Brans-Dicke manifolds with closed timelike curves.

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Abstract

The paper contains, along with a brief review of concepts of Brans-Dicke theory, a new solution of Brans-Dicke field equations for a rotating geometry with cylindrical symmetry. It turns out that under these constraints the resulting manifold contains closed timelike curves.

1 Introduction

In the last two decades, there has been a great deal of activity on the problem of causal anomalies in general relativity, a subject which had lain rather dormant since the discovery of closed timelike curves (CTCs) in rapidly rotating geometries [11, 7]. There are two main branches in the current revival, that involving the use of wormholes for backward time travel [8] - which we will not discuss further here - and that concerning the study of metrical properties associated with compact and non-compact manifolds [12, 5, 6, 10, 9].

It has long been known that in rotating geometries with cylindrical symmetry particles are held out against their own gravity by centrifugal forces, and their rotation drags inertial frames so strongly that the light cones tilt over in the circumferential direction causing the appearence of CTCs. Thus, in order to look for 4-dimensional spacetimes with CTCs we need to work in models with rotating cylindrical metric, since, as aforementioned, such a kind of metric generates inercial forces that can turn aside the light cone, keeping locally the timelike condition (but loosing the global time-orientation). Moreover, to obtain new solutions one realize the demand of working in more general spaces. We shall concentrate here on Brans-Dicke theory (BD), introduced in 1961 [4]. It is noteworthy that BD geometries have already displayed some bizarre effects on stellar configurations through a local modification of the gravitational constant by the matter energy distribution, see for instance, [1, 13, 14, 2]. Let us start with a recapitulation of the peculiar features of BD gravity.

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2 Preliminaries

The BD theory, originally formulated in a representation in which the equation of motion for test particles is identical to that of general relativity, can be expressed in units in which the local value of the Newtonian "gravitational constant" (G) is a function of a scalar field, actually, $G = 1/\phi$. The BD scalar is in turn determined by the trace of the energy-momentum tensor of all the other nongravitational fields. The field equations in the general form are given by

$$\begin{aligned} G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= \frac{8\pi}{\phi} T_{\mu\nu} + \frac{\omega}{\phi^2}\left(\phi_{;\mu}\phi_{;\nu} - \frac{1}{2} g_{\mu\nu} \sum_{\alpha=0}^{3} \phi^{;\alpha}\phi_{;\alpha}\right) \\ &+ \frac{1}{\phi}\left(\phi_{;\mu\nu} - g_{\mu\nu} \sum_{\alpha=0}^{3} \phi^{;\alpha}\phi_{;\alpha}\right) \end{aligned} \tag{1}$$

and

$$\Box\phi = \frac{8\pi T}{3+2\omega} \tag{2}$$

where ω is the coupling constant, $R_{\mu\nu}$ the Ricci tensor and R the Ricci scalar. $T_{\mu\nu}$ is the usual energy-momentum tensor with trace T. Semicolons denote covariant derivative with respect to the metric $g_{\mu\nu}$, and $\Box\phi$ stands for the D'Alembertian of ϕ.

We want to solve (1) and (2) under the condition

$$ds^2 = dt^2 - e^{2\lambda(r)}\left(dr^2 + dz^2\right) - \ell(r) d\theta^2 + 2m(r)\, d\theta dt \tag{3}$$

which represents the more general ultrastatic[1] metric tensor with cylindrical symmetry.

We specialize our problem to that of an energy momentum tensor with anisotropic pressure tensor[2] $\Pi_{\mu\nu}$ such that, is symmetric ($\Pi_{\mu\nu} = \Pi_{\nu\mu}$), trace free ($\Pi = 0$), and orthogonal to the comoving observer ($\Pi_{0\mu} = 0$). So,

$$T_{\mu\nu} = \rho u_\mu u_\nu + \Pi_{\mu\nu} \tag{4}$$

with ρ the energy density, and $u^\mu = \delta_0^\mu$ the four velocity in the comoving system. This means that $\Pi_{\mu\nu} = \mathrm{diag}(0, \alpha, \beta, -(\alpha+\beta))$, and thus, $T_{\mu\nu}$ in the matrix form reads,

$$(T_{\mu\nu}) = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & -(\alpha+\beta) \end{pmatrix}$$

As usual, the inverse matrix $g^{\mu\nu}$ is given by

$$g^{00} = \frac{\ell}{\ell+m^2},\ g^{03} = \frac{m}{\ell+m^2},\ g^{11} = -\frac{1}{e^{2\lambda}},\ g^{22} = -\frac{1}{e^{2\lambda}},\ g^{33} = -\frac{1}{\ell+m^2}.$$

[1]An ultrastatic spacetime is described by a manifold wherein the metric is simply given by $ds^2 = -dt\otimes dt + \sum_{ij} g_{ij} dx^i \otimes dx^j$.

[2]Recall that such a tensor has the radial pressure different from the lateral and tangential stresses.

Using that $\Pi^\nu_\mu = \sum_\alpha g^{\alpha\nu}\Pi_{\mu\alpha}$,

$$T^\nu_\mu = \sum_\alpha g^{\nu\alpha}T_{\alpha\mu} = \sum_\alpha g^{\nu\alpha}\rho u_\alpha u_\mu + \Pi^\nu_\mu,$$

we get, $\Pi^1_1 = -\alpha$, $\Pi^2_2 = -\beta$, $\Pi^3_3 = \alpha+\beta$, and $u_\nu = (1,0,0,m)$. Then, $T^0_0 = \rho$, $T^1_1 = -\alpha$, $T^2_2 = -\beta$, $T^3_3 = \alpha+\beta$, $T^0_3 = \rho m$, and $T^3_0 = 0$.

Hereafter, differentiations with respect to r are indicated by accents; with this in mind, the non-zero Christoffel symbols (computed from the derivatives of the metric tensor) are

$$\Gamma^0_{01} = \frac{mm'}{2\,(\ell+m^2)},\ \Gamma^3_{01} = -\frac{m'}{2\,(\ell+m^2)},\ \Gamma^1_{03} = \frac{1}{2}\frac{m'}{e^{2\lambda}},$$

$$\Gamma^1_{11} = \lambda',\ \Gamma^2_{12} = \lambda',\ \Gamma^1_{22} = -\lambda'$$

$$\Gamma^0_{13} = \frac{1}{2}\left(\frac{\ell m' - m\ell'}{\ell+m^2}\right), \Gamma^3_{13} = \frac{1}{2}\left(\frac{mm'+\ell'}{\ell+m^2}\right), \Gamma^1_{33} = -\frac{1}{2}\frac{\ell'}{e^{2\lambda}}.$$

Now, the nonvanishing components of the Ricci tensor are

$$R_{00} = \tfrac{1}{2}\left(\tfrac{m'^2}{e^{2\lambda}(\ell+m^2)}\right)$$

$$R_{03} = \tfrac{1}{4}\left(\tfrac{2m''\ell+2m''m^2-m'\ell'}{\ell+m^2}\right)$$

$$R_{11} = \tfrac{1}{4(\ell+m^2)^2}\{2\ell m'^2 + 4mm''\ell + 4m''m^3 - 4mm'\lambda'\ell - 4m'\lambda' m^3 -2m^2m'^2 - 4mm'\ell' + 4\lambda''\ell^2 + 8\lambda''\ell m^2 + 4\lambda'' m^4 +2\ell\ell'' + 2\ell'' m^2 - 2\ell'\lambda'\ell - 2\ell'\lambda' m^2 - \ell'^2\}$$

$$R_{22} = \tfrac{1}{2(\ell+m^2)}\left\{2mm'\lambda' + 2\lambda''\ell + 2\lambda'' m^2 + \ell'\lambda'\right\}$$

$$R_{33} = \tfrac{1}{4e^{2\lambda}(\ell+m^2)}\left\{2\ell m'^2 + 2\ell''\ell + 2\ell'' m^2 - 2mm'\ell' - \ell'^2\right\}.$$

Applying contraction of index, the field equation can also be written as

$$R^\mu_\nu = \frac{8\pi}{\phi}\left(T^\mu_\nu - \frac{1}{2}T\delta^\mu_\nu\right) + \frac{\omega}{\phi^2}\phi^{;\mu}\phi_{;\nu} + \phi^{;\mu}_{;\mu} + \frac{1}{2}\delta^\mu_\nu\frac{\Box\phi}{\phi}. \tag{5}$$

In particular,

$$R^0_0 = g^{0\mu}R^0_\mu = g^{00}R_{00} + g^{03}R_{30} =$$

$$= \frac{\ell}{(\ell+m^2)}\frac{1}{2}\frac{m'^2}{e^{2\lambda}(\ell+m^2)} + \frac{m}{(\ell+m^2)}\frac{\left(2m''(\ell+m^2) - m'\ell'\right)}{4(\ell+m^2)e^{2\lambda}}.$$

If $D^2 = \ell + m^2 \Rightarrow$

$$R_0^0 = \frac{\ell m'^2}{2D^4 e^{2\lambda}} + \frac{2mm''D^2 - mm'\ell'}{4D^4 e^{2\lambda}}.$$

Now,

$$\ell' = 2DD' - 2mm'$$

and

$$\ell'' = 2D'^2 + 2DD'' - 2m'^2 - 2mm''.$$

Introducing $\sqrt{-g} = e^{2\lambda}D$,

$$R_0^0 = \left(\frac{m'^2}{2D^2} + \frac{1}{2D^2}mm'' - \frac{1}{2D^3}m'mD'\right)\frac{1}{e^{2\lambda}}$$

or since,

$$\frac{d}{dr}\left(\frac{mm'}{2D}\right) = \frac{(m'^2 + mm'')2D - 2D'mm'}{4D^2}$$

we obtain

$$R_0^0 = \frac{1}{e^{2\lambda}D}\frac{d}{dr}\left(\frac{mm'}{2D}\right) = \frac{1}{\sqrt{-g}}\frac{d}{dr}\left(\frac{mm'}{2D}\right).$$

For R_1^1 we get,

$$\begin{aligned} R_1^1 &= g^{11}R_{11} = \frac{(-1)}{e^{2\lambda}}\frac{1}{4D^4}\Big(2\ell m'^2 + 4mm''D^2 - 2m^2m'^2 - 4mm'\lambda'D^2 + 4\lambda''\ell^2 \\ &\qquad -4mm'\ell' + 8\lambda''\ell m^2 + 4\lambda''m^4 + 2\ell''D^2 \\ &\qquad -\ell'^2 - 2\ell'\lambda'D^2\Big) \\ &= \frac{1}{4e^{2\lambda}D^4}\Big(-2m'^2(D^2 - m^2) - 4mm''D^2 + 2m'^2m^2 + 4mm'(2DD' - 2mm') \\ &\qquad -2\left(2D'^2 + 2DD'' - 2m'^2 - 2mm''\right)D^2 + (2DD' - 2mm')^2 \\ &\qquad -4\lambda''D^4 + 4D'\lambda'D^2\Big) \\ &= \frac{1}{4D^4e^{2\lambda}}\Big(-2m'^2D^2 + 2m'^2m^2 - 4mm''D^2 + 2m'^2m^2 + 8mm'DD' - 8m^2m'^2 \\ &\qquad -4D'^2D^2 - 4D^3D'' + 4m'^2D^2 \\ &\qquad +4mm''D^2 + 4D^2D'^2 \\ &\qquad -8DD'mm' + 4m^2m'^2 - 4\lambda''D^4 + 4D'\lambda'D^2\Big) \end{aligned}$$

and finally,

$$R_1^1 = \left\{\frac{m'^2}{2D} - D'' - \lambda''D + D'\lambda'\right\}\frac{1}{\sqrt{-g}}.$$

The corresponding expression for R_2^2 is easily obtained,

$$R_2^2 = g^{22}R_{22} = \frac{1}{2D^2e^{2\lambda}}\left(2mm'\lambda' + 2\lambda''D^2 + \ell'\lambda'\right)$$

or equivalently,

$$R_2^2 = \frac{1}{\sqrt{-g}}\left(\lambda'' D + D'\lambda'\right).$$

For R_3^3 we obtain at once,

$$\begin{aligned} R_3^3 &= g^{3\mu}R_{\mu 3} = g^{30}R_{03} + g^{33}R_{33} \\ &= \frac{m}{4\left(\ell+m^2\right)^2 e^{2\lambda}}\left(2m''\left(\ell+m^2\right) - m'\ell' + 2\ell' m'^2 + 2\ell''\left(\ell+m^2\right) - 2mm'\ell' - \ell'^2\right) \\ &= -\frac{m}{4D^4e^{2\lambda}}\left(2m''D^2 - m'\ell'\right) - \frac{1}{4D^4e^{2\lambda}}\left(2\ell m'^2 + 2\ell'' D^2 - 2mm'\ell' - \ell'^2\right) \\ &= -\frac{1}{4D^4e^{2\lambda}}\left(2mm''D^2 - mm'\ell' + 2\ell m'^2 + 2\ell'' D^2 - 2mm'\ell' - l'^2\right) \end{aligned}$$

and since,

$$mm' + \frac{\ell'}{2} = DD',$$

we have,

$$\begin{aligned} R_3^3 = \frac{1}{e^{2\lambda}} \{ & -\tfrac{mm''}{2D^2} + \tfrac{mm'}{4D^4}\left(2DD' - 2mm'\right) - \tfrac{m'^2}{2D^4}\left(D^2 - m^2\right) - \tfrac{\ell''}{2D^2} \\ & + \tfrac{D'}{2D^3}\left(2DD' - 2mm'\right)\}. \end{aligned}$$

Recalling that, $2DD' - 2mm' = \ell'$, and

$$\frac{d}{dr}\left(\frac{\ell' + mm'}{2D}\right) = \left[\left(\ell'' + m'^2 + mm''\right)2D - 2D'\left(\ell' + mm'\right)\right],$$

it becomes,

$$R_3^3 = -\frac{1}{\sqrt{-g}}\frac{d}{dr}\left(\frac{\ell' + mm'}{2D}\right).$$

It is straightforward to obtain the expression for R_0^3

$$\begin{aligned} R_0^3 &= g^{3\mu}R_{\mu 0} = g^{30}R_{30} = \left(-\frac{1}{2}\right)\frac{mm'^2}{\left(\ell+m^2\right)^2 e^{2\lambda}} + \frac{2m''\left(l+m^2\right)}{4\left(\ell+m^2\right)} - m'\ell' \\ &= -\frac{1}{e^{2\lambda}D^4}\left(\frac{mm'^2}{2} - \frac{1}{4}\left(2m''D^2 - m'\ell'\right)\right) \\ &= -\frac{mm'^2}{2D^4e^{2\lambda}} + \frac{m''}{2D^2e^{2\lambda}} - \frac{m'\ell'}{4D^4e^{2\lambda}} \\ &= \frac{1}{e^{2\lambda}}\left\{-\frac{mm'^2}{2D^4} + \frac{1}{2D^2}m'' - \frac{m'}{4D^4}\left(2DD' - 2mm'\right)\right\} \end{aligned}$$

$$R_0^3 = \frac{1}{\sqrt{-g}}\frac{d}{dr}\left(\frac{m'}{2D}\right).$$

Finally,

$$R^0_3 = g^{0\mu}R_{\mu 3} = g^{00}R_{03} + g^{03}R_{33}$$

$$\begin{aligned}
R^0_3 &= -\frac{\ell}{4\left(\ell+m^2\right)^2 e^{2\lambda}}\left(2m''\left(\ell+m^2\right) - m'\ell'\right) \\
&\quad + \frac{m}{4\left(\ell+m^2\right)^2 e^{2\lambda}}\left(2\ell m'^2 + 2\ell''\left(\ell+m^2\right) - 2mm'\ell' - \ell'^2\right) \\
&= \frac{1}{4D^4e^{2\lambda}}\left(-2\ell m''D^2 + m'\ell'\ell + 2\ell mm'^2 + 2\ell''mD^2 - 2m^2m'\ell' - m\ell'^2\right) \\
&= \frac{1}{4D^4e^{2\lambda}}\left(-2\ell m''D^2 + 2\ell''mD^2\right) + \frac{1}{4D^4e^{2\lambda}}\left(m'\ell'\ell + 2\ell mm'^2 - 2m^2m'\ell' - m\ell'^2\right) \\
&= \frac{1}{2D^2e^{2\lambda}}\left(\ell''m - m''\ell\right) + \frac{1}{4D^4e^{2\lambda}}\left(m'\ell'\ell + 2m\ell'\left(-mm' - \frac{\ell'}{2}\right) + 2\ell mm'^2\right)
\end{aligned}$$

again

$$-mm' - \frac{\ell'}{2} = -DD'$$

obtaining

$$\begin{aligned}
&= \frac{1}{2D^2e^{2\lambda}}\left(\ell''m - m''\ell\right) + \frac{1}{4D^4e^{2\lambda}}\left(m'\ell'\ell - 2m\ell'DD' + 2\ell mm'^2\right) \\
&= \frac{1}{2D^2e^{2\lambda}}\left(\ell''m - m''\ell\right) + \frac{1}{4D^4e^{2\lambda}}\left(m'\ell'\ell + 2\ell m'\left(mm'\right) - 2mDD'\ell'\right)
\end{aligned}$$

if

$$mm' = DD' - \frac{\ell'}{2}$$

we get

$$\begin{aligned}
&= \frac{1}{2D^2e^{2\lambda}}\left(\ell''m - m''\ell\right) + \frac{1}{4D^4e^{2\lambda}}\left(m'\ell'\ell + 2\ell m'DD' - \ell\ell'm' - 2mDD'\ell'\right) \\
&= \frac{1}{2D^2}\left(\ell''m - m''\ell\right) + \frac{1}{2D^3}D'\left(\ell m' - m\ell'\right)
\end{aligned}$$

Now, since

$$\frac{d}{dr}\left(\frac{m\ell' - m'\ell}{2D}\right) = \frac{\left(m'\ell' + m\ell'' - m''\ell - m'\ell'\right)}{2D} - \frac{2D'}{4D^2}\left(m\ell' - m'\ell\right)$$

we obtain

$$R^0_3 = \frac{1}{\sqrt{-g}}\frac{d}{dr}\left(\frac{m\ell' - m'\ell}{2D}\right).$$

Roughly speaking, we use to say that a 4-dimensional spacetime manifold in general relativity is characterized by Einstein equation, now, in the same sense, we say that a BD manifold satisfies its general field equations.

3 Results

Lemma. Let $\mathcal{M}$ be a differential manifold with the line element given by $ds^2 = dt^2 - e^{2\lambda(r)}\left(dr^2 + dz^2\right) - \ell(r)d\theta^2 + 2m\left(r\right)d\theta dt$. If $\lim_{r\to 0}\lambda = 0$, $\ell(0)$ is non singular, $\ell(0) \neq 0$, and $\lim_{r\to 0} m = 0$, then, there exists a coordinate system where the coordinate θ behaves like an angular coordinate.

Proof:

We begin introducing the change of coordinates

$$x^{\sigma} = \{t, r, z, \theta\} \rightarrow x'^{\mu} = \{T, R, Z, \varphi\}$$

given by

$$g'_{\mu\nu}\left(x'\right) = \sum_{\alpha,\beta} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}\left(x\right) \tag{6}$$

with $x^{\alpha}, x^{\beta} \in x^{\sigma}$ and $x'^{\mu}, x'^{\nu} \in x'^{\sigma}$. Thus, one possible selection compatible with (6) is

$$g_{tt} = g'_{TT},\ g'_{RR} = -e^{2\Lambda(R)},\ g'_{zz} = -e^{2\Lambda(R)},\ g'_{\varphi\varphi} = -F^2(R)L\left(R\right), g'_{T\varphi} = F(R)M\left(R\right),$$

or equivalently, $T = t$, $R = r$, $g_{zz} = g'_{zz}$, $e^{2\lambda(r)} = e^{2\Lambda(R)}$, $z = Z$, $\theta = \varphi$,

$$-\ell\left(r\right) = -F^2 L\left(R\right),$$

and

$$m\left(r\right) = M\left(R\right)F.$$

With this in mind, the metric tensor can be re-written as,

$$ds^2 = dT^2 - e^{2\lambda(R)}\left(dR^2 + dZ^2\right) - F^2 L\left(R\right)d\varphi + 2MFd\varphi dT, \tag{7}$$

now, one could always choose F such that the product of L times F goes to zero as r^2,so that around the origin, the line element (7) reduces to that of the cylinder, and φ $(= \theta)$ behaves like an angular coordinate.

Theorem. There exists at least one BD manifold with CTCs.

Proof:

For all above, to find such a manifold we have to solve the following system:

$$\frac{d}{dr}\left(\frac{mm'}{2D}\right) = \frac{4\pi}{\phi}\sqrt{-g}\rho - \frac{mm'}{2D}\frac{\phi'}{\phi} + \frac{1}{2}\frac{\Box\phi}{\phi}\sqrt{-g} \tag{8}$$

$$\frac{d}{dr}\left(\frac{\ell' + mm'}{2D}\right) = -\frac{4\pi}{\phi}\sqrt{-g}(\rho + 2\alpha + 2\beta) - \frac{mm' + \ell'}{2D}\frac{\phi'}{\phi} + \frac{1}{2}\frac{\Box\phi}{\phi}\sqrt{-g} \tag{9}$$

$$\frac{d}{dr}\left(\frac{m'}{2D}\right) = -\frac{m'}{2D}\frac{\phi'}{\phi} \tag{10}$$

$$\frac{d}{dr}\left(\frac{m\ell' - m'\ell}{2D}\right) = -\frac{8\pi}{\phi}\sqrt{-g}m\rho + \frac{m'\ell - m\ell'}{2D}\frac{\phi'}{\phi} \tag{11}$$

$$\begin{aligned} -D\lambda'' + \frac{m'^2}{2D} + D'\lambda' - D'' &= \frac{4\pi}{\phi}\sqrt{-g}\,(\rho - 2\alpha) + \omega D\frac{\phi'^2}{\phi^2} + D\frac{\phi''}{\phi} \\ &\quad -\frac{1}{2}\frac{\Box\phi}{\phi}\sqrt{-g} - D\lambda'\frac{\phi'}{\phi} \end{aligned} \tag{12}$$

$$-D\lambda'' - D'\lambda' = \frac{4\pi}{\phi}\sqrt{-g}\,(-\rho - 2\beta) + D\lambda'\frac{\phi'}{\phi} - \frac{1}{2}\frac{\Box\phi}{\phi}\sqrt{-g} \tag{13}$$

$$\Box\phi = \frac{8\pi}{3 + 2\omega}\rho. \tag{14}$$

To do it, we first observe that Eq. (10) is integrable,

$$\frac{d}{dr}\ln\left(\frac{m'}{D}\right) = -\frac{\phi'}{\phi} \Rightarrow \frac{m'}{2D} = \frac{b}{\phi} \tag{15}$$

where b is a constant to be determined. Multiplying (8) by ϕ, we obtain at once

$$\phi\frac{d}{dr}\left(\frac{mm'}{2D}\right) = 4\pi\sqrt{-g}\rho - \frac{mm'}{2D}\phi' + \frac{1}{2}\sqrt{-g}\Box\phi$$

or equivalently,

$$\frac{d}{dr}\left(\phi\frac{mm'}{2D}\right) = 4\pi\sqrt{-g}\rho + \frac{1}{2}\sqrt{-g}\Box\phi. \tag{16}$$

Since $\sqrt{-g}\Box\phi = -\left(D\phi'\right)'$, from Eqs. (14), (15), 16), we get

$$\begin{aligned} \frac{d}{dr}(mb) &= -(2+\omega)\left(D\phi'\right)' \\ \frac{2Db^2}{\phi} &= -(2+\omega)\left(D\phi'\right)' \\ 2Db^2 &= -(2+\omega)\left(D'\phi'\phi + D\phi\phi''\right) \\ D\left(2b^2 + (2+\omega)\,\phi\phi''\right) &= -(2+\omega)\,D'\phi'\phi \\ \frac{D'}{D} &= \frac{2b^2 + (2+\omega)\,\phi\phi''}{-(2+\omega)\,\phi'\phi}dr, \end{aligned}$$

and after integration,

$$D(r) = e^{D_1(r)},$$

where

$$D_1(r) = \int \frac{2b^2 + (2+\omega)\,\phi(r)\left(\frac{\partial^2}{\partial r^2}\phi(r)\right)}{-(2+\omega)\left(\frac{\partial}{\partial r}\phi(r)\right)\phi(r)} dr.$$

We propose,

$$\phi(r) = a \cosh^n r \tag{17}$$

and replace in Eq. (11) so as to get $a = \sqrt{\frac{-2b^2}{2+\omega}}$ and $n = 1$.

Now it is straightforward to obtain

$$\begin{aligned}
\phi(r) &= \frac{\sqrt{2}b}{\sqrt{2+\omega}}\mathrm{coshr} \\
\ell(r) &= \frac{1 - 2(2+\omega)\sinh^2\mathrm{r}}{\cosh^2\mathrm{r}} \\
m(r) &= \sqrt{4+2\omega}\,\mathrm{tanhr}
\end{aligned}$$

Note that Eq. (9) holds only if $\alpha + \beta = 0$. Adding (12) and (13), we obtain an integrable expression for λ,

$$\lambda'' = \frac{\sqrt{-g}(\omega+1)}{D}\frac{\Box\phi}{\phi} - \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 - \frac{\phi''}{\phi} + \frac{m'^2}{4D^2} - \frac{D''}{2D}$$

which after integration becomes,

$$\lambda(r) = (1+\omega)\ln(\mathrm{coshr}) - \frac{r^2}{4}(\omega+2).$$

Finally, we calculate $\rho(r)$ from Eq. (14)

$$4\pi\rho = \frac{b}{\sqrt{2(|\omega|-2)}}(2|\omega|-3)(\cosh r)^{2|\omega|-3}\exp\left\{-\frac{|\omega|-2}{2}r^2\right\}. \tag{18}$$

and from (12), α comes out as

$$4\pi\alpha = -b\,\frac{\sqrt{2(|\omega|-2)}}{4}(\cosh r)^{2|\omega|-1}\exp\left\{-\frac{|\omega|-2}{2}r^2\right\} \tag{19}$$

As a help to discuss the singularities, we introduce (as usual) the orthonormal basis [3] given by the differential forms

$$\Theta^0 = -m\,d\theta + dt \qquad \Theta^1 = e^\lambda dr \qquad \Theta^2 = e^\lambda dz \qquad \Theta^3 = D\,d\theta \tag{20}$$

[3]For details, see the Appendix of Ref. [3]

with the corresponding basis vectors

$$e^i_{\hat{0}} = \delta^i_0 \qquad e^i_{\hat{1}} = e^{-\lambda}\,\delta^i_1 \qquad e^i_{\hat{2}} = e^{-\lambda}\,\delta^i_2 \qquad e^i_{\hat{3}} = D^{-1}\,\delta^i_3 + mD^{-1}\,\delta^i_0. \tag{21}$$

Now, it is easy to compute the behaviour of certain scalars built from the Riemann tensor

$$R_{\hat{\mu}\hat{\nu}\hat{\eta}\hat{\delta}} = e^{\alpha}{}_{\hat{\mu}}\, e^{\beta}{}_{\hat{\nu}}\, e^{\epsilon}{}_{\hat{\eta}}\, e^{\psi}{}_{\hat{\delta}}\, R_{\alpha\beta\epsilon\psi},$$

and afterwards, to verify that all the above physical parameters remain finite and regular for the entire range of variables, $\omega \in (-\infty, -2)$. This indicates clearly that the model is free of physical singularities, *viz.*, singularities at a finite proper distance of the origin.

Finally, applying the Lemma, it follows that any curve with constant t, r, and z is closed (in our case we could set, for instance, $F = r^2$). In particular, such closed curves are timelike if $r > \text{arcsinh}[1/4 + 2\omega]$.

References

[1] - L. A. Anchordoqui-D. F. Torres-S. E. Perez Bergliaffa, "Brans–Dicke wormholes in nonvacuum spacetime", Phys. Rev. D, vol. 55, 5226 (1997).

[2] - O. G. Benvenuto-L. G. Althaus-D. F. Torres, "Evolution of white dwarfs as a probe of theories of gravitation: The case of Brans-Dicke", Mon. Not. R. Ast. Soc., to be published.

[3] - W. Bonnor, "The rigidely rotating relativistic dust-cylinder", J. Physic A: Math. Gen., vol 13, 2121 (1980).

[4] - C. Brans-R. H. Dicke, "Mach's principle and a relativistic theory of gravitation" Phys. Rev., vol. 123, 925 (1961).

[5] - G. J. Galloway, "Closed timelike geodesics ", Trans. Amer. Math. Soc., vol. 285, 379 (1984).

[6] - G. J. Galloway, "Compact Lorentzian manifolds without closed non spacelike geodesics", Proc. Amer. Math. Soc., vol. 98, 119 (1986).

[7] - K. Gödel, "An example of a new type of cosmological solutions of Einstein's field equations of gravitation ", Rev. Mod. Phys., vol 21, 477 (1949).

[8] - M. S. Morris-K. S. Thorne-U. Yurtsever, "Wormholes, time machines, and the weak energy condition", Phys. Rev. Lett., vol. 61, 1446 (1988).

[9] - M. Novello-M. J. Reboucas, "Rotating universe with successive causal and noncausal regions ", Phys. Rev. D, vol. 19, 2850 (1979).

[10] - A. K. Raychaudhuri-S. N. Guha Thakurta, "Homogeneous space-times of the Gödel type", Phys. Rev. D, vol. 22, 802 (1980).

[11] - W. van Stockum, "The gravitational field of a distribution of particles rotating about an axis of symmetry", Proc. R. Soc. Edin., vol. 57, 135 (1937).

[12] - F. Tipler, "Existing of closed timelike geodesics in Lorentz Space", Proc. Amer. Math. Soc., vol. 76, 145 (1979).

[13] - D. F. Torres, "Boson stars in general scalar-tensor gravitation: Equilibrium configurations", Phys. Rev. D, vol. 56, 3478 (1997).

[14] - D. F. Torres-A. R. Liddle-F. E. Schunck, " Gravitational memory of boson stars", Phys. Rev. D, vol. 57, 4821 (1998).