $\langle A, B\rangle$ and $\langle C, D\rangle$ are allowable.

The arithmetic fragment of the forcing relation is defined as usual for the standard logical signs as in [25]. To interpret formal statements of set membership, one requires that

$$\alpha \Vdash a \in \langle A, B\rangle \text{ if and only if } \mathfrak{A} \vDash a \in A,$$

and

$$\beta \Vdash a \in \langle A, B\rangle \text{ if and only if } \mathfrak{B} \vDash a \in B.$$

For universal set quantification, the relevant conditions are

$$\alpha \Vdash \forall X.\ \phi(X) \text{ if and only if } \forall \langle A, B\rangle\ [\alpha \Vdash \phi(\langle A, B\rangle) \text{ and } \beta \Vdash \phi(\langle A, B\rangle)],$$

while

$$\beta \Vdash \forall X.\ \phi(X) \text{ if and only if } \forall \langle A, B\rangle\ .\ \beta \Vdash \phi(\langle A, B\rangle).$$

For existential quantification, the condition

$$\alpha \Vdash \exists X.\ \phi(X) \text{ if and only if } \exists \langle A, B\rangle\ .\ \alpha \Vdash \phi(\langle A, B\rangle)$$

holds, and analogously for $\beta$. The partially ordered model so defined we call $\mathscr{M}$.

That forcing in $\mathscr{M}$ is natural or persistent from $\alpha$ to $\beta$ is as expected.

**Proposition 1.** For all formulae $\phi$ in the language of **HAS** and all pairs $\langle A,B\rangle$, if $\alpha \Vdash \phi(\langle A,B\rangle)$, then $\beta \Vdash \phi(\langle A,B\rangle)$.

*Proof.* By induction on formulae. For example, let $\alpha \Vdash \forall X.\ \phi(X,\langle A,B\rangle)$. Then, by definition, for all pairs $\langle C,D\rangle$,

$$\alpha \Vdash \phi(\langle C,D\rangle,\langle A,B\rangle),$$

and

$$\beta \Vdash \phi(\langle C,D\rangle,\langle A,B\rangle).$$

So, for all pairs $\langle C,D\rangle$, $\beta \Vdash \phi(\langle C,D\rangle,\langle A,B\rangle)$. Therefore, once again from the forcing definition, one sees that $\beta \Vdash \forall X.\ \phi(X,\langle A,B\rangle)$. □

For pure second-order sentences, forcing at $\beta$ and truth in the structure $\mathfrak{B}$ coincide extensionally.

**Proposition 2.** For all formulae $\phi$ in the language of **HAS** and all pairs $\langle A,B\rangle$, $\beta \Vdash \phi(\langle A,B\rangle)$ if and only if $\mathfrak{B} \models \phi(B)$.

*Proof.* First, by definition, $\beta \Vdash a \in (\langle A,B\rangle)$ if and only if $\mathfrak{B} \models a \in B$. Second, $\beta$ is the terminal node in $\mathbb{S}$. □

It remains only to check that the axioms of **HAS** are forced at $\alpha$.

**Proposition 3.** For any axiom $\phi$ of **HAS**, $\alpha \Vdash \phi$.

*Proof.* For the sake of example, we verify the forcing of complete induction and full comprehension.

*For Induction*: **Proposition 2** already tells us that $\beta$ forces Induction. Let $Sx$ be the formal expression, in the language of **HAS**, for the successor of $x$. As for $\alpha$, assume

$$\alpha \Vdash [0 \in \langle A,B\rangle \wedge \forall x(x \in \langle A,B\rangle \to Sx \in \langle A,B\rangle)].$$

Then, by definition of forcing,

$$\mathfrak{A} \models 0 \in A$$

and

$$\mathfrak{A} \models \forall x(x \in A \to Sx \in A).$$

Because $\mathfrak{A} \models$ **PAS**,

$$\mathfrak{A} \models \forall x.\ x \in A.$$

From **Proposition 1**, it follows that

$$\beta \Vdash 0 \in \langle A,B\rangle \wedge \forall x(x \in \langle A,B\rangle \to Sx \in \langle A,B\rangle).$$

Thanks to **Proposition 2** and $\mathfrak{B} \models$ **PAS**, we know that

$$\beta \Vdash \forall x.\ x \in \langle A,B\rangle.$$

Therefore, again by the definition of $\Vdash$,

$$\alpha \Vdash \forall x.\ x \in \langle A,B\rangle.$$

*For Comprehension*: Since $\mathfrak{B} \vDash$ **PAS**, for any formula $\phi$ not containing the second-order variable $X$ and any pair $\langle C,D\rangle$, we know that

$$\beta \Vdash \exists X \forall x(x \in X \leftrightarrow \phi(x, \langle C,D\rangle)).$$

Consider the pair $\langle A,B\rangle$ where

$$A = \{a \in |\mathfrak{A}| : \alpha \Vdash \phi(a, \langle C,D\rangle)\},$$

and

$$B = \{b \in |\mathfrak{B}| : \beta \Vdash \phi(b, \langle C,D\rangle)\}.$$

By **Proposition 1**, $\langle A,B\rangle$ is allowable. From the definition of $\langle A,B\rangle$, one concludes that, for all $a \in |\mathfrak{A}| \cup |\mathfrak{B}|$,

$$\alpha \Vdash a \in \langle A,B\rangle \text{ if and only if } \alpha \Vdash \phi(a, \langle C,D\rangle),$$

and

$$\beta \Vdash a \in \langle A,B\rangle \text{ if and only if } \beta \Vdash \phi(a, \langle C,D\rangle).$$

Therefore,

$$\alpha \Vdash \exists X \forall x(x \in X \leftrightarrow \phi(x, \langle C,D\rangle)).$$ □

In the model $\mathscr{M}$ for **HAS**,

$$\forall p\, \exists s(Tr(s) \leftrightarrow 0 \in p)$$

obtains. As for the range of the variable '$p$' above, '$p$' has at most three allowable values: $\langle\emptyset,\emptyset\rangle$, $\langle\emptyset,\{0\}\rangle$, and $\langle\{0\},\{0\}\rangle$. For the first of those, '$0 = 1$' is an arithmetic sentence whose code number $s_1$ is such that

$$\alpha \Vdash (Tr(s_1) \leftrightarrow 0 \in \langle\emptyset,\emptyset\rangle).$$

For the second, '**PAS** is inconsistent' is an arithmetic sentence whose code $s_2$ is such that

$$\alpha \Vdash (Tr(s_2) \leftrightarrow 0 \in \langle\emptyset,\{0\}\rangle).$$

Finally, '$0 = 0$' is an arithmetic sentence whose code $s_3$ is such that

$$\alpha \Vdash (Tr(s_3) \leftrightarrow 0 \in \langle\{0\},\{0\}\rangle).$$

Plainly, $\alpha$ does not force '**PAS** is inconsistent or **PAS** is not inconsistent.' So, **EA** and *a fortiori* **S** $\Rightarrow$ **V** do not imply, in intuitionistic second-order arithmetic, **TND**. □

**Notes:**

(1) As remarked, the proof of this theorem calls upon a conventional metametalogic, at least in the assumption of model completeness for predicate logic, namely, that every consistent set of formal sentences has a model. Such an assumption is not correct intuitionistically; it implies the nonintuitionistic Law of Testability or **TEST**:

$$\forall p(\neg p \vee \neg\neg p)$$

[18], [4], and [5]. However, all is not lost. Friedman [7] has shown, consistently with intuitionistic mathematics, that classical set theory is conservative over intuitionistic set theory for $\Pi^0_2$ sentences of arithmetic. The statement that **EA** does not imply **TND** is a formal nonderivability statement. It is therefore arithmetic, specifically $\Pi^0_1$. Therefore, by Friedman's result, there must be an intuitionistically allowable proof of the theorem.

Alternatively, one could conduct the reasoning of the above proof within the hereditarily stable fragment of intuitionistic third-order arithmetic. That fragment is the image of the Gödel–Gentzen negative translation [11], [9], so the necessary classical reasoning will go through. Again, because the nonderivability statement is

$\Pi^0_1$, it is absolute for the negative translation. As a result, it is provable in intuitionistic arithmetic of the third order.

(2) For later use, we now mention that **TEST** holds true under the forcing interpretation described in the preceding proof.

## 5. EA AND THE THEOREMS OF PROPOSITIONAL LOGIC

In **HAS**, one cannot show that **EA** or $\mathbf{S} \Rightarrow \mathbf{V}$ implies the validity (either sentential or valuational) of any scheme of propositional logic not already a theorem of Heyting's propositional logic. Therefore, although the intuitionist can demonstrate neither **EA** nor that Hilbert and Quine's definition of the logical truth of a scheme implies Russell and Tarski's, it is open to the intuitionist to adopt both the assumption and the inference without damaging the range of intuitionistic validity, formally delimited.

**Theorem.** Let $\phi(p,q)$ be any propositional scheme in the atoms $p$ and $q$. If $\phi(p,q)$ is not a theorem of Heyting's propositional logic, then "$\phi(p,q)$ is a logical truth valuationally" and "$\phi(p,q)$ is a logical truth sententially" are both underivable in **HAS** either from **EA** or from $\mathbf{S} \Rightarrow \mathbf{V}$.

*Proof.* It is entirely for the sake of example that the scheme $\phi(p,q)$ is limited to two proposition letters. The theorem and its proof apply to any propositional scheme. The argument here echoes Smorynski's [25] model-theoretic proof of de Jongh's Theorem.

Assume $\phi(p,q)$ to be a propositional scheme in $p$ and $q$ that is not a theorem of Heyting's formal propositional logic. Then, there is a finite frame $\langle K, \leqq \rangle$ in the Jaskowski sequence, and hence a tree, with bottom node $\alpha$ and number of terminal nodes $n \in \mathbb{N}$ such that $\alpha \nVdash \phi(p,q)$ [14]. Let the terminal nodes of $\langle K, \leqq \rangle$ be numbered 0 through $n-1$. In frames of the Jaskowski sequence, each node is precisely characterized by the set of terminal nodes above it. Thanks to Myhill's proof of the Theorem of Gödel, Rosser, and Mostowski [21], there are $n$ sentences $\phi_i$, $i < n$, in the language of **HAS** that are mutually independent over the classical second-order arithmetic **PAS**. By the conventional completeness theorem of Henkin, for each $i < n$, there is a model $\mathfrak{A}_i$ of $\mathbf{HAS} \cup \{\phi_i \wedge \bigwedge_{j \neq i} \neg\phi_j\}$.

We use these models to construct, on the frame $\langle K, \leqq \rangle$, a forcing model $\mathscr{M}_K$ for **HAS** that satisfies **EA**, but does not satisfy $\phi(p,q)$. For decorating the frame, I attach model $\mathfrak{A}_i$ to the terminal node numbered $i$. To all other nodes, the standard model gets attached. For values of the second-order variables of the forcing model, I take all families

$$F = \lambda\beta \in K.\, A_\beta$$

of sets with members drawn from the domains of the models that are allowable in that, for all $\beta$, $\gamma \in K$, such that $\beta \leqq \gamma$,

$$F(\beta) \subseteq F(\gamma).$$

Forcing is defined as usual except that, for atomic formulae of the form $a \in F$, for $F$ an allowable family as above and $\beta \in K$,

$$\beta \Vdash a \in F \text{ if and only if } a \in F(\beta).$$

Then, as in the proof of the theorem of the last section, it is straightforward to check that $\mathscr{M}_K$ is a model for **HAS**.

To see that $\mathscr{M}_K$ forces **EA** and therefore $\mathbf{S} \Rightarrow \mathbf{V}$, let $\beta$ be any node of $\langle K, \leqq \rangle$. Let $j$ and $k$, $j \neq k < n$, be all the terminal nodes that lie above $\beta$. ($\beta$ was selected with two terminal nodes above it for the sake of example only.) Take $\psi_\beta$ to be the arithmetic formula $\neg\neg(\phi_j \vee \phi_k)$. $\psi_\beta$ is forced only at $\beta$ and those nodes of $K$ that lie above $\beta$.

In the model $\mathscr{M}_K$, the truth-values are the appropriate families $L_0$ where $L$ is an $\leqq$-upward-closed subset of $K$ and

$$0 \in L_0(\beta) \text{ if and only if } \beta \in L.$$

Now, let $L$ be an $\leqq$-upward-closed subset of the frame. Since the frame is finite, $L$ is determined by its $\langle K, \leqq \rangle$-minimal nodes $\gamma_j$, $j < m$ for some $m$. Take $\psi_L$ to be the disjunction $\bigvee \psi_{\gamma_j}$ for $j < m$. Then, $\psi_L$ is such that

$$Tr(\psi_L) \leftrightarrow 0 \in L_0.$$

Therefore,

$$\alpha \Vdash \forall p\, \exists s (Tr(s) \leftrightarrow 0 \in p).$$

The propositional formula $\phi(p,q)$ fails to be sententially valid in $\mathscr{M}_K$ because

$$\alpha \nVdash \phi(\psi_M, \psi_N),$$

where $M$ is the upward closed set that is the value of the atom $p$ in the original frame and $N$ is that for $q$. Also, the construction shows that there are arithmetic sentences $s_M$ and $s_N$ that are forced precisely at the nodes of $M$ and $N$, respectively. □

**Note.** The metametamathematics of the proof as it stands is conventional. Once again, as above, we know that there is an intuitionistic proof of the underivability results thanks to the conservative extension theorem of [7].

## 6. More on $\mathbf{S} \Rightarrow \mathbf{V}$ and EA in intuitionistic metamathematics

It remains to ask if $\mathbf{S} \Rightarrow \mathbf{V}$ is, over intuitionistic formal theories, tantamount to **EA**.

**Theorem.** In the intuitionistic second-order arithmetic **HAS**, $\mathbf{S} \Rightarrow \mathbf{V}$ does not derive **EA**.

*Proof.* For each frame $K$ of the Jaskowski sequence $\mathscr{S}$, form the model $\mathscr{M}_K$ as in the preceding proof. Also, construct two extra identical copies $\mathscr{M}_1$ and $\mathscr{M}_2$ of the forcing model $\mathscr{M}$ over Sierpinski space employed in Section 4 *supra*. Then, glue these models together in the familiar fashion over a single bottom node $\alpha$—as in [25]—to obtain the infinite joint model

$$\mathscr{M}_3 = \Sigma_{K \in \mathscr{S}} M_K + M_1 + M_2.$$

$\mathscr{M}_3$ cannot satisfy **EA**, since the top nodes in each of $\mathscr{M}_1$ and $\mathscr{M}_2$ determine truth values of $\mathscr{M}_3$ that cannot be distinguished by first-order arithmetic sentences.

To see that $\mathbf{S} \Rightarrow \mathbf{V}$ holds in $\mathscr{M}_3$, note first that, in virtue of the preceding proof, the inference $\mathbf{S} \Rightarrow \mathbf{V}$ is forced at every node in the frame except (perhaps) for $\alpha$. Now, let $\Theta(p)$ be any propositional scheme. Since derivability in Heyting's propositional logic is decidable, either $\vdash \Theta(p)$ or $\nvdash \Theta(p)$. If the former, then **HAS** proves that $\Theta(p)$ is valid in the valuational sense. Therefore,

$$\alpha \Vdash \forall p\, \Theta(p)$$

and this instance of $\mathbf{S} \Rightarrow \mathbf{V}$ holds at $\alpha$. On other hand, if $\Theta(p)$ is not a theorem of Heyting's propositional logic, then there is a frame $K$ in $\mathscr{S}$ whose the bottom node $\beta$ in the frame of $\mathscr{M}_3$ is such that

$$\beta \nVdash \forall s (Tr(s, \ulcorner p \urcorner, \ulcorner \Theta(p) \urcorner)).$$

Therefore,

$$\alpha \nVdash \forall s (Tr(s, \ulcorner p \urcorner, \ulcorner \Theta(p) \urcorner)),$$

and the proof is complete. □

Again, via the kinds of argument set out in the note at the end of Section 4, one can show intuitionistically that there are fully intuitionistic proofs of the underivability results of this section.

## 7. Completeness and logical truth: Questions from Dana Scott

Now, we return to working within strictly intuitionistic metatheories.

At first glance, there might seem to be theoretical ties, even intuitionistically, between **EA** and versions of logical completeness. Some readers may recall Quine's discussion [22] of the conventional mathematical implications between (what we here call) $\mathbf{S} \Rightarrow \mathbf{V}$ and the Hilbert-Bernays form of the classical Completeness Theorem for first-order predicate logic. Dana Scott has asked if there are any such relations of significance obtaining, over **IZF** or **HAS**, between the Expressibility Assumption **EA** and either strong completeness or model completeness for intuitionistic propositional logic. (Scott posed his questions during the Scottfest Colloquium at the Department of Computer Science of Carnegie-Mellon University on 11 October 2013.)

**Definitions.** Here, $\vdash$ stands for the formal *derivability* relation of Heyting's intuitionistic propositional logic.

(1) Formal intuitionistic propositional logic is *strongly complete* just in case, for all propositional schemes $\phi$ and all sets $\Gamma$ of schemes, if $\Gamma \vDash \phi$ then $\Gamma \vdash \phi$. In this definition, '$\vDash$' can be defined with respect to either assignments of truth-values à la Tarski or frames.

(2) Formal intuitionistic propositional logic is *weakly complete* just in case, for all propositional schemes $\phi$, if $\vDash \phi$, then $\vdash \phi$. This time, '$\vDash$' is defined with respect to frames exclusively.

(3) Formal intuitionistic propositional logic is *model complete* just in case, for all sets $\Gamma$ of propositional schemes, if $\Gamma$ is consistent (relative to intuitionistic propositional logic), then there is a model $\mathscr{M}$ such that $\mathscr{M} \vDash \Gamma$. Once again, either frames or Tarskian assignments can determine the relevant models.

We note in advance that, because strong completeness entails **TND** in **HAS** [19], it also entails model completeness. Model completeness does not entail strong completeness, since the Law of Testability **TEST** is tantamount to model completeness [4], but does not imply **TND**. Heyting's $\vdash$ is decidable when restricted to single formulae; hence, weak completeness, as enunciated above, is a theorem of **HAS** and **IZF**. Therefore, weak completeness cannot be used to deduce strong completeness or model completeness within **IZF**.

The following results, relevant to answering Scott's questions, are provable.

**Theorems.** Over **HAS**,

(1) **EA** does not derive model completeness,
(2) **EA** does not derive strong completeness,
(3) Strong completeness derives **EA**, but
(4) Model completeness does not derive **EA**, and
(5) Weak completeness does not derive **EA**.

**Proofs.**

(1) As proved in Section 4, **EA** does not derive **TEST**.
(2) This follows from the theorems of Section 4 because strong completeness derives **TND**.
(3) Again, since strong completeness implies **TND** in **HAS**, it entails **EA**.

(4) **TEST** holds in the forcing model $\mathscr{M}$ over Sierpinski space of Section 4, but **EA** does not. Therefore, model completeness—equivalent to **TEST** over **HAS**—cannot derive **EA**.
(5) As proved in Section 3, **EA** is not a theorem of **HAS**; so, weak completeness cannot derive **EA** in **HAS**. □

## 8. EA AND THE BROUWER–KRIPKE SCHEME

**EA** bears a more than superficial resemblance to a(n in)famous reduction principle of intuitionistic higher mathematics, the Brouwer–Kripke Scheme or **BKS**:

$$\forall p\ \exists f \in (\mathbb{N} \Rightarrow \mathbb{N})\ (p \leftrightarrow \exists n \in \mathbb{N}\,.\, f(n) = 0),$$

where $\mathbb{N} \Rightarrow \mathbb{N}$ is the set of all total, natural-number-valued functions on the natural numbers $\mathbb{N}$. John Myhill [20] first recommended adding **BKS** to systems of intuitionistic analysis to capture thereby Brouwer's theory of the creating subject [2]. Like **BKS**, **EA** fails in the Kleene realizability interpretation extended to **HAS** or **IZF** [16], [17]. More generally, **EA** will not hold under any interpretation of **IZF**—such as Kleene realizability—that satisfies the Uniformity Principle **UP**, where **UP** is the statement that every total, natural-number-indexed covering of $\mathscr{P}$ is univalent:

$$\forall R\ [\forall p \in \mathscr{P}\ \exists n \in \mathbb{N}\,.\, R(p,n) \to \exists n \in \mathbb{N}\ \forall p \in \mathscr{P}.\, R(p,n)].$$

Please recall that the '$s$' variable in **EA** ranges over natural numbers that Gödelize sentences of arithmetic.

Brouwer's use of the creating subject can be recovered by using **BKS** to transform assertions of invalidity in logic into strong counterexamples in analysis. For example, from the fact that **TEST** is invalid, Brouwer argued via **BKS** (in effect) that it is false that every real number different from zero is also at some positive distance apart from zero [29, pp. 842–843]. In similar fashion, one could reason via **EA** from general results about invalidity to their arithmetical instantiations. For example, given **EA**, it follows from the invalidity of **TEST**,

$$\neg\forall p(\neg p \vee \neg\neg p),$$

that

$$\neg\forall s.\ Tr(\ulcorner \neg s \vee \neg\neg s \urcorner).$$

The totality of such reductive inferences from **EA** yields an obvious valuational correlate to de Jongh's Maximality Theorem and its extensions [15]. Since the relevant completeness theorems, e.g., for all schemes $\Theta(p)$,

$$\vdash \Theta(p) \text{ if and only if } \forall p\ \in \mathscr{P}.\ \Theta(p)$$

are underivable in **IZF**, de Jongh's result is not itself obtainable in a straightforward fashion from its valuational correlate.

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