ACTAS DEL XIV CONGRESO
DR. ANTONIO A. R. MONTEIRO (2017)
2019, Páginas 63–71

CONSTRUCTION OF NELSON ALGEBRAS

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ABSTRACT. The Vakarelov construction of Nelson algebras up from Heyting ones is generalized to obtain De Morgan algebras from distributive lattices. Necessary and sufficient conditions for these De Morgan algebras to be Nelson algebras are shown, and a characterization of the join-irreducible elements in the finite case is given.

1. INTRODUCTION

The calculus of constructive logic with strong negation was introduced by N. N. Vorobyeb [24] following ideas suggested by D. Nelson [14] and A. A. Markov [6]. H. Rasiowa introduced N-lattices ([17]) or *Nelson algebras* as an adequate semantics for this calculus. D. Vakarelov [22] showed how a refinement of a construction due to Kalman allows a better intuitive connection between the original ideas of Nelson and Markov and their algebraic counterpart, associating pairs of elements in a Heyting algebra to propositions and their refutation by a counter-example. This kind of refutation will represent the strong negation of the proposition, while the weak negation will be represented by $x \to 0$, as it happens in the intuitionistic calculus. We believe that this connection is worthy of further exploration. Some further applications of this construction may be found in [23].

The results in this article were communicated in 1997 in the IV Congreso Antonio Monteiro [13], but have remained unpublished until now. Meanwhile, this article has been cited in [19], and the idea of the construction has appeared also in [15], with the name of *twist structure*.

Definition 1.1. A *Heyting algebra* (A. Monteiro [7], see also R. Balbes and P. Dwinger [1]) is an algebra $(A, \wedge, \vee, \Rightarrow, 0, 1)$ of type $(2,2,2,0,0)$ that satisfies:

$H_0)$ $0 \wedge x \approx 0$	$H_1)$ $x \Rightarrow x \approx 1$
$H_2)$ $(x \Rightarrow y) \wedge y \approx y$	$H_3)$ $x \wedge (x \Rightarrow y) \approx x \wedge y$
$H_4)$ $x \Rightarrow (y \wedge z) \approx (x \Rightarrow z) \wedge (x \Rightarrow y)$	$H_5)$ $(x \vee y) \Rightarrow z \approx (x \Rightarrow z) \wedge (y \Rightarrow z)$

Definition 1.2. A *Nelson algebra* is an algebra $(A, 1, \sim, \vee, \wedge, \to)$ of type $(0,1,2,2,2)$ which satisfies the identities:

N1) $x \vee 1 \approx 1$	N2) $x \wedge (x \vee y) \approx x$
N3) $x \wedge (y \vee z) \approx (z \wedge x) \vee (y \wedge x)$	N4) $\sim\sim x \approx x$
N5) $\sim (x \wedge y) \approx \sim x \vee \sim y$	N6) $x \wedge \sim x \approx (x \wedge \sim x) \wedge (y \vee \sim y)$
N7) $x \to x \approx 1$	N8) $x \wedge (x \to y) \approx x \wedge (\sim x \vee y)$
N9) $(x \wedge y) \to z \approx x \to (y \to z)$	N10) $(x \to y) \wedge (\sim x \vee y) \approx \sim x \vee y$
N11) $x \to (y \wedge z) \approx (x \to y) \wedge (x \to z)$	

This definition is equivalent to the one given by H. Rasiowa [17, 18]. A. and L. Monteiro [11] proved that the axioms N1), N10) and N11) follow from N2)-N9), and that the latter are independent. These results were obtained in 1973 (see [11] and also D. Brignole and A. Monteiro [3, 2]) and published only in 1995. From axioms N2) and N3) it follows that A

2010 *Mathematics Subject Classification.* Primary: 03G25; Secondary: 06D20.

is a distributive lattice ([21]). Furthermore, by N4)-N6), A is a Kleene algebra. A. Monteiro [9, 10] showed that in every Nelson algebra:

N12) $(x \vee y) \to z \approx (x \to z) \wedge (y \to z)$.

2. The Kalman construction

The following construction of Nelson algebras was given by J. Kalman [5] for Kleene algebras.

Given a bounded distributive lattice $(L, \wedge, \vee, 0, 1)$, and an element $a \in L$, let

$$L(a) = \{(x,y) \in H \times H : x \wedge y \leq a \leq x \vee y\}.$$

It is clear that:

K1) $(0,1), (1,0) \in L(a)$.

J. Kalman defines the following operations on $L(a)$:

K2) $(x_1,x_2) \cup (y_1,y_2) = (x_1 \vee y_1, x_2 \wedge y_2)$;
K3) $(x_1,x_2) \cap (y_1,y_2) = (x_1 \wedge y_1, x_2 \vee y_2)$;
K4) $\sim (x_1,x_2) = (x_2,x_1)$;

and shows that $(L(a), \cap, \cup, (1,0))$ is a Kleene algebra.

We derive the following generalization of this construction:

For a given ideal I and a filter F of a bounded distributive lattice $(L, \wedge, \vee, 0, 1)$, we consider the set:

$$M(L,I,F) = \{(a,b) \in L \times L : a \wedge b \in I \text{ and } a \vee b \in F\}.$$

It is clear that if $(a,b) \in M(L,I,F)$ then $(b,a) \in M(L,I,F)$. As $0 \wedge 1 = 0 \in I$ and $0 \vee 1 = 1 \in F$, then $(0,1), (1,0) \in M(L,I,F)$.

$M(L,I,F)$ is closed under the operations $\cup$ and $\cap$ defined by K2) and K3): Let (a_1,a_2), $(b_1,b_2) \in M(L,I,F)$, then

(1) $a_1 \wedge a_2 \in I$ (2) $a_1 \vee a_2 \in F$ (3) $b_1 \wedge b_2 \in I$ (4) $b_1 \vee b_2 \in F$.

As $a_1 \wedge a_2 \wedge b_2 \leq a_1 \wedge a_2$, by (1) we obtain (5) $a_1 \wedge a_2 \wedge b_2 \in I$. Analogously, from (3) we get (6) $b_1 \wedge b_2 \wedge a_2 \in I$. Thus, by (5) and (6), $(a_1 \vee b_1) \wedge (a_2 \wedge b_2) = (a_1 \wedge a_2 \wedge b_2) \vee (b_1 \wedge a_2 \wedge b_2) \in I$.

In a similar way, from (2) and (4), $(a_1 \vee b_1) \vee (a_2 \wedge b_2) = (a_1 \vee a_2 \vee b_1) \wedge (a_1 \vee b_1 \vee b_2) \in F$ yields. Therefore $(a_1 \vee b_1, a_2 \wedge b_2) = (a_1,a_2) \cup (b_1,b_2) \in M(L,I,F)$.

In a similar fashion we may prove that $(a_1 \wedge b_1, a_2 \vee b_2) = (a_1,a_2) \cap (b_1,b_2) \in M(L,I,F)$.

Next, we show that $(M(L,I,F), \cap, \cup, (0,1), (1,0))$ is a bounded distributive lattice, using the axioms given by M. Sholander [21].

N0) $(a,b) \cap (0,1) = (a \wedge 0, b \vee 1) = (0,1)$.
N1) $(a,b) \cup (1,0) = (a \vee 1, b \wedge 0) = (1,0)$.
N2) $(a,b) \cap ((a,b) \cup (c,d)) = (a,b) \cap (a \vee c, b \wedge d) = (a \wedge (a \vee c), b \vee (b \wedge d)) = (a,b)$.
N3) $(a,b) \cap ((c,d) \cup (e,f)) = ((e,f) \cap (a,b)) \cup ((c,d) \cap (a,b))$ since $(a,b) \cap ((c,d) \cup (e,f)) = (a,b) \cap (c \vee e, d \wedge f) = (a \wedge (c \vee e), b \vee (d \wedge f))$, and on the other hand, $((e,f) \cap (a,b)) \cup ((c,d) \cap (a,b)) = (e \wedge a, f \vee b) \cup (c \wedge a, d \vee b) = ((e \wedge a) \vee (c \wedge a), (f \vee b) \wedge (d \vee b)) = (a \wedge (c \vee e), b \vee (d \wedge f))$.

Theorem 2.1. *If we define the operation $\sim$ on $M(L,I,F)$ by K4) then $(M(L,I,F), \cap, \cup, \sim)$ is a De Morgan algebra.*

Proof. Indeed:

N4) $\sim\sim (a,b) = \sim (b,a) = (a,b)$.

N5) $\sim ((a,b) \cap (c,d)) = \sim (a \wedge c, b \vee d) = (b \vee d, a \wedge c) = (b,a) \cup (d,c) = \sim (a,b) \cup \sim (c,d)$. □

Remark 2.1. *An element c of a De Morgan algebra is a* center *if $c = \sim c$. Then if $c = (x,y) \in M(L,I,F)$ is a center: $(x,y) = \sim (x,y) = (y,x)$, so $x = y$ and since $x \wedge y \in I$, $x \vee y \in F$, then $x \in I$, $x \in F$ and therefore $x \in I \cap F$. If now $x \in I \cap F$, then $(x,x) \in M(L,I,F)$ and it is a center.*

Remark 2.2. *The order relation "$\leq$" induced on $M(L,I,F)$ by the operation $\cap$ is such that*

$$(a_1,a_2) \leq (b_1,b_2) \text{ iff } a_1 \leq b_1 \text{ and } b_2 \leq a_2.$$

If we denote with L^* the dual lattice of L, it is clear that $M(L,I,F) \subseteq L \times L^*$.

M. Fidel and D. Brignole define in [4] the following notion:

Definition 2.1. A *P-De Morgan algebra* is a non-empty subset M of $L \times L^*$ such that:

P1) $(0,1),(1,0) \in M$,
P2) If $(x_1,x_2) \in M$ then $(x_2,x_1) \in M$,
P3) If $(x_1,x_2),(y_1,y_2) \in M$, then $(x_1 \wedge y_1, x_2 \vee y_2) \in M$,

in which the operations $\cap, \cup$ and $\sim$ are defined by K2)-K4).

If $(x_1,x_2),(y_1,y_2) \in M$ then by $P2),(x_2,x_1),(y_2,x_2) \in M$ and by $P3)$, $(x_2 \wedge y_2, x_1 \vee y_1) \in M$; then, using again $P2)$, we obtain $(x_1 \vee y_1, x_2 \wedge y_2) \in M$.

From the remarks made above it follows that the construction of $M(L,I,F)$ is a particular case of P-De Morgan algebras. We indicate an example of a P-De Morgan algebra which is not of the form $M(L,I,F)$: if $\mathbf{T} = \{0,a,1\}$ is a chain $(0 < a < 1)$ then $\mathbf{T}$ is a distributive lattice. $M = \{(0,1),(a,a),(1,0)\}$ is a P-De Morgan algebra, but there does not exist an ideal I nor a filter F in $\mathbf{T}$ such that $M(\mathbf{T},I,F) = M$.

Lemma 2.1. *$M(L,I,F)$ is a Kleene algebra if and only if for all $i \in I$ and $f \in F$, $i \leq f$ is satisfied.*

Proof. If $M(L,I,F)$ is a Kleene algebra, then

N6) $(a,b) \cap \sim (a,b) \leq (c,d) \cup \sim (c,d)$.

If $i \in I$ and $f \in F$ then all the elements $(i,1),(0,f) \in M(L,I,F)$, because $i \wedge 1 = i \in I$, $i \vee 1 = 1 \in F$ and $0 \wedge f = 0 \in I$, $0 \vee f = f \in F$. Then,

$$(i,1) = (i,1) \cap (1,i) = (i,1) \cap \sim (i,1) \leq (0,f) \cup \sim (0,f) = (0,f) \cup (f,0) = (f,0),$$

and then $i \leq f$.

If now for all $i \in I$, $f \in F$, $i \leq f$ is satisfied, then for $(a,b),(c,d) \in M(L,I,F)$, we have

$$(a,b) \cap \sim (a,b) = (a,b) \cap (b,a) = (a \wedge b, a \vee b),$$

$$(c,d) \cup \sim (c,d) = (c,d) \cap (d,c) = (c \vee d, c \wedge d).$$

As $a \wedge b \in I, c \vee d \in F$, then $a \wedge b \leq c \vee d$ and as $c \wedge d \in I, a \vee b \in F$, $c \wedge d \leq a \vee b$. Therefore, $(a,b) \cap \sim (a,b) \leq (c,d) \cup \sim (c,d)$. □

Remark 2.3. *If L is a finite distributive lattice, then every ideal I is generated by a single element, $I = I(a) = \{x \in L : x \leq a\}$ and the same happens for every filter $F = F(b) = \{y \in L : b \leq y\}$. In this case, the condition indicated in Lemma 2.1 is satisfied if and only if $a \leq b$. Furthermore, $M(L,I,F)$ has a center, necessarily unique, if and only if $a = b$.*

It is easy to see that:

Lemma 2.2. *1) If I_1, I_2 are ideals of L and $I_1 \subseteq I_2$ then $M(L, I_1, F)$ is a subalgebra of $M(L, I_2, F)$.*

2) If F_1, F_2 are filters of L and $F_1 \subseteq F_2$ then $M(L, I, F_1)$ is a subalgebra of $M(L, I, F_2)$.

Lemma 2.3. *If B is a boolean algebra then $M(B, \{0\}, \{1\})$ is a boolean algebra isomorphic to B.*

Proof. $(a,b) \in M(B, \{0\}, \{1\})$ if and only if $a \wedge b = 0$ and $a \vee b = 1$, so $a = -b$.

Let $h : B \to M(B, \{0\}, \{1\})$ defined by $h(a) = (a, -a)$. It is clear that h is well-defined and is bijective. Furthermore, $h(1) = (1,0); h(0) = (0,1)$ and $h(x \wedge y) = (x \wedge y, -(x \wedge y)) = (x \wedge y, -x \vee -y) = (x, -x) \cap (y, -y) = h(x) \cap h(y)$. $h(-x) = (-x, x) = \sim (x, -x) = \sim h(x)$ so $h(x \vee y) = h(x) \cup h(y)$ is also satisfied. □

Remark 2.4. *The previous lemma suggests a natural injection of a De Morgan algebra A in $M(A, A, A)$ by means of the application $b \mapsto (b, \sim b)$. However, it is easy to find cases where it does not exist a filter nor an ideal such that the corresponding subalgebra of $M(A, A, A)$ isomorphic to A is of the form $M(A, I, F)$.*

Example 2.1. *If I and F are, respectively, a prime ideal and a prime filter of a distributive lattice L then the elements of $M(L, I, F)$ are those in the set*

$$[(F \cap I) \times L] \cup [F \times I] \cup [I \times F] \cup [L \times (F \cap I)].$$

Indeed, if $(a,b) \in M(L, I, F)$ then $a \vee b \in F$ and as F is a prime filter, $a \in F$ or $b \in F$. In a similar way, from $a \wedge b \in I$, we deduce that $a \in I$ or $b \in I$. If $a \in F$, then we have $a \in I$ and $(a,b) \in [(F \cap I) \times L]$ or $b \in I$ and $(a,b) \in F \times I$. If $b \in F$ and $b \in I$, then $(a,b) \in L \times (F \cap I)$, but if $a \in I$, then $(a,b) \in I \times F$. On the other hand, if $(a,b) \in (F \cap I) \times L$, then $a \wedge b \leq a \in I$ and $a \vee b \geq a \in F$ so $(a,b) \in M(L, I, F)$. Following similar reasonings one may check that $(F \times I) \cup (I \times F) \cup (L \times (F \cap I)) \subseteq M(L, I, F)$.

If $F \cap I = \emptyset$, then the elements in the set $M(L, I, F)$ are simply those of $(F \times I) \cup (I \times F)$, and the union is disjoint. Actually there is an isomorphism between the ordered sets $M(L, I, F)$ and $(F \times I^*) \oplus (I \times F^*)$, where $\oplus$ denotes the ordinal sum of ordered sets.

When $F = F(b)$ is a principal filter and $I = I(a)$ is a principal ideal, the ideal generated in $M(L, I, F)$ by the element (a,b) is isomorphic, as an ordered set, to the set $I(a) \times (F(b))^*$, which is dually isomorphic to $F(b) \times (I(a))^*$. There is also an isomorphism from $F(b) \times (I(a))^*$ to the filter generated by the element (b,a) in $M(L, I, F)$.

The following construction of Nelson algebras was introduced in 1977 by D. Vakarelov [22] and allows us to construct Nelson algebras from Heyting ones. Given a Heyting algebra $(H, \wedge, \vee, \Rightarrow, 0, 1)$, let $V(H)$ be the set:

$$V(H) = \{(a,b) \in H \times H : a \wedge b = 0\}.$$

In addition to the operations defined on $V(H) = M(H, \{0\}, H)$ by K2)-K4), Vakarelov defines $\to$ by:

K5) $(x_1, x_2) \to (y_1, y_2) = (x_1 \Rightarrow y_1, x_1 \wedge y_2)$,

and proves it is a binary operation on $V(H)$.

Let p_1 be the projection over the first coordinate of pairs in $M(L, I, F)$. As p_1 is a lattice homomorphism, $L' = p_1(M(L, I, F))$ is a sublattice of L containing I and F, we may restrict our attention to constructions such that $p_1(M(L, I, F)) = L$, because if this were not the case, we could consider $M(L', I, F)$.

Lemma 2.4. *Let H be a Heyting algebra and $(a,b), (c,d) \in M(H, I, F)$. Then $(a,b) \to (c,d) = (a \Rightarrow c, a \wedge d) \in M(H, I, F)$ if and only if $(a \Rightarrow c) \vee a \in F$.*

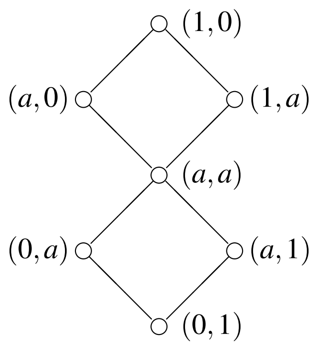
Proof. The condition $(a,b) \to (c,d) = (a \Rightarrow c, a \wedge d) \in M(H,I,F)$ is equivalent to say that (i) $(a \Rightarrow c) \wedge a \wedge d \in I$ and (ii) $(a \Rightarrow c) \vee (a \wedge d) \in F$. As $a \wedge (a \Rightarrow c) \wedge d = a \wedge c \wedge d \leq c \wedge d \in I$, then (i) is always valid. The condition (ii) is equivalent to $(a \Rightarrow c) \vee a \in F$ and $(a \Rightarrow c) \vee d \in F$. Since $c \vee d \in F$ and $c \leq a \Rightarrow c$, $(a \Rightarrow c) \vee d \in F$. □

Lemma 2.5. *Let H be a Heyting algebra with an ideal I and a filter F such that $p_1(M(H,I,F)) = H$. Then K5) defines a binary operation on $M(H,I,F)$ if and only if $Ds(H) = \{y \in H : y = x \vee \neg x\} \subseteq F$.*

Proof. Using Lemma 2.4 we only have to prove that for every $a, c \in H$, $(a \Rightarrow c) \vee a \in F$ if and only if $Ds(H) \subseteq F$. If we assume that for every $a, c \in H$, $(a \Rightarrow c) \vee a \in F$, then in particular $\neg a \vee a = (a \Rightarrow 0) \vee a \in F$ for every $a \in H$.

From $0 \leq c$ we get $a \Rightarrow 0 \leq a \Rightarrow c$ and $(a \Rightarrow c) \vee a \geq a \vee \neg a \in F$, so if $Ds(H) \subseteq F$ then $(a \Rightarrow c) \vee a \in F$. □

Remark 2.5. *If $\mathbf{T}$ is the chain regarded above, then $\mathbf{T}$ can be algebrized as a Heyting algebra. The Kleene algebra $M(\mathbf{T}, I(a), F(a))$ has the following diagram:*



Although $p_1(M(\mathbf{T}, I(a), F(a))) = \mathbf{T}$ and therefore the operation $\to$ is well defined, one may see that $(1,a) \to (1,a) = (1,a) \neq (1,0)$ so it is not a Nelson algebra. A. Monteiro showed in [8] *that it is not possible to endow this lattice with the structure of a Nelson algebra.*

Lemma 2.6. *If H is a Heyting algebra then $(M(H,\{0\},F),(1,0),\sim,\cap,\cup,\to)$ is a Nelson algebra.*

Proof. We must prove that $M(H,\{0\},F)$ is a Nelson subalgebra of $V(H)$. Regarding Lemma 2.4, it is sufficient to prove that for every $(a,b),(c,d) \in M(H,\{0\},F)$, $(a \Rightarrow c) \vee a \in F$. From $0 = a \wedge b \leq c$ by basic properties of Heyting algebras, $b \leq a \Rightarrow c$ and $a \vee b \leq a \vee (a \Rightarrow c) \in F$. □

Notice also that if $(M(H,I,F),(1,0),\sim,\cap,\cup,\to)$ is a Nelson algebra then $I = \{0\}$. Indeed, if $i \in I$, then $(i,1) \in M(H,I,F)$ and by N7),

$$(i,1) \to (i,1) = (i \Rightarrow i, i \wedge 1) = (1,i) = (1,0),$$

so $i = 0$.

Thus, we have proved the following theorem:

Theorem 2.2. *Let H be a Heyting algebra. Then $(M(H,I,F),\cap,\cup,\to,\sim,(1,0))$ is a Nelson algebra if and only if $I = \{0\}$.*

The previous theorem suggests the following:

Definition 2.2. If H is a Heyting algebra, and F a filter in H, we denote by $N(H,F)$ the Nelson algebra $(M(H,\{0\},F),\cap,\cup,\rightarrow,\sim,(1,0))$. Then $V(H)=N(H,H)$.

Remark 2.6. *If B is a boolean algebra, then $V(B)$ is a three-valued Post algebra* (see A. Monteiro [12, p. 202]). *More generally, $N(B,F)$ is a three-valued Łukasiewicz algebra.*

Lemma 2.7. *Given a Heyting algebra H, $p_1(N(H,F))=H$ if and only if $Ds(H)\subseteq F$.*

Proof. If $p_1(N(H,F))=H$ then for any $x\in H$ there exists $x'\in H$ such that $(x,x')\in N(H,F)$. Therefore, $\neg(x,x')=(x,x')\rightarrow(0,1)=(\neg x,x)\in N(H,F)$, so $x\vee\neg x\in F$.

If $Ds(H)\subseteq F$ then for any $x\in H, x\vee\neg x\in F$. As $x\wedge\neg x=0$, we can conclude that $(x,\neg x)\in N(H,F)$ and $x\in p_1(N(H,F))$. □

We can only give a partial converse of Theorem 2.2:

Theorem 2.3. *Let $(H,\wedge,\vee,\Rightarrow,0,1)$ be an algebra of type $(2,2,2,0,0)$. If $(N(H,F),(1,0),\sim,\cap,\cup,\rightarrow)$ is a Nelson algebra, where the operations are defined by K2)-K5) and $p_1(N(H,F))=H$, then H is a Heyting algebra.*

Proof. Note in the first place that for every $x\in H$, there is an element $x'\in H$ such that (x,x') and (x',x) belong to $N(H,F)$.

H_0): $0\wedge x=0$.

By N1), using the commutativity of the distributive lattice $N(H,F)$, $(1,0)=(x',x)\cup(1,0)=(1,0)\cup(x',x)=(x'\vee 1,0\wedge x)$. Then, $0\wedge x=0$.

Furthermore, as $N(H,F)$ is a distributive lattice with first element, $\sim(1,0)=(0,1)$, we have $(y,y')\cup(0,1)=(y\vee 0,y'\wedge 1)=(y,y')$, from where (1) $y\vee 0=y$. In a similar way it is proved that (2) $x\wedge 0=0$ and (3) $0\vee y=y$.

H_1): $x\Rightarrow x=1$.

By N7), $(1,0)=(x,x')\rightarrow(x,x')=(x\Rightarrow x,x\wedge x')$. Hence $x\Rightarrow x=1$.

H_2): $(x\Rightarrow y)\wedge y=y$.

From N10), it is easy to see that for every a,b in a Nelson algebra A we have $b\leq\sim a\vee b\leq a\rightarrow b$. Therefore,

$$((x,x')\rightarrow(y,y'))\cap(y,y')=(y,y').$$

Then we have $((x\Rightarrow y)\wedge y,(x\wedge y')\vee y')=(y,y')$ so $(x\Rightarrow y)\wedge y=y$.

H_3): $x\wedge(x\Rightarrow y)=x\wedge y$.

From N8),

$$(x,x')\cap((x,x')\rightarrow(y,y'))=(x,x')\cap(\sim(x,x')\cup(y,y')).$$

We get: $(x\wedge(x\Rightarrow y),x'\vee(x\wedge y'))=(x\wedge(y\vee x'),x'\vee(x\wedge y'))$ and as $x\wedge(y\vee x')=(x\wedge y)\vee(x\wedge x')=(x\wedge y)\vee 0=x\wedge y$ (the necessary identities for this are easily derived from the fact that $N(H,F)$ is a distributive lattice), $x\wedge(x\Rightarrow y)=x\wedge y$.

H_4): $x\Rightarrow(y\wedge z)=(x\Rightarrow z)\wedge(x\Rightarrow y)$.

From N11),

$$(x,x')\rightarrow((y,y')\cap(z,z'))=((x,x')\rightarrow(y,y'))\cap((x,x')\rightarrow(z,z'))$$

So, we obtain

$$(x\Rightarrow(y\wedge z),x\wedge(y'\vee z'))=((x\Rightarrow z)\wedge(x\Rightarrow y),(x\wedge z')\vee(x\wedge y')).$$

Thus, $x\Rightarrow(y\wedge z)=(x\Rightarrow z)\wedge(x\Rightarrow y)$.

H_5): $(x \vee y) \Rightarrow z = (x \Rightarrow z) \wedge (y \Rightarrow z)$.

As $N(H,F)$ is a Nelson algebra, by N12):

$$((x,x') \cup (y,y')) \to (z,z') = ((x,x') \to (z,z')) \cap ((y,y') \to (z,z')).$$

Using again the aforementioned results we get:

$$((x \vee y) \Rightarrow z, (x \vee y) \wedge z') = ((x \Rightarrow z) \wedge (y \Rightarrow z), (x \wedge z') \vee (y \wedge z')).$$

Then, $(x \vee y) \Rightarrow z = (x \Rightarrow z) \wedge (y \Rightarrow z)$. □

Lemma 2.8. *Let H be a Heyting algebra and S a subalgebra of the Nelson algebra $V(H)$ such that $p_1(S) = H$. Then there is a filter F in H such that $S = N(H,F)$* ([23]).

Proof. Let $F = \{x \in H : x = a \vee b$ for some pair $(a,b) \in S\}$. We claim that F is a filter.

As $1 = 0 \vee 1$ and $(0,1) \in S$, $1 \in F$.

If $x \leq y$ and $x \in F$, then there is a pair $(a,b) \in S$ such that $a \vee b = x$. As $y \in H = p_1(S)$, there is also an element $y' \in H$ such that $(y,y') \in S$. We have then $(a,b) \cup (y,y') = (a \vee y, b \wedge y') \in S$ and $a \vee y \vee (b \wedge y') = (a \vee b \vee y) \wedge (a \vee y \vee y') = (x \vee y) \wedge (a \vee y \vee y') = y \wedge (y \vee (a \vee y')) = y$, so $y \in F$.

If $x,y \in F$, then there are pairs $(a,b),(c,d) \in S$ such that $a \vee b = x$ and $c \vee d = y$. We calculate now $((a,b) \cup (b,a)) \cap ((c,d) \cup (d,c)) = (a \vee b, a \wedge b) \cap (c \vee d, c \wedge d) = (x,0) \cap (y,0) = (x \wedge y, 0) \in S$ and since $x \wedge y = (x \wedge y) \vee 0$, $x \wedge y \in F$.

We prove now that $N(H,F) = S$. It is clear that $S \subseteq N(H,F)$. If $(a,b) \in N(H,F)$, then $a \wedge b = 0$ and $a \vee b \in F$, and therefore there exists $(c,d) \in S$ such that $c \vee d = a \vee b$. Then $(c,d) \cup (d,c) = (c \vee d, d \wedge c) = (a \vee b, 0) \in S$. Since $p_1(S) = H$, there is some $b' \in H$ such that $(b,b') \in S$ and therefore $(b,b') \to (0,1) = (\neg b, b) \in S$. So we can write $(a \vee b, 0) \cap (\neg b, b) = ((a \vee b) \wedge \neg b, 0 \vee b) = (a,b) \in S$, because from $a \wedge b = 0$, $a \leq \neg b$ and then $(a \vee b) \wedge \neg b = (a \wedge \neg b) \vee (b \wedge \neg b) = a \vee 0 = a$. □

3. Join-irreducible elements in $V(H)$

A. Sendlewski determined in [20] the Priestley topological space corresponding to $N(H,F)$ for any Heyting algebra H and F one of its filters containing $Ds(H)$, which is the intersection of all the maximal filters of H.

Definition 3.1. An element p of a lattice L with first element 0 is *join-irreducible* if $p \neq 0$ and if $p = a \vee b$ implies $p = a$ or $p = b$. We shall denote with $\pi(L)$ the set of join-irreducible elements of L.

Definition 3.2. If x is an element of a Heyting algebra H, the *pseudocomplement of x* or *intuitionistic negation of x*, is the element $\neg x = x \Rightarrow 0$. It is easy to prove that $x \wedge \neg x = 0$, for every $x \in H$.

Theorem 3.1. *Given a finite nontrivial Heyting algebra H, the join-irreducible elements of $V(H)$ are the elements of the form $(x, \neg x)$ with $x \in \pi(H)$ or $(0,y)$ with $y \in \pi(H^*)$.*

Proof. Let $p = (q_1,q_2) \in \pi(V(H))$. Then,

$$p \neq (0,1), \tag{*}$$

so $q_1 \neq 0$ or $q_2 \neq 1$.

If $q_1 \neq 0$, let us see that $q_1 \in \pi(H)$ and $q_2 = \neg q_1$.

If $a_1, b_1 \in H$ and $q_1 = a_1 \vee b_1$, then, as $(q_1,q_2) \in V(H)$, $0 = q_1 \wedge q_2 = (a_1 \vee b_1) \wedge q_2 = (a_1 \wedge q_2) \vee (b_1 \wedge q_2)$, thus $a_1 \wedge q_2 = 0$, $b_1 \wedge q_2 = 0$ and therefore $(a_1,q_2),(b_1,q_2) \in V(H)$. Then, from $(q_1,q_2) = (a_1 \vee b_1, q_2) = (a_1,q_2) \cup (b_1,q_2)$, as $(q_1,q_2) \in \pi(V(H))$, we have $q_1 = a_1$ or $q_1 = b_1$.

From $q_1 \wedge q_2 = 0$ follows $q_2 \leq \neg q_1$. Then, $(q_1, q_2) = (q_1 \vee 0, \neg q_1 \wedge q_2) = (q_1, \neg q_1) \cup (0, q_2)$, which yields $(q_1, q_2) = (q_1, \neg q_1)$ or $(q_1, q_2) = (0, q_2)$, but as $q_1 \neq 0, (q_1, q_2) = (q_1, \neg q_1)$.

If $q_1 = 0$, then, by $(*)$, $q_2 \neq 1$. Let us prove that $q_2 \in \pi(H^*)$. If $q_2 = a_2 \wedge b_2$, then $(0, q_2) = (0, a_2 \wedge b_2) = (0, a_2) \cup (0, b_2)$ and as $(q_1, q_2) = (0, q_2) \in \pi(V(H))$, this yields $q_2 = a_2$ or $q_2 = b_2$.

Conversely, if $p = (q_1, \neg q_1)$ with $q_1 \in \pi(H)$, then:

(1) $q_1 \wedge \neg q_1 = 0$, so $p \in V(H)$.
(2) $q_1 \neq 0$, and therefore $p \neq (0,1)$.
(3) If $p = (a_1, a_2) \cup (b_1, b_2)$, with $(a_1, a_2), (b_1, b_2) \in V(H)$ then $q_1 = a_1 \vee b_1$, which yields $q_1 = a_1$ or $q_1 = b_1$.

   If $q_1 = a_1$, $q_1 \wedge a_2 = a_1 \wedge a_2 = 0$, so $a_2 \leq \neg q_1$. As, on the other hand, $\neg q_1 = a_2 \wedge b_2 \leq a_2$, we have $a_2 = \neg q_1$.

   Similarly, if $q_1 = b_1$, then $b_2 = \neg q_1$, from where we conclude that $p = (a_1, a_2)$ or $p = (b_1, b_2)$.

If $p = (0, q_2)$, with $q_2 \in \pi(H^*)$, then $q_2 \neq 1$ and therefore $p \neq (0,1)$. If now $p = (a_1, a_2) \cup (b_1, b_2)$, then $a_1 \vee b_1 = 0$, so $a_1 = a_2 = 0$ and $q_2 = a_2 \wedge b_2$, therefore, $q_2 = a_2$ or $q_2 = b_2$, this is, $p = (a_1, a_2)$ or $p = (b_1, b_2)$. □

Remark 3.1. *In the previous proof we only used the fact that H is a distributive pseudocomplemented lattice with first and last element, this is, a bounded distributive lattice in which for each element x there exists the greatest of the elements z such that,* $z \wedge x = 0$. *Therefore, the theorem is valid for* $M(L, \{0\}, L)$ *with L in those conditions.*

Remark 3.2. *Let us denote with* $F_1(u)$ *the special filter of the first kind (H. Rasiowa* [18, p. 90]*) generated by an element u in a Nelson algebra. A. Monteiro proved that* $F_1(u) = F(\sim \neg u)$, *where* $\neg x = x \to 0$. *If we now consider the Nelson algebras of the form* $V(H)$, *we have that*

$$F_1((x,y)) = F(\sim \neg(x,y)) = F(\sim ((x,y) \to (0,1)) = F(\sim (x \Rightarrow 0, x \wedge 1))$$
$$= F(\sim (\neg x, x)) = F((x, \neg x)).$$

$F_2(a) = \{x \in A : \sim x \to \sim a = 1\}$ *is the special filter of the second kind generated by the element a. Let us see that* $F_2((x,y)) = F((0,y))$:

If $(u,v) \in F_2((x,y))$ *then* $\sim (u,v) \to \sim (x,y) = (1,0)$, *which is equivalent to* $(v,u) \to (y,x) = (1,0)$, *this is,* $(v \Rightarrow y, v \wedge x) = (1,0)$. *We have then that* $v \Rightarrow y = 1$ *from where* $v \leq y$ *and as* $0 \leq u$, *then* $(0,y) \leq (u,v)$.

If now $(0,y) \leq (u,v), v \leq y$. *As* $(x,y) \in V(H), x \wedge y = 0$, *which implies that* $y \leq \neg x$ *and therefore* $v \leq y = y \wedge \neg x$, *from where it results that* $v \leq y$ *and* $v \leq \neg x$, *and then* $\sim (u,v) \to \sim (x,y) = (v,u) \to (y,x) = (v \Rightarrow y, v \wedge x) = (1,0)$, *so* $(u,v) \in F_2((x,y))$.

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