THE SET OF GAUSSIAN FRACTIONS

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ABSTRACT. The set F of complex numbers that have a binary representation in the base b= -1+i with integer part zero has a boundary which is a Jordan curve J. We exhibit a parametric representation of J. It is the union of six similar Jordan arcs and each of them is a selfsimilar set that satisfies Moran's open set condition (cf.[1] or [3]). The interior domain of J is a uniform domain. The convex hull of J is an octagon and those of the mentioned arcs are decagons.

1.INTRODUCTION. A complex number will be called a *gaussian fraction,* or simply a fraction, if it belongs to the set (see Fig. 1):

$$F = \{f \in C : f = \sum_{-\infty}^{-1} a_j b^j, \ a_j \in D\}$$

where b= -1+i, D= {0,1}. E will denote the set of *gaussian integers* : E = {u+iv : u,v ∈ Z}. For g ∈ E we write $F_g := g + F$. Therefore $F = F_0$. A theorem of Kátai and Szabó ([10]) asserts that $C = \cup \{F_g : g \in E\}$. Our first step is to prove that $J = \partial F$ is a Jordan curve. In relation with this cf. S. Ito [9]. We use in an essential way results of the fundamental work of W. Gilbert on bases for number systems ([4]-[8]). Most of them are stated in the next two paragraphs and a few are proved again in the text. Proofs of the needed statements may also be found in [13]. For other minor but useful results see the Appendix.

1.1. NUMBERS WITH TWO REPRESENTATIONS. For $z = \sum_{-\infty}^{L} p_j b^j$, $p_j \in D$, let us define the state k of this representation of z as $p(k) := (\sum_{j=k}^{L} p_j b^j) \cdot b^{-k} \in E$. If $z = \sum_{-\infty}^{L} q_j b^j$ is another representation of z then for any k ≤ L, $p(k) - q(k) \in S := \{0,1,-1,i,-i,1+i,-1-i\}$, (cf.[5]). We call *state k of the representations* the pair (p(k),q(k)) and describe its *type* symbolically as follows :

|pq| if p(k)-q(k)= 0 ; p|q if p(k)-q(k)= -1 ; $\frac{p}{q}$ if p(k)-q(k) =i; $\frac{|q}{p|}$ if p(k)-q(k)= -1-i, etc.

If z is a number with two different representations we can

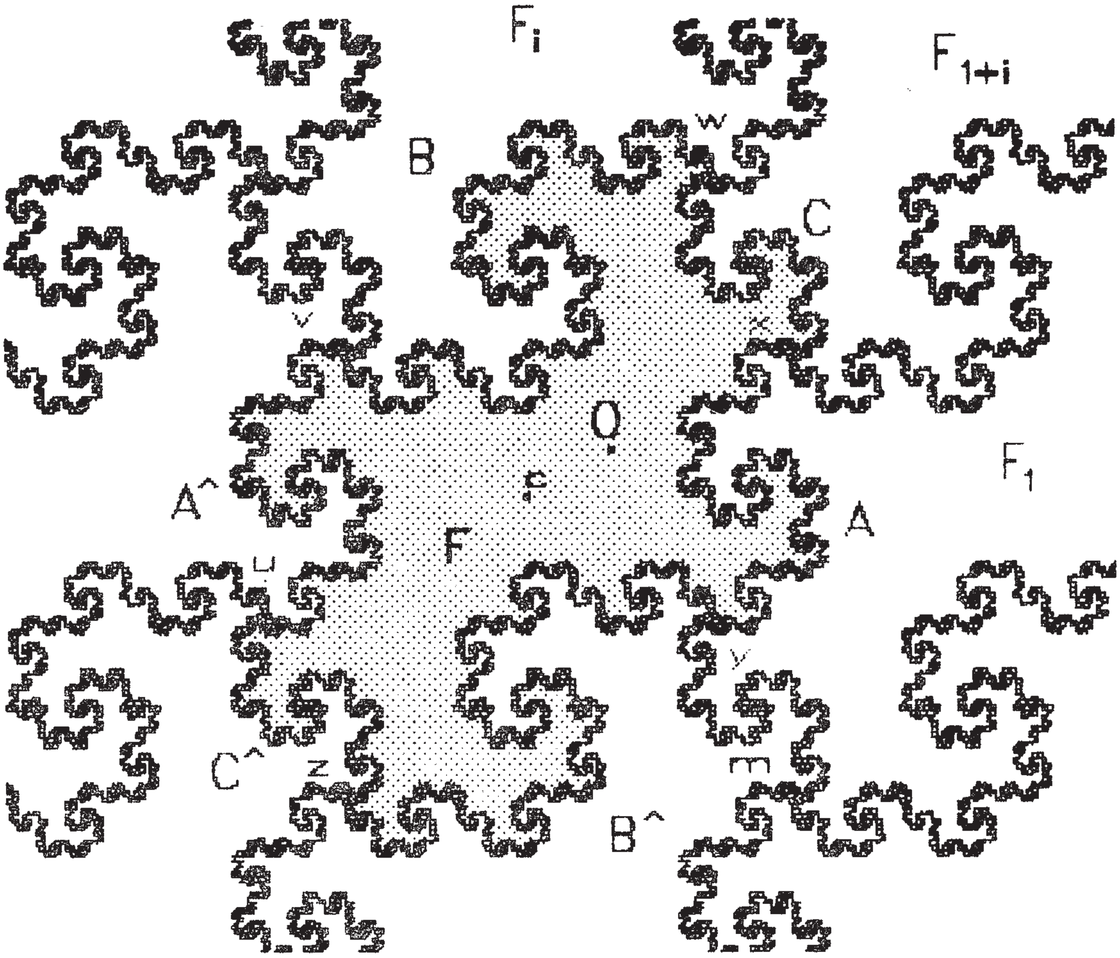


Fig. 1

$$x = \frac{2+i}{5} = 0.\overline{011} \qquad u = \frac{-4-2i}{5} = 0.\overline{100} \qquad m = x-i = \frac{2-4i}{5} = v.b$$

$$y = \frac{1-2i}{5} = 0.\overline{001} \qquad v = \frac{-3+i}{5} = 0.\overline{110} \qquad u = w.b$$

$$z = \frac{-3-4i}{5} = 0.\overline{101} \qquad w = \frac{1+3i}{5} = 0.\overline{010} \qquad w = y.b$$

$$z_0 = 0.\overline{1} \qquad c = \frac{z_0}{2} = 0.00\overline{01} = \frac{-2-i}{10}$$

always choose L great enough to have $p_L = q_L = 0$ and therefore the state L for them is $|pq|$. As k decreases from L to $-\infty$, $(p(k),q(k))$ leaves that type in a certain moment n and reaches the type $p|q$ or $q|p$. We can assume, may be after relabeling the representations, that $(p(n),q(n)) \in p|q$. For $k=n,n-1,\ldots$ $(p(k),q(k))$ follows the arrows of the graph Γ, (see Fig. 2). If $(p(k),q(k))$ is of type of a node of Γ, the column beside an arrow that goes from this node to another one has entries equal to p_{k-1} and q_{k-1}. These ciphers produce the transition from state $(p(k),q(k))$ to state $(p(k-1),q(k-1))$ via the equation

$$(*) \qquad p(k-1) - q(k-1) = (p(k) - q(k))b + (p_{k-1} - q_{k-1}).$$

THEOREM 1. a) Each number with two different representations is associated to an infinite string in the graph Γ that starts in a node of the graph. Conversely, each such an infinite string is associated to a number $z \in F$ with more than one representation that is uniquely determined if $p(0)= 0$, $q(0) \in \mathcal{S}\backslash\{0\}$.
b) $F_g \cap F$ is not void if and only if $g \in \mathcal{S}$.
c) Numbers of the form wb^m, $w \in E$, m an integer, have only one representation.
The numbers described in c) will be called *rational numbers*.

1.2. NUMBERS WITH THREE REPRESENTATIONS. These numbers are associated to infinite strings of the graph τ in Fig. 2, (cf. [5]). More explicitly : if z is a number with three different representations

$$(**) \qquad z = \sum_{-\infty}^{L} p_i b^i = \sum_{-\infty}^{L} q_i b^i = \sum_{-\infty}^{L} r_i b^i ,$$

there exists an n such that the states $p(n)$, $q(n)$, $r(n)$ are all different. They belong pairwise to types of graph Γ only if they are related as in one of the nodes of τ. Then, the ciphers $(p_{n-1}, q_{n-1}, r_{n-1})$ are uniquely determined by the graph Γ and appear in the column beside the arrow in τ that points to the node of the type of the next state $(p(n-1), q(n-1), r(n-1))$.
THEOREM 2. a) Let z be a number with three different representations (**) and $p(0) = 0 \neq q(0) \neq r(0) \neq 0$. Then

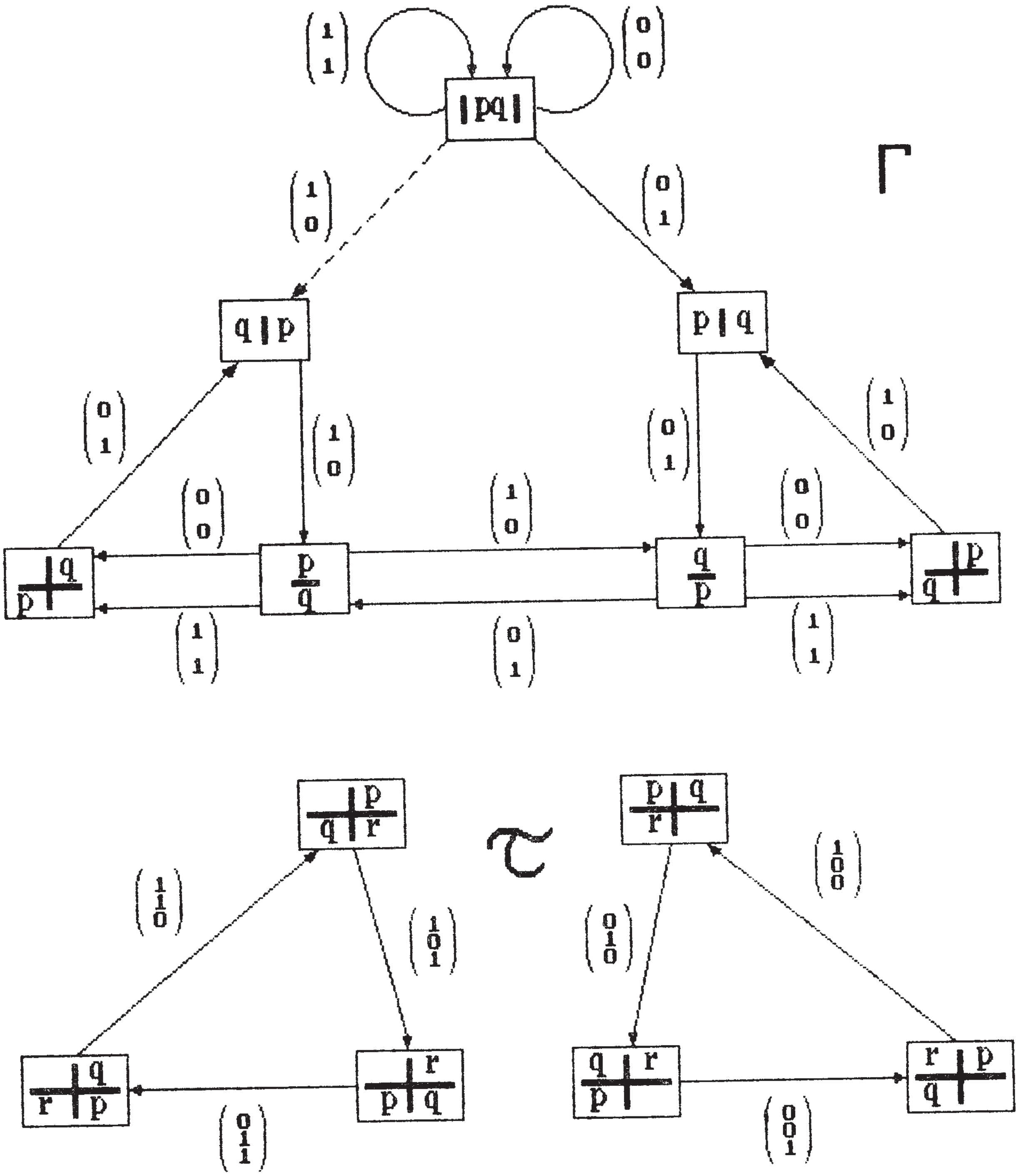


Fig. 2

$p(0), q(0), r(0)$ are related as in one of the nodes of τ and the successive ciphers of these representations can be read, following the graph, from the columns beside the arrows.
b) Each infinite string of τ that starts in one of its nodes defines a unique complex number $z \in F$ if $p(0)= 0$. The ciphers of the three representations of z are the entries in the columns beside the arrows.
c) There is no number with four representations.
Observe that if we add a non-null gaussian integer to a number z with three representations we obtain another number w with three representations; z and w are associated to the same infinite string of τ. These numbers are ultimately periodic with period 001 or 110.

1.3. AUXILIARY LEMMAS. In this paragraph we isolate some results that are used in what follows.

LEMMA 1. i) If $g \in E$, $|g| \leq 2^{k/2} + 3$, k a nonnegative integer, then g has a representation with no more than $k + 10$ ciphers.

ii) If $z \in C$, $|z| < 2^{-5}$ then $z \in F$.

PROOF. i) holds for $k=0$: the gaussian integer of modulus less than or equal to four with longest radix representation is $-3 + 2i$. It has ten ciphers (cf.table 1, Appendix). Assume i) is true for k. If $|g| \leq 2^{(k+1)/2} + 3$, $g= r + bg_0$, $r \in D$, $g_0 \in E$, then we have

$$|g_0| \leq (|g|+1)/|b| \leq (2^{(k+1)/2}+ 4)/ \sqrt{2} < (\sqrt{2})^k + 3.$$

Therefore, g_0 needs no more than $k+ 10$ ciphers and g no more than $k+ 1+ 10$.

ii) Suppose $|z| < 2^{-5}$. For $j \geq 10$ let $g_j \in E$ be such that

$|zb^j - g_j| < 1$. Then $|g_j| \leq 1 + |b|^{j-10} < 2^{(j-10)/2}+ 3$.

Because of i), $g_j = \sum_{k=0}^{j-1} a_{k,j} b^k$. Therefore, the sequence $\{z_j\}$,

$z_j = g_j b^{-j} = \sum_{r=-j}^{-1} a_{r+j,j} b^r \in F$, converges to z and contains a

subsequence $\{z_j'\}$ such that the ciphers with the same fixed

subindex in its elements are constant from some moment on.
In consequence, z has a representation of the form $0.p_{-1}p_{-2}\cdots$ and belongs to F, QED.

We call a sequence like $\{z_j'\}$ a *telescopic* sequence. It has the property that the representations of its elements converge pointwise to the reperesentation of $\lim z_j'$, that surely exists.

We denote with F^* the set of *rational* numbers in F :

$$F^* := \{ \sum_{-N}^{-1} a_r b^r \}.$$

LEMMA 2. 1) The set F^* is contained in the interior of F, F°.
2) $F = \overline{F^\circ}$.

PROOF. Because of Lemma 1,ii), $0 \in F^\circ$. Let us define

(1) $\qquad \Phi_0(z) = z/b \ , \ \Phi_1(z) = z/b + 1/b.$

Then, $F = \Phi_0(F) \cup \Phi_1(F)$ and $\Phi_0(F^\circ) \cup \Phi_1(F^\circ)$ is contained in F°. Since $0 \in F^\circ$, if $z=0.a_{-1}\ldots a_{-r}$ then $z = \Phi_{a_{-1}} \circ \ldots \circ \Phi_{a_{-r}}(0) \in F^\circ$ and 1) follows. 2) is a consequence of $F = \overline{(F^*)}$, QED.

LEMMA 3. Let $g \in S \backslash \{0\}$. Then, $z \in F \cap F_g$ if and only if z is associated to an infinite string of the graph Γ that starts at the node corresponding to the type of the state $(0,g)$.

PROOF. In fact, $z = 0.p_{-1}p_{-2}\cdots = (g)_b.q_{-1}q_{-2}\cdots$, QED.

COROLLARY 1. If $0 \neq g \in S$ and $z \in F \cap F_g$ then neither radix representation of z has more than four consecutive equal ciphers after the point.

PROOF. To prove this it is sufficient to examine the diagram of graph Γ and observe that the state $(p(0),q(0)) = (0,g)$ is not of type $|pq|$, QED.

COROLLARY 2. $F^\circ \cap F_g = \emptyset$.

PROOF. In fact, for $z \in F_g$, $z = \lim_{N \to \infty} z_N$, $z_N = (g)_b.a_{-1}a_{-2}\ldots a_{-N}$

Since rational numbers have a unique representation, $z_N \notin F$. In consequence, $z \notin F^\circ$, QED.

1.4. THE BOUNDARY OF F. We call J the boundary of the closed set F, $J = \partial F$.

DEFINITION 1. $A = F \cap F_1$; $A^\wedge = F \cap F_{-1}$; $B = F \cap F_i$; $B^\wedge = F \cap F_{-i}$
$C = F \cap F_{1+i}$; $C^\wedge = F \cap F_{-1-i}$.

From Th.1,b) and Cor. 2 it follows that

$$J = A \cup A^\wedge \cup B \cup B^\wedge \cup C \cup C^\wedge.$$

LEMMA 4. i) $z \in A$ if and only if $zb \in B$,

ii) $z \in C$ if and only if $zb + 1 \in A$,

iii) $z \in B$ if and only if $zb + i \in C \cup B$ or $zb + (1+i) \in C$,

iv) $y \in F_0 \cap F_1 \cap F_{-i} = A \cap B^\wedge$ if and only if $y = 0.\overline{001} =$
$= (1 - 2i)/5 = 1.\overline{100}$,

$z \in F_0 \cap F_{-i} \cap F_{-1-i} = B^\wedge \cap C^\wedge$ if and only if $z = 0.\overline{101} =$
$= (-3 - 4i)/5$,

$x \in F_0 \cap F_1 \cap F_{1+i} = A \cap C$ if and only if $x = 0.\overline{011} =$
$= (2 + i)/5 = 1.\overline{110}$,

v) $u \in F \cap F_{-1} \cap F_{-1-i} = A^\wedge \cap C^\wedge$ iff $u = (-4 -2i)/5 = 0.\overline{100}$,

$v \in F \cap F_{-1} \cap F_i = B \cap A^\wedge$ iff $v = (-3 + i)/5 = 0.\overline{110}$,

$w \in F \cap F_i \cap F_{1+i} = B \cap C$ iff $w = (1 + 3i)/5 = 11.\overline{001}$,

vi) $\emptyset = A \cap (B \cup A^\wedge \cup C^\wedge) = C \cap (A^\wedge \cup C^\wedge \cup B^\wedge) =$
$= B \cap (C^\wedge \cup B^\wedge \cup A)$.

PROOF. i) $z \in A$ if and only if $z = 0.p_{-1}p_{-2}\ldots = 1.q_{-1}q_{-2}\ldots$. Then, $(p(0),q(0)) \in p|q$ and from the graph Γ it follows that $p_{-1} = 0$, $q_{-1} = 1$. Therefore, $zb = 0.p_{-2}p_{-3}\ldots = 11.q_{-2}\ldots \in B$.

Conversely, from these equalities we get

(2) $$z = 0.0p_{-2}p_{-3}\ldots = 1.1q_{-2}q_{-3}\ldots \in A.$$

ii) If $z \in C$ then $z = 0.p_{-1}p_{-2}\ldots = 1110.q_{-1}q_{-2}\ldots$. The infinite string starts from the node $\frac{|q}{p|}$. Then $p_{-1} = 0$, $q_{-1} = 1$ and $zb = 0.p_{-2}\ldots = 11101.q_{-2}\ldots$. Therefore, $zb + 1 = 1.p_{-2}\ldots =$
$= 0.q_{-2}\ldots \in A$, (cf. Appendix).

Conversely, it follows from these equalities that $z \in C$.

iii) $z \in B$ is equivalent to $z = 0.p_{-1}\ldots = 11.q_{-1}\ldots$; then $(p(0),q(0)) \in \frac{q}{p}$. There are three nodes that can be reached from this one in the first step. Therefore, we have the following possibilities :

$$(3) \qquad zb = \left< \begin{array}{ll} 0.p_{-2}\ldots = 111.q_{-2}\ldots & \text{or} \\ 0.p_{-2}\ldots = 110.q_{-2}\ldots & \text{or} \\ 1.p_{-2}\ldots = 111.q_{-2}\ldots . & \end{array} \right.$$

That is, $zb \in F \cap F_{-i}$ or $zb \in F \cap F_{-1-i}$ or $zb \in F_1 \cap F_{-i}$.
Conversely, from (3) it follows that $z \in B$, (this means that the arc wv verifies : b.arc wv = arc um , cf. Fig. 1).
iv) and v) are the contents of Cor. 4 in [5]. For example, $y = 0.p_{-1}\ldots = 1.q_{-1}\ldots = 111.r_{-1}\ldots$; looking at the graph τ we see that $(p(0),q(0),r(0)) \in \begin{array}{c|c} p & q \\ \hline r & \end{array}$. Then $y = 0.\overline{001}$, QED.

COROLLARY 3. i) $A = \Phi_0(B)$,

ii) $C = \Phi_0(A) - 1/b$,

iii) $B = (\Phi_0(C) - i/b) \cup (\Phi_0(C) - (1+i)/b) \cup (\Phi_0(B) - i/b)$.

1.5. **THE SET A.** We define next three similarities :

$$(4) \qquad \Omega_1(z) = z/b + 1/2 \ ; \qquad \Omega_2(z) = z/b^3 + (1+i)/4 \ ;$$
$$\Omega_3(z) = z/b^3 + (1-i)/4.$$

From table 1 we get : $1/2 = 1.11$, $(1+i)/4 = 0.011$, $(1-i)/4 = = 0.001$, (cf. also [5]).

LEMMA 5. i) $A = \Omega_1(A) \cup \Omega_2(A) \cup \Omega_3(A)$

ii) $\Omega_2(A) \cap \Omega_3(A) = \emptyset$

iii) $\Omega_1(A) \cap \Omega_2(A) = \{\Omega_1(y)\}$; $\Omega_1(y) = \Omega_2(y) = (2+i)/10 = = 0.01\overline{100}$;

iv) $\Omega_1(A) \cap \Omega_3(A) = \{\Omega_1(x)\}$; $\Omega_1(x) = \Omega_3(x) = (4-3i)/10 = = 0.00\overline{101}$;

v) A is a self-similar set whose similarity dimension is $s = = 2\log\mu/\log 2$, where μ is the positive root of the polynomial $\mu^3 - \mu^2 - 2$, ($\mu = |b|^s = 2^{s/2} \approx 1.6956$, $s \approx 1.52$).

vi) $x = \Omega_2(x)$, $y = \Omega_3(y)$.

PROOF. i) We obtain from Cor.3 that

$B = (\Phi_0^2(A) - 1/b^2 - i/b) \cup (\Phi_0^2(A) - 1/b^2 - (1+i)/b) \cup (A - i/b)$,

$A = \Phi_0(B) = (\Phi_0^3(A) + \Phi_0(-1/2)) \cup (\Phi_0^3(A) + \Phi_0(i/2)) \cup$
$\cup (\Phi_0(A) + \Phi_0(i/2 - 1/2))$.

Then $A = \Omega_2(A) \cup \Omega_3(A) \cup \Omega_1(A)$.

ii) The action of each function Ω_i on the radix representation is as follows (cf.(2)) where in each formula the points represent the same string of digits :

$$(5)\begin{cases} \Omega_1(0.0p_{-2}\ldots) = 1.11p_{-2}\ldots \quad ; \; \Omega_1(1.1q_{-2}\ldots) = 0.00q_{-2}\ldots \\ \Omega_2(0.0p_{-2}\ldots) = 0.0110p_{-2}\ldots \; ; \; \Omega_2(1.1q_{-2}\ldots) = 1.1101q_{-2}\ldots \\ \Omega_3(0.0p_{-2}\ldots) = 0.0010p_{-2}\ldots \; ; \; \Omega_3(1.1q_{-2}\ldots) = 1.1001q_{-2}\ldots \end{cases}$$

If $z_1 = 0.0p_{-2}\ldots$ and $z_2 = 1.1q_{-2}\ldots$ belong to A and $z = \Omega_2(z_1) = \Omega_3(z_2)$ then $0.0110p_{-2}\ldots = 1.1001q_{-2}\ldots$. But this is impossible because there is no infinite string in Γ with this beginning.

iii) Assume $z \in \Omega_1(A) \cap \Omega_2(A)$. Then, there exist z_1, $z_2 \in A$ such that $z = \Omega_1(z_1) = \Omega_2(z_2)$. By (2) we can assume that $z_1 = 0.0p_{-2}\ldots = 1.1q_{-2}\ldots$ and $z_2 = 0.0P_{-2}P_{-3}\ldots$. Then, using (5), $z = 1.11p_{-2}\ldots = 0.0110P_{-2}\ldots = 0.00q_{-2}\ldots$.

In this case b^2z has the following three representations :
$b^2z = 111.p_{-2}\ldots = 1.10P_{-2}\ldots = 0.q_{-2}\ldots$. Therefore, $b^2z \in F_0 \cap F_1 \cap F_{-i}$. From Lemma 4 we get $b^2z = (1 - 2i)/5$, that is $z = (2 + i)/10$.

iv) Let $z = \Omega_1(z_1) = \Omega_3(z_2)$, $z_1 = 0.0p_{-2}\ldots = 1.1q_{-2}\ldots$, $z_2 = 0.0P_{-2}\ldots = 1.1Q_{-2}\ldots$. Then $z = 1.11p_{-2}\ldots = 0.00q_{-2}\ldots = 0.0010P_{-2}\ldots = 1.1001Q_{-2}\ldots$. Therefore,
$b^2z = 111.p_{-2}\ldots = 0.q_{-2}\ldots = 110.01Q_{-2}\ldots \in F_{-i} \cap F_0 \cap F_{-1-i}$.
In consequence $b^2z = (-3 - 4i)/5 = 0.\overline{101}$ (cf.Lemma 4,iv)) and $z = (4-3i)/10 = 0.00\overline{101}$. v) and vi) follow immediately. QED.

2. SELF-SIMILARITY OF A. We wish to prove that A is an s-set i.e., that $0 < H^s(A) < \infty$, where s is the similarity dimension of A. In view of Hutchinson's theorem it suffices to show that Moran's open set condition holds and this is the content of next theorem 3. We generalize our earlier notation as follows : let $a_L \dots a_0 . a_{-1} \dots a_k$, $a_j \in D$, be a rational number. We write :

$$F_{a_L \dots a_0 . a_{-1} \dots a_k} := \{z = a_L \dots a_0 . a_{-1} \dots a_k c_{k-1} c_{k-2} \dots ;\ c_m \in D\}$$

A set of this kind is termed a *tile of order k*. Then, from Cor. 2 of Lemma 3, Th. 1,b), we get

(6) $F^\circ_{a_0 . a_{-1} \dots a_k} \cap F^\circ_{b_0 . b_{-1} \dots b_h} \neq \emptyset$, $k \geq h$, implies that

$\quad a_i = b_i$ for $i \geq k$.

Observe that (6) holds even in the case when a_0 and b_0 are not binary digits but denote gaussian integers.

THEOREM 3. Let $f := F^\circ_{0.0011111}$ and $V := \bigcup_{r=1}^{\infty} \bigcup \{\Omega_{i_r} \circ \dots \circ \Omega_{i_1}(f) : i_j \in \{1,2,3\}\}$. Then, V is a bounded open set such that,

i) $\Omega_i(V) \subset V$, $i = 1,2,3$,

ii) $\Omega_i(V) \cap \Omega_j(V) = \emptyset$ for $i \neq j$.

To prove the theorem we shall use two auxiliary lemmas.

LEMMA 6. Assume that $F^\circ_{a_0 . a_{-1} \dots a_h} = \Omega_{i_r} \circ \dots \circ \Omega_{i_1}(f)$. Then, $a_0 = a_{-1}$ and the sequence $\{a_{-1}, \dots, a_h\}$ does not contain five consecutive ciphers 1, except for the last ones.

PROOF. The action of the similarities on tiles is shown in the following formulae (cf.(5)):

$$(7)\quad \begin{cases} \Omega_1(F_{0.0\dots}) = F_{1.11\dots} & ; \quad \Omega_1(F_{1.1\dots}) = F_{0.00\dots} \\ \Omega_2(F_{0.0\dots}) = F_{0.0110\dots} & ; \quad \Omega_2(F_{1.1\dots}) = F_{1.1101\dots} \\ \Omega_3(F_{0.0\dots}) = F_{0.0010\dots} & ; \quad \Omega_3(F_{1.1\dots}) = F_{1.1001\dots} \end{cases}$$

In each equality the points represent the same sequence of ciphers. The case r=1 is proved by the left formulae in (7). Assume it holds for $r-1 \geq 1$.

Let $\Omega_{i_r}\circ\ldots\circ\Omega_{i_1}(f) = \Omega_{i_r}(F^\circ_{b_0.b_{-1}\ldots b_h})$ with $\{b_j\}$ verifying the inductive hypothesis. From (7) we see that if i_r=2,3, a sequence of five consecutive digits 1 cannot appear after the point, except in the case in which exactly five digits equal to 1 appear at the end of the sequence $\{b_{-1}b_{-2}\ldots b_h\}$. If i_r=1 to have five consecutive digits equal to 1 not at the end it is necessary that $F^\circ_{b_0.b_{-1}\ldots b_h} = F^\circ_{0.0111\ldots b_h}$. But this last tile cannot be of the form $\Omega_{i_{r-1}}\circ\ldots\circ\Omega_{i_1}(f)$ since neither right hand side in (7) is of the form $F_{0.0111\ldots}$, QED.

LEMMA 7. Let m,k < 0 and $b_0.b_{-1}\ldots b_k$, $a_0.a_{-1}\ldots a_m$ be the representations of two complex numbers with $b_0 = b_{-1}$. If there exist $j_1,\ldots,j_n \in \{1,2,3\}$ such that

$$\Omega_{j_n}\circ\ldots\circ\Omega_{j_1}(F^\circ_{b_0.b_{-1}\ldots b_k}) = F^\circ_{a_0.a_{-1}\ldots a_m}$$

then $j_1,\ldots,j_n$ are uniquely determined.

PROOF. Let us call $G_0 = F^\circ_{b_0.b_{-1}\ldots b_k}$ and $G_h = \Omega_{j_h}(G_{h-1})$, h = 1,2,...,n. Then

$$F^\circ_{a_0.a_{-1}\ldots a_m} = \Omega_{j_n}\circ\ldots\circ\Omega_{j_{h+1}}(G_h).$$

G_0 is a data. Assume we know G_h, $G_h = F^\circ_{b'_0.b'_{-1}\ldots b'_L}$, $b'_0 = b'_{-1}$. In view of (7) it follows that,

$a_{m-(L+1)} \neq b'_{-1}$ implies $j_{h+1} = 1$,

$a_{m-(L+1)} = b'_{-1}$, $a_{m-L+1} = 1$ implies $j_{h+1} = 2$,

$a_{m-(L+1)} = b'_{-1}$, $a_{m-L+1} = 0$ implies $j_{h+1} = 3$.

Then, j_{h+1} is uniquely determined and so is G_{h+1}, QED.

PROOF OF TH.3. i) is obvious. In view of (7), $V \subset F \cup F_1$ and V is bounded. Let us prove ii). It is sufficient to show that $\Omega_{i_r}\circ\ldots\circ\Omega_{i_1}(f) \cap \Omega_{j_h}\circ\ldots\circ\Omega_{j_1}(f) = \emptyset$ if $(i_r\ldots i_1) \neq (j_h\ldots j_1)$.

Suppose that

(8) $\Omega_{i_r}\circ\ldots\circ\Omega_{i_1}(f) = F^\circ_{a_0.a_{-1}\ldots a_k}$; $\Omega_{j_h}\circ\ldots\circ\Omega_{j_1}(f) = F^\circ_{a'_0.a'_{-1}\ldots a'_m}$

have a nonvoid intersection and that $k \geq m$. Then $a_j = a'_j$ for $j=0,-1,\dots,k$ (cf.(6)). Since $a_{k+4} = a_{k+3} = a_{k+2} = a_{k+1} = a_k = 1$, we must have $k=m$ because of Lemma 6. Now, Lemma 7 implies that $(i_r \dots i_1) = (j_h \dots j_1)$, QED.

3.REGULARITY OF A. We call P the middle point of the segment x,y (cf.Lemma 4). That is, $P = (x+y)/2 = (3-i)/10 = ((-3+i)/5)\bullet(-1/2) = v\bullet(-1/2)$. Then, $P = 0.\overline{110} \bullet 0.11 = 0.00\overline{10} = 1.1\overline{10} \in A$.

We denote with S the rotation around the point P :

$$S(z) = ((3-i)/5) - z \; .$$

PROPOSITION 1. $S(A) = A$.

PROOF. $-v = (3-i)/5 = 1.\overline{1}$. Let z be as in (2). Then $S(z) = 1.1p'_{-2}p'_{-3}\dots = 0.0q'_{-2}q'_{-3}\dots$ where $p'_i = 1 - p_i$, $q'_i = 1 - q_i$ and $S(A) \subset A$. Applying S to both sides, we obtain : $A = S^2(A) \subset S(A)$, QED.

We have introduced the transformation S to change one of the similarities that define A without changing this compact invariant set. We define

$$(9) \quad \begin{cases} \tau_0(z) := \Omega_2(z) = z/b^3 + (1+i)/4, \\ \tau_1(z) := \Omega_1(S(z)) = -z/b + (1-2i)/10, \\ \tau_2(z) := \Omega_3(z) = z/b^3 + (1-i)/4. \end{cases}$$

Because of Prop.1, Lemma 5 can be rewritten as follows.

PROPOSITION 2. $A = \tau_0(A) \cup \tau_1(A) \cup \tau_2(A)$; $\tau_0(A) \cap \tau_1(A) = \{\tau_0(y)\} = \{\tau_1(x)\} = \{(2+i)/10\}$; $\tau_0(A) \cap \tau_2(A) = \emptyset$; $\tau_1(A) \cap \tau_2(A) = \{\tau_1(y)\} = \{\tau_2(x)\} = \{(4-3i)/10\}$.

3.1. AUXILIARY PROPOSITIONS. Let us assume that

$$(10) \quad t = \sum_1^\infty a_j \, 3^{-j} \;, \quad t' = \sum_1^\infty a'_j \, 3^{-j} \;; \qquad a_j, a'_j \in \{0,1,2\}.$$

PROPOSITION 3. i) Let $a_j = a'_j$ for $j < K$ and $a'_K > a_K$. If $|t - t'| < 3^{-N}$, $N \geq K$, then $a'_K = a_K + 1$ and $a'_j + 2 = a_j = 2$ for j such that $K + 1 \leq j \leq N$.

ii) If $t = t'$ then $t' = (0.a_1 \dots a_{K-1}(a_K+1))_3 =$

$= t = (0.a_1 \dots a_{K-1} a_K 22\dots)_3$.

PROOF. ii) follows from i).

i) $3^{-N} > \sum_1^\infty (a_j' - a_j)3^{-j} = (a_K' - a_K)3^{-K} + \sum_{K+1}^\infty (a_j' - a_j)3^{-j} =$

$= (a_K' - a_K - 1)3^{-K} + \sum_{K+1}^\infty (2 + a_j' - a_j)3^{-j}$.

Taking into account that the last parentheses are nonnegative and at least equal to 1 if not 0, i) follows, QED.

PROPOSITION 4. The diameter of F is less than 2.

If $B_2 = \{|z| < 2\}$ then $\tau_i(B_2) \subset B_2$. Besides, if z_1, $z_2 \in B_2$ then

i) $|z_1 - z_2||b|^{-3N} \le |\tau_{a_1} \dots \tau_{a_N}(z_1) - \tau_{a_1} \dots \tau_{a_N}(z_2)| \le$

$\le |z_1 - z_2||b|^{-N}$

ii) $|\tau_1 \tau_0^{N-1}(z_1) - \tau_0 \tau_2^{N-1}(z_2)| \le 8|b|^{2-3N}$,

$|\tau_2 \tau_0^{N-1}(z_1) - \tau_1 \tau_2^{N-1}(z_2)| \le 8|b|^{2-3N}$.

PROOF. Let $z = z_1 - z_2$ with $z_i \in F$. Then $z = \sum_{-\infty}^{-1} d_j b^j$, $d_j \in \{0,1,-1\}$. Observe that

(11) $|d_{3j} + d_{3j+1}b + d_{3j+2}b^2| \le |1 - b + b^2| = \sqrt{13}$.

In consequence, $|z| \le \sqrt{13} \sum_1^\infty 2^{-3j/2} < 2$ and $F \subset B_2$. It is easy to see that $\tau_i(B_2) \subset B_2$, $i=0,1,2$. i) follows by induction from :

(12) $|\tau_j(z_1) - \tau_j(z_2)| = |z_1 - z_2||b|^{-3}$, $j=0,2$;

$|\tau_1(z_1) - \tau_1(z_2)| = |z_1 - z_2||b|^{-1}$.

ii) First observe that $\tau_{h+1}\tau_0^{N-1}(x) = \tau_h \tau_2^{N-1}(y)$ for $h=0,1$. The inequalities we are looking for are obtained then from the triangle inequality and the following estimates, which are corollaries of (12) :

$$(13) \quad \begin{cases} |\tau_k \tau_0^{N-1}(z_1) - \tau_k \tau_0^{N-1}(x)| \le |z_1 - x||b|^{2-3N} \le 4|b|^{2-3N}, \\ |\tau_h \tau_2^{N-1}(z_2) - \tau_h \tau_2^{N-1}(y)| \le |z_2 - y||b|^{2-3N} \le 4|b|^{2-3N}, \end{cases}$$

where $k = 1, 2$ and $h = 0, 1$. QED.

3.2. THE BOUNDARY OF F IS A SIMPLE CLOSED CURVE. The purpose of this paragraph is to prove next theorem.

THEOREM 4. A is a Jordan arc with initial point $x = 0.\overline{011}$ and terminal point $y = 0.\overline{001}$.

Assuming the last statement, a description of the boundary of F follows immediately.

THEOREM 5. i) $J = \partial F$ is the union of six similar consecutive arcs A, $B^\wedge$, $C^\wedge$, $A^\wedge$, B, C (see Fig. 1) which join the points x, y, z, u, v, w,

ii) J is a Jordan curve with Hausdorff dimension s and $0 < H^s(J) < \infty$, (cf.[7],[8]),

iii) For any $p \in J$ and any ball $B_r(p)$, $H^s(B_r(p) \cap J) > 0$.

PROOF. i) and ii) follow from Lemma 4 i), ii), iv), v), Lemma 5 v) and Th. 1. iii) follows from the self-similarity of A, QED.

Let us recall that $s = 2\log\mu / \log 2$ and $\mu = |b|^s$ is the positive root of $\mu^3 - \mu^2 - 2 = 0$ (cf. Lemma 5). At this stage it is not difficult to prove next theorem where m(.) denotes the plane Lebesgue measure.

THEOREM 6. $H^s(J) = 2(1 + \mu + 1/\mu)H^s(A)$; $m(F^\circ) = m(F) = 1$.

The last statement is a consequence of Th. 5 and the fact that $\{F_g : g \in E\}$ tiles the plane (cf.Cor. 2 and Prop. 4).

PROOF OF THEOREM 4. Assume $t \in [0,1]$. Let us define f :

$$(14)\quad t = \sum_1^\infty a_j 3^{-j} \longrightarrow f(t) = \lim_{n \to \infty} \tau_{a_1} \ldots \tau_{a_n}(0)$$

where $a_j \in \{0,1,2\}$. To prove the theorem it will be sufficient to prove the following statements :

i) f(t) is well defined,

ii) $f([0,1]) \subset A$,

iii) $f(t) = x$ if and only if $t = 0$; $f(t) = y$ iff $t = 1$,

iv) f is one-to-one and continuous,

v) $A \subset f([0,1])$.

i) Prop. 4 i) implies that the limit (14) exists. Let t and t′ be as in Prop. 3 ii). Then, for $N \geq K$, from Prop. 4 i) and ii), we get

$$|\tau_{a_1}\cdots\tau_{a_N}(0) - \tau_{a_1'}\cdots\tau_{a_N'}(0)| \le |b|^{1-K}\, |\tau_{a_K}\tau_2^{N-K}(0) - \tau_{a_K'}\tau_0^{N-K}(0)| \le$$

$\le 8|b|^{2K-3N}$. Consequently, for $N \longrightarrow \infty$, $|f(t) - f(t')| = 0$.

ii) $f(t) \in A$ since A is the invariant set of the similarities τ_i.

iii) Given $t = (0.a_1a_2\ldots)_3$ call $t_1 := 3t - a_1 = (0.a_2a_3\ldots)_3$. Then

$$(15) \qquad f(t) = \tau_{a_1}(f(t_1)).$$

Assume that $f(t) = x$. Since $x \notin \tau_1(A) \cup \tau_2(A)$, from $x = \tau_{a_1}(f(3t - a_1))$ it follows that $a_1 = 0$ (cf. Prop. 2). But x is the fixed point of τ_0 and so $f(3t) = x$. Continuing in this way we obtain $a_j = 0$ for all j. Conversely, $f(0) = x$. With the same procedure one proves the second part of iii).

iv) Suppose $f(t) = f(t')$, t and t' as in (10). Suppose $a_j = a_j'$ for $j < K$, $a_K < a_K'$. Define

$$t_1 := \sum_K^\infty a_j\, 3^{K-j-1}, \quad t_1' := \sum_K^\infty a_j'\, 3^{K-j-1}.$$

From (15) we get :

$$f(t) = \tau_{a_1}\cdots\tau_{a_{K-1}}(f(t_1)) \ , \ f(t') = \tau_{a_1}\cdots\tau_{a_{K-1}}(f(t_1'))$$

and therefore $f(t_1) = f(t_1')$. Using again (15) it follows that $\tau_{a_K}(f(3t_1 - a_K)) = \tau_{a_K'}(f(3t_1' - a_K'))$. Therefore $a_K' = a_K + 1$ and $y = f(3t_1 - a_K)$, $x = f(3t_1' - a_K'))$, (cf. ii) and Prop. 2).

But iii) implies $3t_1 - a_K = 1$, $3t_1' - a_K' = 0$. Then, $3(t_1 - t_1') = 1 + a_K - a_K' = 0$, and $t = t'$.

The continuity of f : assume that $0 < |t - t'| < 3^{-N}$. Then, if $N \ge K$, from Prop. 3 i) and Prop. 4, we get

$$|f(t) - f(t')| = |\tau_{a_1}\cdots\tau_{a_{K-1}}(f(t_1)) - \tau_{a_1}\cdots\tau_{a_{K-1}}(f(t_1'))| \le$$

$$\le |f(t_1) - f(t_1')||b|^{1-K} = |\tau_{a_K}\tau_2^{N-K}(z_1) - \tau_{a_K'}\tau_0^{N-K}(z_2)||b|^{1-K} \le$$

$$\le 8|b|^{2K-3N} \le 2^{3-N/2} .$$

But, if $N \le K-1$, $|f(t) - f(t')| \le |f(t_1) - f(t_1')||b|^{-N} \le 2^{2-N/2}$.

v) Let $z \in A$. Then, $z = \tau_{a_1}(z_1)$ with $z_1 \in A$ (Prop. 2), whence

$z = \tau_{a_1}\cdots\tau_{a_N}(z_N)$, $z_N \in A$. But

$$|\tau_{a_1}\cdots\tau_{a_N}(z_N) - \tau_{a_1}\cdots\tau_{a_N}(0)| \le |z_N||b|^{-N} \le 2^{1-N/2} .$$

This means that $z = f(t)$, $t = (0.a_1a_2\ldots)_3$, QED.

4. F° IS A UNIFORM DOMAIN. Our purpose now is to prove that the interior domain of the curve J is a quasi-disk. For this we show in Theorem 7 that J satisfies Ahlfors' condition ([11], Ch. 1).

PROPOSITION 5. There exists $K > 0$ such that if $0 \le t_1 < t \le t_2 \le 1$ then

$$(16) \qquad |f(t) - f(t_1)| \le K|f(t_2) - f(t_1)|.$$

PROOF. From (9) we obtain

$$(17)\ \tau_0\tau_2(z) = \tau_0^{\,2}(z) + (4b)^{-1} \ ; \quad \tau_1\tau_0(z) = \tau_0\tau_1(z) + (4b)^{-1},$$

and from (17) and (14),

$$(18)\ f(t + 2/9) = f(t) + (4b)^{-1} \quad \text{whenever } 0 \le t \le 1/9 + 1/27.$$

Then, *the subarc of* A *with parameter* $t \in [1/3 - 1/9, 1/3 + 1/27]$ *is a translation of* $f([0, 1/9 + 1/27])$. Because of the symmetry of A (cf. Prop.1 and Lemma 5 i)), or after checking as before, we also obtain that $f([2/3 - 1/27, 2/3 + 1/9])$ is a translation of $f([1 - 1/9 - 1/27, 1])$. We cover now the unit interval with five subintervals : $[0,1] = I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5 =$
$= [0,1/3] \cup [1/3 -1/9, 1/3 +1/27] \cup [1/3, 2/3] \cup$
$\cup [2/3 - 1/27, 2/3 + 1/9] \cup [2/3, 1]$.

Accordingly, $A = W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5$ where $W_j := f(I_j)$.

To prove (16) we consider two cases : a) there exists $j \in \{1,2,3,4,5\}$ such that $f(t_1), f(t_2) \in W_j$; b) there is no such j. Let d be the infimum of the distances between pairs of points of A that verify b). d is positive. So, in case b),
$|f(t_1) - f(t_2)| \ge d$ and (16) holds with $K = (\operatorname{diam} A)/d$.

In case a), because of the comment below formula (18), it is enough to consider $j \in \{1,3,5\}$. Let $j = 2k + 1$, $k \in \{0,1,2\}$, $t_1, t_2 \in I_j$, $t_1 \le t \le t_2$. We define $t_1' = 3t_1 - k$; $t_2' = 3t_2 - k$; $t' = 3t - k$. Then, because of (15), $f(t') = \tau_k^{-1}(f(t))$.

Therefore, (16) holds if and only if

(19) $|f(t') - f(t_1')| \le K|f(t_2') - f(t_1')|$.

This new pair $\{t_1',t_2'\}$ verifies :

(20) $\sqrt{8}\ |f(t_1)-f(t_2)| \ge |f(t_1')-f(t_2')| \ge \sqrt{2}\ |f(t_1)-f(t_2)| > 0.$

Repeating this procedure if $\{t_1',t_2'\}$ verifies a), we reach after a finite number of steps (and for the first time) a pair $\{\tilde{t}_1, \tilde{t}_2\}$ with $|f(\tilde{t}_1) - f(\tilde{t}_2)| \ge d$, QED.

REMARK. We have constructed in this way a similarity u such that $u(\{f(t);\ t_1 \le t \le t_2\})$ is contained in A and

$$|u(f(t_1)) - u(f(t_2))| \ge d.$$

4.1.CENTER OF SYMMETRY OF F. Consider the numbers $z_0 = -(2+i)/5 = = 0.\bar{1}$, $c = z_0/2$ and define

(21) $W(z) := z_0 - z.$

If $z = 0.a_{-1}a_{-2}\ldots$ and $w = W(z)$ then $w = 0.(1-a_{-1})\ldots$ and $W(w) = z$. That is, $z \in F$ iff $w \in F$. Because of $w - c = -(z - c)$, c is the center of symmetry of F. One can also easily verify that

(22) $W(A) = A\hat{}$; $W(B) = B\hat{}$; $W(C) = C\hat{}$.

4.2. J IS A QUASI-CIRCLE. J(s) will denote the positively oriented Jordan curve composed of the succesive arcs A,C,B,A^,C^,B^. J is a simple curve because of Lemma 4. These arcs are all similar to A and verify :

$$(23)\qquad \left\{\begin{array}{lll} B\hat{} = F_0 \cap F_{111} & = bA - i \\ C\hat{} = F_0 \cap F_{110} & = A/b + 1/b \\ A\hat{} = F_0 \cap F_{11101} & = A - 1 \\ B = F_0 \cap F_{11} & = bA \\ C = F_0 \cap F_{1110} & = A/b - 1/b \end{array}\right.$$

Assume that $\alpha, \beta \in J$. $\ulcorner\alpha\beta\urcorner$ will denote the positively oriented arc on J with endpoints α, β.

THEOREM 7. There exists $K > 0$ such that for any pair $\{\alpha,\beta\} \in J$ it holds that

(24) $\inf\{\mathrm{diam}\ulcorner\alpha\beta\urcorner,\ \mathrm{diam}\ulcorner\beta\alpha\urcorner\} \le K|\beta - \alpha|$.

PROOF.First of all notice that the arcs A^ U C^ U B^, A U C U B,

$B^{\wedge} \cup A$, $B \cup A^{\wedge}$ are similar to subarcs of A. In fact, let $T^{\wedge} := z/b^3 + 1/2$. Writing the effect of $T^{\wedge}$ on the radix representation of z, we have (cf.Appendix)

$T^{\wedge}(a_L \ldots a_0 . a_{-1} a_{-2} \ldots) = [a_L \ldots a_4 (a_3+1).(a_2+1)(a_1+1)a_0 \ldots]$.

Therefore, $T^{\wedge}(F_0) \subset F_1$; $T^{\wedge}(F_{11101} \cup F_{110} \cup F_{111}) \subset F_0$. That is

(25) $\qquad T^{\wedge}(A^{\wedge} \cup C^{\wedge} \cup B^{\wedge}) \subset A$.

Also, from (22) we obtain with $T(z) := T^{\wedge}(W(z))$ that

(26) $\qquad T(A \cup C \cup B) \subset A$.

On the other hand, the mapping $T'(z) := (z - 1) / b^4$ is such that $T'(F_0) \subset F_1$ and $T'(F_1 \cup F_{111}) \subset F_0$. Therefore,

(27) $\qquad T'(B^{\wedge} \cup A) \subset A$.

If $T''(z) := T'(W(z))$, again from (22) we get

(28) $\qquad T''(B \cup A^{\wedge}) \subset A$.

To prove Th.7 we consider the alternative : i) both points α, β belong to one of those four arcs, ii) else. In case i) the theorem follows from (25)-(28) and (16). In case ii), α and β belong to non consecutive arcs. Then, there exists $p > 0$ such that $|\alpha - \beta| \geq p$. In this situation (24) holds with $K = (\text{diam } J)/p$, QED.

COROLLARY 1. i) J is a quasi-circle

ii) there exist a,b and $r > 0$ such that for any set $U \subset J$ with $0 < |U| \leq r$ there is a mapping $u : U \longrightarrow J$ such that

(29) $a|X - Y| \leq |U| \, |u(X) - u(Y)| \leq b|X - Y|$ if $X,Y \in U$.

iii) Moreover, $u(U) \subset A$, $|u(U)| \geq d$.

PROOF. J is a quasi-circle since it is the boundary of a quasi-disk. Let $A°$ denote the open arc $A\setminus\{x,y\}$. Because of Th.5 i) and formulae (25)-(28) it is sufficient to prove (29) for sets $U \subset A°$ of small diameter. Because of Th. 7, it will suffice to show that (29) holds for small closed subarcs of $A°$, say of diameter less than or equal to r. Assume further that r is such that if $|U| \leq r$ then U is included in one of the sets W_j, j=1,2,3,4,5. Such a number exists and because of Prop. 5 we have $r \leq d$. The remark following it assures the existence of a similarity u, $u : U \longrightarrow A$, such that $|u(U)| \geq d$. Then we have,

(30) $d \le |u(U)| \le |A| \le 2$

(31) $|U|/|u(U)| = |X - Y|/|u(X) - u(Y)|$.

Combining these relations one obtains (29), QED.

COROLLARY 2. If E is an s-set satisfying i) and ii) of Corollary 1 and Th. 5 iii) then there is a bilipschitz mapping β such that $\beta(E) = J$.

(For a sketch of proof see [2]).

5. THE CONVEX HULL OF F. We want to show that Q := co(F) is an octagon, (see Fig. 3). First observe that if $d_i \in \{0,1\}$ then

(32) $$-1 \le Re\,(d_0 + d_1 b + d_2 b^2 + d_3 b^3) \le 3.$$

In consequence, if $z = x + iy = \sum_1^\infty d_j b^{-j}$ we have

$$z = \sum_{k=1}^\infty (d_{4k-3}b^3 + d_{4k-2}b^2 + d_{4k-1}b + d_{4k})b^{-4k}$$ Since $b^4 = -4$,

it follows that : $- 13/15 = - 3/4 - 1/4^2 - 3/4^3 - \ldots \le Re\, z \le$ $\le 1/4 + 3/4^2 + 1/4^3 + 3/4^4 + \ldots = 7/15$. Besides we have :

$$\frac{z}{b} = (c_4 + c_3 b + c_2 b^2)b^{-4} + \sum_2^\infty (c_{4k-3}b^3 + c_{4k-2}b^2 + c_{4k-1}b + c_{4k})b^{-4k}.$$

Since $c_i \in \{0,1\}$, the second of the following relations holds. The third and fourth inequalities are easily obtained as the first ones.

(33) $$\begin{cases} -13/15 \le Re\, z = x \le 7/15 \\ -11/30 \le Re\, z/b = (-x+y)/2 \le 7/15 \\ -11/30 \le Re\, z/b^2 = -y/2 \le 7/15 \\ -11/30 \le Re\, z/b^3 = (x+y)/4 \le 13/60 \end{cases}$$

Therefore, F is contained in the region whose boundary is defined by the following lines :

(34) $$\begin{cases} l_1 : x = 7/15 & l_2 : x = -13/15 \\ l_3 : y = 11/15 & l_4 : y = -14/15 \\ l_5 : x+y = 13/15 & l_6 : x+y = -22/15 \\ l_7 : x-y = 11/15 & l_8 : x-y = -14/15 \end{cases}$$

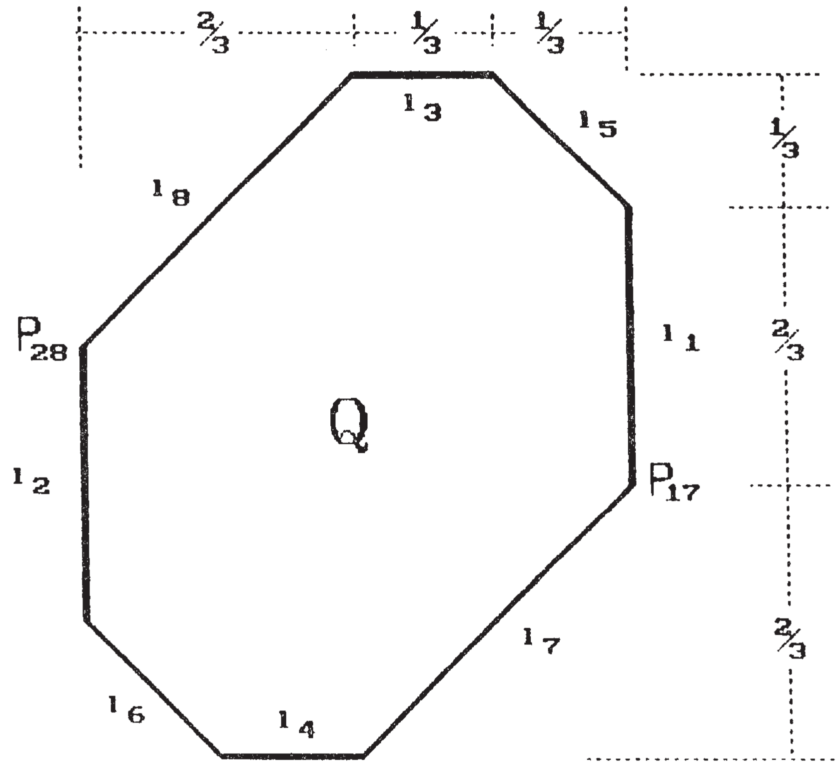
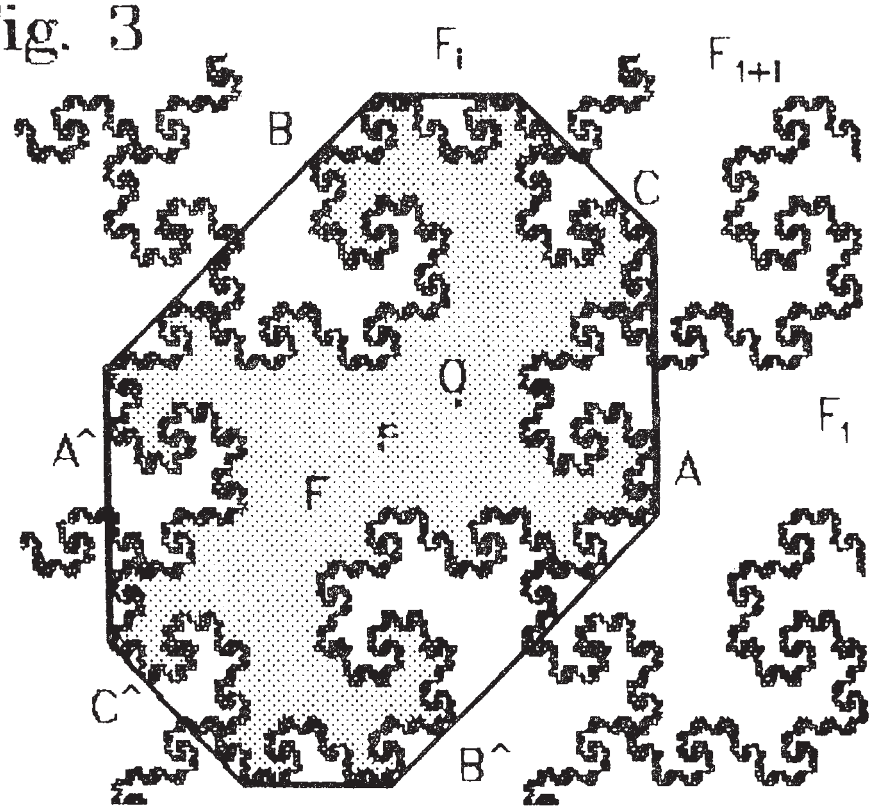

2/3
1/3
1/3
1/3
l_3
l_5
l_8
l_1
2/3
P_{28}
Q
l_2
P_{17}
l_7
2/3
l_6
l_4

Fig. 3


F_i
F_{1+1}
B
C
Q
F_1
A^
A
F
C^
B^

We define : $P_{ij} = P_{ji} = l_i \cap l_j$.

THEOREM 6. $Q = co\,\{P_{17}, P_{15}, P_{35}, P_{38}, P_{28}, P_{26}, P_{46}, P_{47}\}$.

PROOF. It will suffice to prove that

$$(35) \qquad \left[\begin{array}{ll} P_{17} \in A & \qquad P_{28} \in A^{\wedge} \\ P_{15} \in C & \qquad P_{26} \in C^{\wedge} \\ P_{35} \in B & \qquad P_{46} \in B^{\wedge} \\ P_{38} \in B & \qquad P_{47} \in B^{\wedge} \end{array}\right.$$

Since $W(l_i) = l_{i+1}$ if i is odd and $W(l_i) = l_{i-1}$ if i is even, to prove (35) it will be sufficient to show that the four relations on the left hold. From (34) we get,

$$(36) \qquad \left[\begin{array}{ll} P_{17} = (7-4i)/15 & \qquad P_{35} = (2+11i)/15 \\ P_{15} = (7+6i)/15 & \qquad P_{38} = (-3+11i)/15 \end{array}\right.$$

Next we write the points in (36) in positional notation (cf. § 6). Observe that $1/15 = 0.\overline{00000001}$. Also that $7-4i = [-4\ 3]_b = 101101$. Then $P_{17} = 0.\overline{00101101}$. From the graph Γ we get :

$P_{17} = 1.\overline{11110000} \in A$. In fact, this can be checked following in Γ the sequence :

$p|q\ ;\ \frac{q}{p}\ ,\ \frac{p}{q}\ ,\ \frac{\ \ |q}{p|\ \ }\ ,\ q|p\ ,\ \frac{p}{q}\ ,\ \frac{q}{p}\ ,\ \frac{\ \ |p}{q|\ \ }\ ,\ p|q\ ;\ \dots$.

Since $7+6i = [6\ 13] = 1101001$, we get, as before, the equalities

$$P_{15} = 0.\overline{01101001} = 1110.\overline{10000111} \in C.$$

Analogously, $P_{35} = 0.\overline{01001011} = 11.\overline{00111100} \in B$,

$P_{38} = 0.\overline{01011010} = 11.\overline{11100001} \in B$, QED.

5.1. **FIXED POINTS AND THE VERTICES OF Q.** Because of (21) the radix representation of a point in the right column of (35) is obtained from the representation of the point at its left changing zeros by ones and conversely. Then, as next Lemma shows, any extremal point of Q is the fixed point of a composition of four similarities Φ_0 and four Φ_1. In this regard cf. [12].

LEMMA 8. A periodic point $\Omega = 0.\overline{c_1 \ldots c_M}$ such that the cipher $c_i = 1$ for $i = i_1, \ldots, i_N$ where $1 \le i_j < i_{j+1} \le M$, $c_i = 0$ otherwise, is the fixed point of the similarity :

(37) $\Delta := \Phi_0^{i_1-1} \circ \Phi_1 \circ \Phi_0^{i_2-i_1-1} \circ \ldots \circ \Phi_0^{i_N-i_{N-1}-1} \circ \Phi_1 \circ \Phi_0^{M-i_N}$

PROOF. In fact,

(38) $$\Delta(z) = zb^{-M} + (b^{-i_1} + \ldots + b^{-i_N}).$$

Then, $\Delta(z) = z$ if and only if $z = \Omega$, QED.

6. THE SET A AND ITS CONVEX HULL. In Fig. 5 one observes the construction of the attractor A, starting from the segment $[x,y]$, using the similarities τ_i, $i = 0, 1, 2$ (cfr. Prop. 5).

DEFINITION. We say that a selfsimilar set $K = \bigcup_1^n \Phi_i(K)$ has the property $\mathcal{A}$, or shortly $K \in \mathcal{A}$, if there exists $\Delta > 0$ such that for each $X \in K$ and each $r < \Delta$ there are a point $Y \in K$ and a similarity σ of ratio one such that

i) $\sigma(K \cap B_r(X)) = K \cap B_r(Y)$

ii) $\overline{B_r(Y)}$ intersects only one of the sets $\Phi_i(K)$.

THEOREM 7. $A \in \mathcal{A}$.

PROOF. We have observed in the proof of Proposition 5 that the arc $f([0,1/9+1/27])$ that joins x with x' in Fig. 5 and the arc $f([1/3-1/9,1/3+1/27])$ that joins z with $z'$are traslates one of the other. Define $2\delta =$

$= \inf \{\text{dist}\,(f(1/9), A \setminus f([1/9+1/27,1]))\,,$
$\text{dist}\,(f(1/3), A \setminus f([1/3-1/9,1/3+1/27]))$.

Now if $|X-f(1/3)| < \delta$ we define $\sigma(w) = w + f(1/9) - f(1/3)$. Similarly if $|X-f(2/3)| = |S(X)-f(1/3)| < \delta$ we define $\sigma(w) = S(w) + f(1/9) - f(1/3)$. In any other case $\sigma(w) = w$. It is clear that i) holds for $r < \delta$.

Now, if Δ is small enough ii) also holds. QED.

6.1. A CLOSED CONVEX SET H THAT CONTAINS A. Observing the graph Γ it is clear that $A = F_0 \cap F_1 = F_{0.0} \cap F_{1.1}$. In consequence $co(A) \subset co(F_{0.0}) \cap co(F_{1.1}) =: H$. Now $z = x + iy \in F_{0.0}$

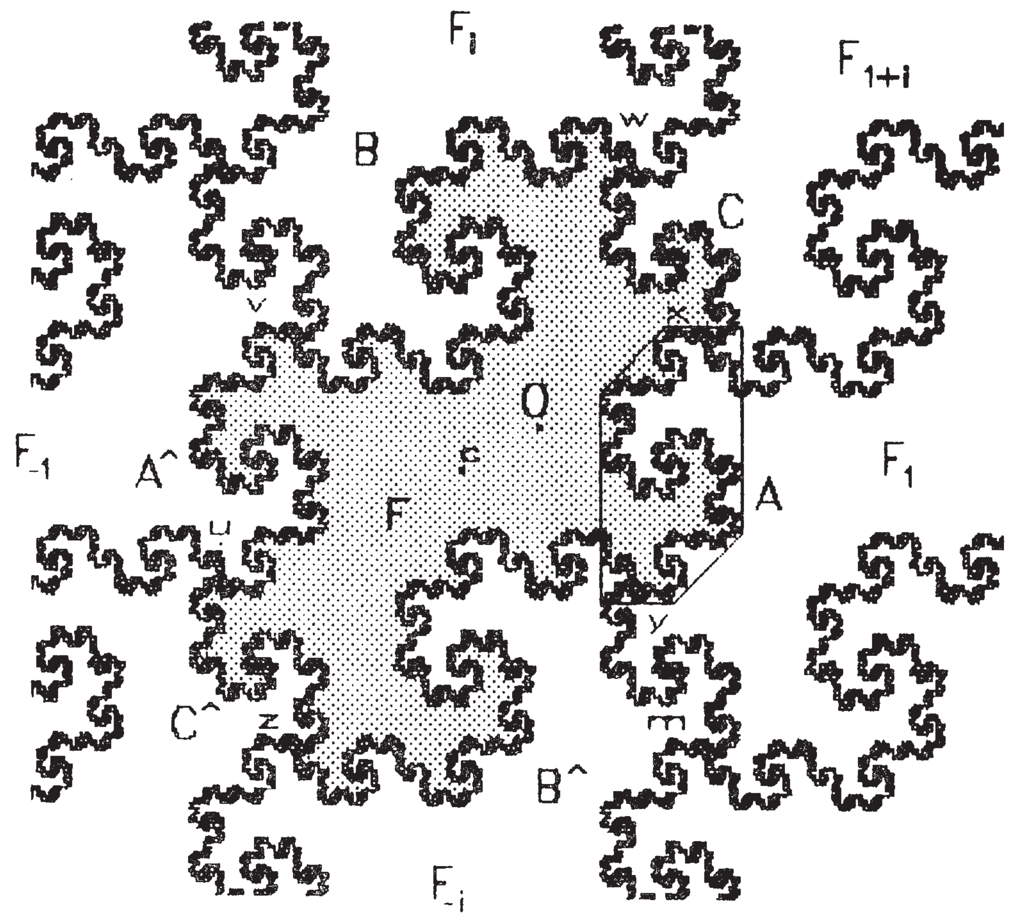
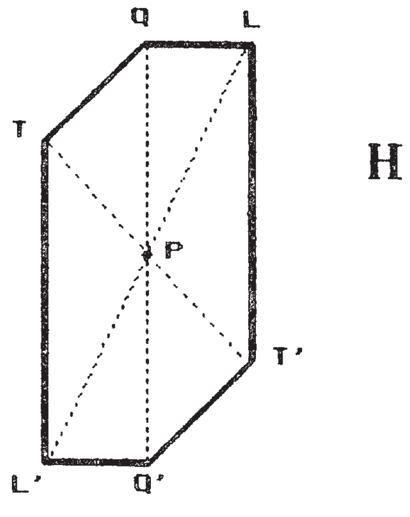


Fig. 4



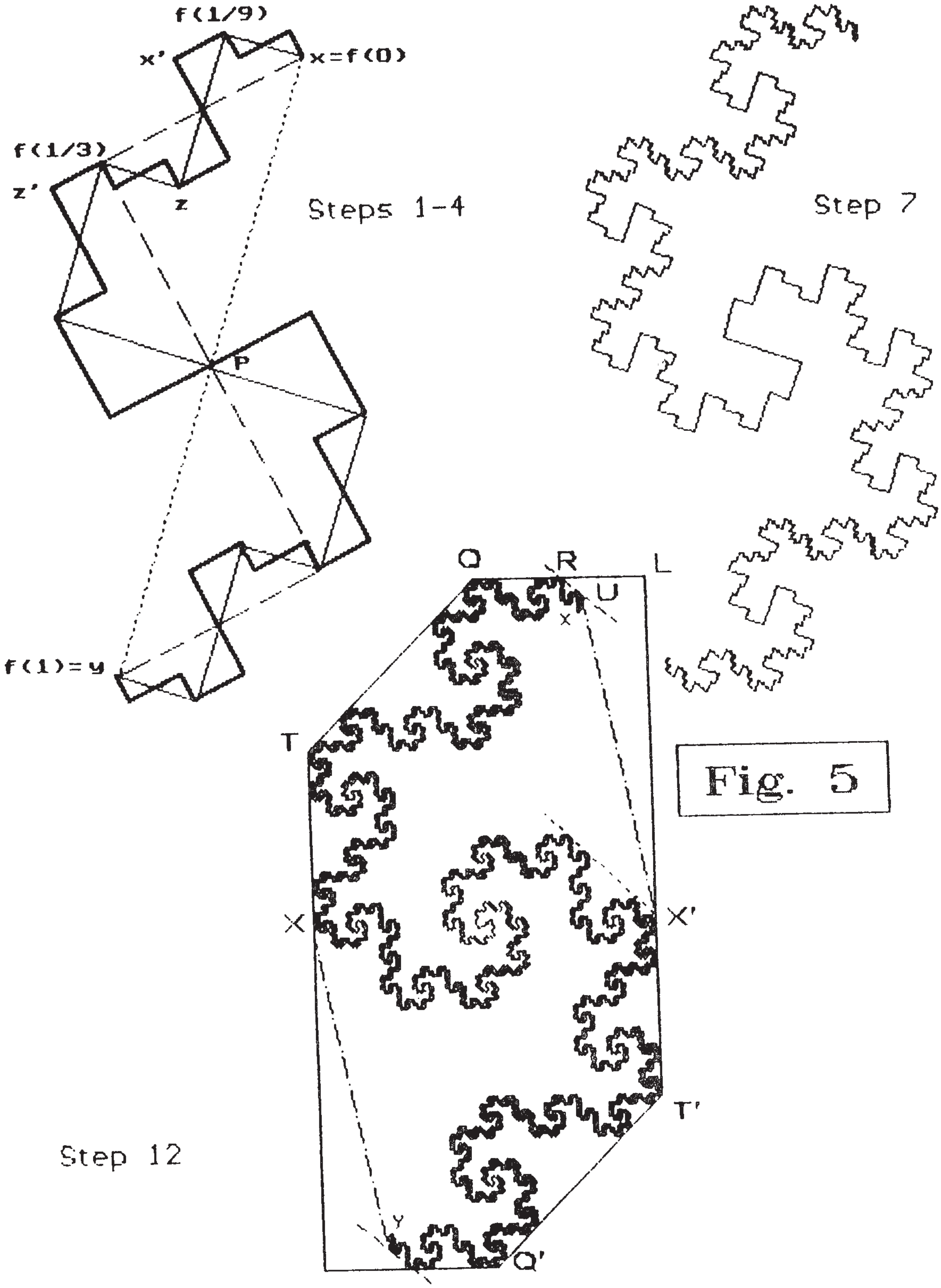


Fig. 5

iff $bz = -(x+y) + i(x-y) \in F_0$. From (33) we get,

$$(37)\quad \left[\begin{array}{ll} -13/15 \le -x-y \le 7/15 & -22/15 \le -2y \le 13/15 \\ -14/15 \le x-y \le 11/15 & -11/15 \le 2x \le 14/15 \end{array}\right.$$

iff $z \in co(F_{0.0})$. Analogously $z \in F_{1.1}$ iff $bz - i = -(x+y) + $ $+i(x-y-1) \in F_0$. So $z \in co(F_{1.1})$ iff

$$(38)\quad \left[\begin{array}{ll} -14/15 \le x-y-1 \le 11/15 & -22/15 \le -2y-1 \le 13/15 \\ -13/15 \le -x-y \le 7/15 & -11/15 \le 2x-1 \le 14/15 \end{array}\right.$$

Combinig (37) and (38) we have: $x+iy \in H$ iff

$$(39)\quad \left[\begin{array}{l} 1/15 \le x-y \le 11/15 \\ -13/30 \le y \le 7/30 \\ 2/15 \le x \le 7/15 \end{array}\right.$$

The region H, defined thus by (39), is a hexagon with vertices T, Q, L, T´, Q´ and L´, and center P (see Fig. 4), where

$$(40)\quad \left[\begin{array}{lll} Q = \dfrac{9+7i}{30}, & T = \dfrac{2+i}{15}, & L = \dfrac{14+7i}{30} \\ Q' = S(Q) = \dfrac{9-13i}{30}, & T' = S(T) = \dfrac{7-4i}{15}, & L' = S(L) = \dfrac{4-13i}{30} \end{array}\right.$$

CLAIM: Q, T, Q´,T´ belong to A.

Since S(A) = A it is enough to show that Q, T $\in$ A. To get the gaussian representation of Q we write Q= (7-9i).(i/2).(1/15) =

$= (110000110)_b.(0.01)_b.(0.\overline{00000001})_b = 0.0\overline{11000011} =$

$= 1.1\overline{10110100} \in A$. Analogously, T = (2 + i)(1/15) =

$= (1111)(0.\overline{00000001})$ belongs to A since T =

$= 0.\overline{00001111} = 1.\overline{11010010}$, and the claim is proved.

6.2. **THE CONVEX HULL OF A**. By the previous result Q, T, Q´ and T´ are extremal points of co(A). To determine the remaining extremal points of co(A), consider the sets :

$F^{\sim} := F_{1.1101101}$, $F^{\#} := F_{0.00}$ and $F^{\approx} = F_{0.01100}$

The graph Γ yields that $A \subset F^{\sim} \cup F^{\#} \cup F^{\approx}$. In fact, if $z \in A$ and

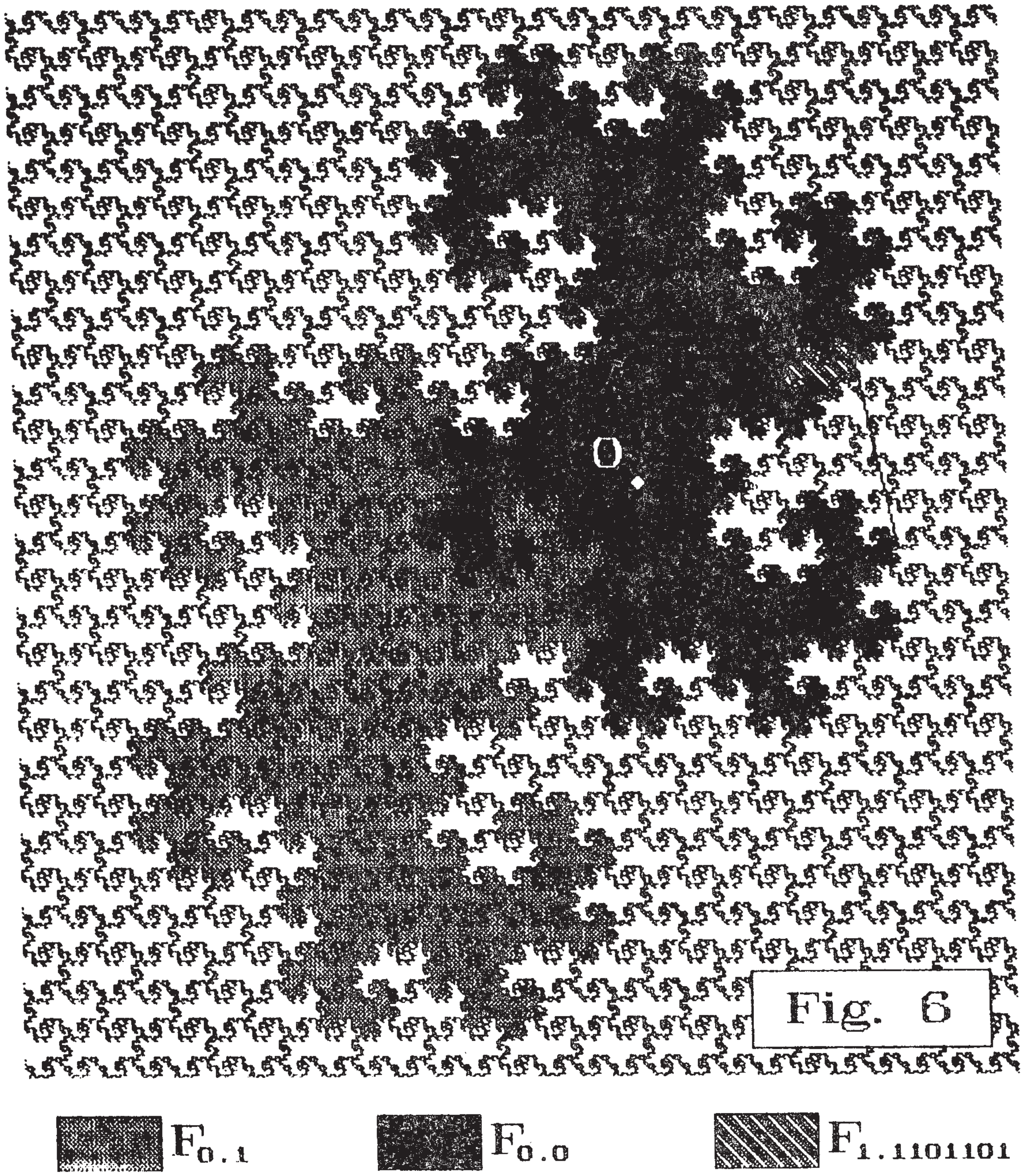


Fig. 6

$z \in F^{\#} \cup F^{\approx}$ then $z \in F_{0.0110110}$. But other representation of z is 1.1101101... and $z \in \tilde{F}$. Since 11101101 = -(1+4i) the extremal points of $co(\tilde{F})$ are (cfr. Theorem 6) :

$$(42) \qquad (P_{ij} + (-1-4i))\, b^{-7} = (P_{ij} -1-4i)(i-1)/16 =: \tilde{P}_{ij}$$

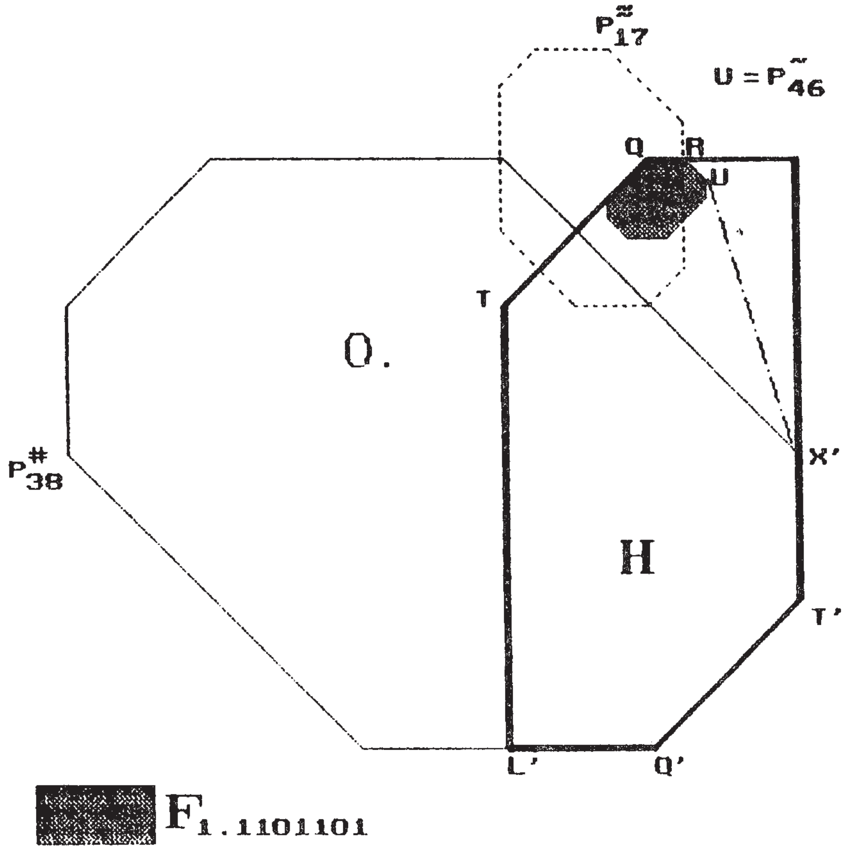


Fig. 7

We have: $\tilde{P}_{17} = \frac{9+7i}{30} = Q$ (cfr. (40)) . Other two are

(43) $\tilde{P}_{47} = \frac{23+14i}{60} =: R$ $\tilde{P}_{46} = \frac{97+51i}{240} =: U$

The radix representations of these numbers are

(44) $\left[\begin{array}{l} R = 1.1101\overline{10110100} = 0.0110\overline{11000011} \\ U = 1.1101101\overline{10110100} = 0.0110110\overline{11000011} \end{array} \right.$

(To obtain these expresions write $R=(-18-74i)b^{-7}(1/15)$

$U = (-23-74i)b^{-7}(1/15)$ and use the fact that $1/15 = 0.\overline{00000001}$)

Therefore R and U belong to A, as well as Q.

Analogously the extremal points of $co(F^{\#})$ are

(45) $P^{\#}_{ij} := P_{ij}.b^{-2} = (P_{ij})(i/2)$

In particular, $P^{\#}_{26} = (9-13i)/30 = Q'$, $P^{\#}_{64} = (14-8i)/30 = T'$,

(46) $\quad P^{\#}_{47} = (14-3i)/30 =: X'$.

The radix representations of these numbers are:

$$(47)\quad \begin{cases} Q' = S(Q) = 0.0\overline{01001011} = 1.1\overline{00111100} \\ T' = S(T) = 0.0\overline{01011010} = 1.1\overline{11100001} \\ X' = 0.00\overline{10100101} = 1.11\overline{00011110} \end{cases}$$

Therefore, $X' \in A$ just as Q' and T'.

Finally the extremal points of $co(F^{\approx})$ are

(48) $\quad P^{\approx}_{ij} := (P_{ij} + 2)b^{-5} = (P_{ij} + 2)(1+i)/8.$

For the sake of completeness we write down the extremal points of the convex hulls of F_0, $F^{\#}$, $F^{\sim}$ and $F^{\approx}$:

$P_{17}=(7-4i)/15$ $\qquad P_{15}=(7+6i)/15$ $\qquad P_{35}=(2+11i)/15$

$P_{38}=(-3+11i)/15$ $\qquad P_{28}=((-13+i)/15$ $\qquad P_{26}=(-13-9i)/15$

$P_{46}=(-8-14i)/15$ $\qquad P_{47}=(-3-14i)/15$

$P^{\#}_{17}=(4+7i)/30$ $\qquad P^{\#}_{15}=(-6+7i)/30$ $\qquad P^{\#}_{35}=(-11+2i)/30$

$P^{\#}_{38}=(-11-3i)/30$ $\qquad P^{\#}_{28}=(-1-13i)/30$ $\qquad P^{\#}_{26}=(9-13i)/30$

$P^{\#}_{46}=(14-8i)/30$ $\qquad P^{\#}_{47}=(14-3i)/30$

$P^{\sim}_{17}=(72+56i)/240$ $\qquad P^{\sim}_{15}=(62+46i)/240$ $\qquad P^{\sim}_{35}=(62+36i)/240$

$P^{\sim}_{38}=(67+31i)/240$ $\qquad P^{\sim}_{28}=(87+31i)/240$ $\qquad P^{\sim}_{26}=(97+41i)/240$

$P^{\sim}_{46}=(97+51i)/240$ $\qquad P^{\sim}_{47}=(92+56i)/240$

$P^{\approx}_{17}=(41+33i)/120$ $\qquad P^{\approx}_{15}=(31+43i)/120$ $\qquad P^{\approx}_{35}=(21+43i)/120$

$P^{\approx}_{38}=(16+38i)/120$ $\qquad P^{\approx}_{28}=(16+18i)/120$ $\qquad P^{\approx}_{26}=(26+8i)/120$

$P^{\approx}_{46}=(36+8i)/120$ $\qquad P^{\approx}_{47}=(41+13i)/120$

We have,

(49) $\qquad \mathrm{Re}\,(P^{\approx}{}_{ij}) \le \mathrm{Re}\,(P^{\approx}{}_{47}) = 41/120 < \mathrm{Re}\,(R),$

(50) $\qquad \mathrm{Re}\,X' + \mathrm{Im}\,X' = 11/30 < 37/60 = \mathrm{Re}\,U + \mathrm{Im}\,U.$

These facts are illustrated in Fig. 7. In consequence, A lies to the left of the line through X′ and U, (see also Fig. 6).

From $A \subset H \cap \{ co(\tilde{F}) \cup co(F^{\#}) \cup co(F^{\approx})\}$, we obtain that $A \subset co\,(L', Q', T', X', U, R, Q, T)$. Because of A = S(A), we have moreover that

(51) $\quad A \subset co\,(U', R', Q', T', X', U, R, Q, T, X) =: \triangle$

where X = S(X′) , U′= S(U), R′= S(R).

Since all the points in the bracket in (51) belong to A (cfr. (42) and (44)), we have proved

THEOREM 8. $\triangle$ = co (A).

7. **APPENDIX.** Next table contains the radix representation in Gauss′ base of the complex numbers (a,c) := a + ic of modulus less than or equal to 4.

TABLE 1.

(-4, 0)=	10000	(0, 1)=	11
(-3,-2)=	10101	(0, 2)=	1110100
(-3,-1)=	11101010	(0, 3)=	1110111
(-3, 0)=	10001	(0, 4)=	1110000
(-3, 1)=	11110	(1,-3)=	111110
(-3, 2)=	1110100101	(1,-2)=	101
(-2,-3)=	11101111	(1,-1)=	111010
(-2,-2)=	11101000	(1, 0)=	1
(-2,-1)=	11101011	(1, 1)=	1110
(-2, 0)=	11100	(1, 2)=	1110101
(-2, 1)=	11111	(1, 3)=	1010
(-2, 2)=	11000	(2,-3)=	111111
(-2, 3)=	11011	(2,-2)=	111000
(-1,-3)=	110010	(2,-1)=	111011
(-1,-2)=	11101001	(2, 0)=	1100
(-1,-1)=	110	(2, 1)=	1111
(-1, 0)=	11101	(2, 2)=	1000
(-1, 1)=	10	(2, 3)=	1011
(-1, 2)=	11001	(3,-2)=	111001
(-1, 3)=	1110110	(3,-1)=	111010110
(0,-4)=	110000	(3, 0)=	1101
(0,-3)=	110011	(3, 1)=	111010010
(0,-2)=	100	(3, 2)=	1001
(0,-1)=	111	(4, 0)=	111010000

Powers of b : $b^3 = 2+2i$, $b^4 = -4$, $b^5 = 4-4i$, $b^6 = 8i$, $b^7 = -8-8i$,

$b^8 = 16$. The radix representation is usually written between parentheses, for example : $-1 = (11101)$, $(1 + i)/4 = (0.011)$, $(1 - i)/4 = = (0.001)$ or if one wishes to indicate the base $b = = -1 + i$: $1/2 = (1.11)_b$; $3b^3 = (1101000)_{-1+i}$. Sometimes it is convenient to use brackets : $3b^3 = [3000]$; this means that instead of a legitimate cipher 0 or 1 there is an integer, not necessarily a binary digit, which is the coefficient of the corresponding power of b. For example : $0 = [122] = = b^2 + 2b + 2 = [1220] = b^3 + 2b^2 + 2b + 0$; $0 = -1 + 1 = = (11101) + (1) = [11102]$. In general, we drop the parentheses. In numbers like $(1 - 2i)/5 = 0.\overline{001}$, the vinculum means that the numbers that are covered repeat periodically. The reader is referred to [6] for a method to perform elementary operations in positional notation.

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