ON THE NUMBER SYSTEM (-2,{0,1,exp2πi/3,exp4πi/3}): NUMBERS WITH TWO REPRESENTATIONS

Agnes Benedek and Rafael Panzone

ABSTRACT. In the number system (-2,D) with base b=–2 and family of ciphers $D=\{0,1,w,\overline{w}\}$ where $w=e^{2\pi i/3}$, $\overline{w}=w^2$, every complex number z is representable: $z=(a_N\ldots a_0.a_{-1}a_{-2}\ldots)_{-2}$, i.e., $z=\sum_{-\infty}^{N} a_j b^j$. (-2,D) has as set of integers $W:=\{a_N\ldots a_1a_0;a_j\in D\}$, the family of Eisenstein numbers $E=\{m+nw:m,n\in Z\}$. The integers of the system are uniquely representable. The set of fractional numbers $F:=\{0.a_{-1}a_{-2}\ldots;a_{-h}\in D\}$ coincides with a copy of the so called Eisenstein set. This set is a fractile. In this paper we study the behaviour of the ciphers in the positional representations of numbers that are not uniquely representable in the system.

I. INTRODUCTION. Let $b\in$**C**, $|b|>1$, $D=\{0,d_1,d_2,\ldots,d_k\}\subset$**C**. α is said *representable* in base b with ciphers D if there exists $\{a_j\in D: j=M,M-1,\ldots\}$ such that $\alpha=\sum_{-\infty}^{M} a_j b^j$. We write $\alpha=a_M\ldots a_0.a_{-1}a_{-2}\ldots=(e.f)_b$ and call (e) the integral part of α and (f) the fractional part of α. G denotes the set of all representable numbers. F is the set of *fractional numbers*, i.e., those numbers in G with a representation such that (e)=0. The set W of *integers* of the system is the subfamily of G with a representation such that (f)=0. A number r will be called a *rational* of the number system (b,D) if it has a finite positional representation, that is, $a_j=0$ for $j<J(r)$. U will denote the set of rationals of the system. We study the number system with base –2 and the set of ciphers D $\not\subset$ **R**, $D:=\{0,1,w,w^2\}$ where $w=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$. D\{0}={third roots of unity}, is a multiplicative group such that $1+w+w^2=0$ (the cyclotomic equation).

DEFINITION I 1. E denotes the Eisenstein's point-lattice: E ≡ [1,w] := ={m.1+n.w: m, n ∈ Z}. Let $\sigma=D\cup(-D)=\{0,\pm1,\pm w,\pm\overline{w}\}$. S :=D–D = $=\{0,\pm1,\pm w,\pm\overline{w},\pm(1-w),\pm(1-\overline{w}),\pm(w-\overline{w})\}$, S':= S\σ = $\{\pm(1-w),\pm(1-\overline{w}),\pm(w-\overline{w})\}$.•

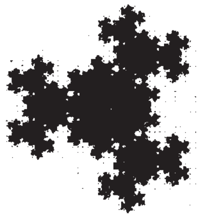
Then, S and σ are subsets of the set E of Eisenstein "integers". It is easy to verify that the numbers in S\{0} can be written in a unique way as a difference of two numbers in D. The numbers in $\sigma\setminus\{0\}$ have modulus equal to 1 and those in S' have modulus equal to $\sqrt{3}$. Besides, $\alpha\in S\Rightarrow|\alpha|\le\sqrt{3}$, $|\mathrm{Re}\,\alpha|\le 3/2$, $|\mathrm{Im}\,\alpha|\le\sqrt{3}$.

NOTATION I 1. x used as a cipher will represent the number $w^2=\overline{w}$. m(A) will denote the plane Lebesgue measure of A⊂**C** and B(z;r) the open ball of center z and radius r. •

The reader will find in [Z] or [P] a detailed proof of each statement in the following Ths. I 1-3. Any number in W, the set of integers of the number system $(-2,\{0,1,w,\overline{w}\})$, belongs to E. This follows from the identity: 1+w+x=0. Moreover,

THEOREM I 1. W=E and m+nw has a unique representation in $(-2,\{0,1,w,x\})$.•

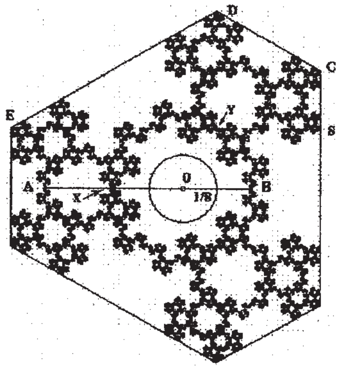
DEFINITION I 2. $F_g := g+F$ where $g \in E$.•



Thus, $F_0 \equiv F$, the fractional set of the number system $(-2,\{0,1,w,x\})$. We shall call it the *Eisenstein set*. The definition I 2 can be extended in the following way: $F_{a_M \ldots a_0 . a_{-1} a_{-2} \ldots a_{-n}} := \{x; x = a_M \ldots a_0 . a_{-1} \ldots a_{-n} \ldots\}$.

THEOREM I 2. The family $\{F_g : g \in E\}$ defines a *tessellation* in the sense that $\mathbf{R}^2 = \bigcup \{F_g : g \in E\}$ and any two different sets of the family have an intersection of plane Lebesgue measure zero.•

DEFINITION I 3. For $j \in D = \{0,1,w,x\}$ let us define $\Phi_j(z) = \frac{z}{b} + \frac{j}{b} = -\frac{z+j}{2}$.•



Then, $F = \bigcup_{i \in D} \Phi_i(F)$. Thus, the 4-rep tile F is the invariant set of the family $\{\Phi_j\}$.

THEOREM I 3. The compact connected set $F \subset B(0;1)$ is the attractor of the family of similarities $\{\Phi_j\}$ that satisfies the open set condition. It holds that $m(F_0) = \sqrt{3}/2$. Besides, if $z \in \mathbf{C}$ and $|z| \le 1/8$ then $z \in F$. The convex hull of F is a hexagon that does not tile the plane. The interior and exterior of F are composed of infinitely many open components.•

II. STATES and TYPES. Since G=C any $\eta \in \partial F$ has at least two representations. A main objective is to make clear the relations among the different representations of a given complex number. For this purpose, let $z = (0.a_1 a_2 \ldots)_b \in F$ and $e \in W \backslash \{0\}$ be such that $e.b_1 b_2 \ldots = 0.a_1 a_2 \ldots$. Then,

(II 1) $$e = \sum_1^{\infty} (a_i - b_i) b^{-i} = \sum_1^{\infty} (-1)^i \frac{a_i - b_i}{2^i}.$$

for $e \in E\backslash\{0\}$. Therefore, $|e|, |\mathrm{Im}\, e| \le \sqrt{3}$ and the bound is reached, for example when $a_i - b_i = (-1)^i (w - \overline{w})$. Besides $|\mathrm{Re}\, e| \le 3/2$, the bound reached for $a_i - b_i = (-1)^i (1-w)$. If $|e| = 1$, (II 1) has several solutions. For example, $1.1w^2\overline{10} = 0.ww\overline{01}$ and $1.\overline{10} = 0.\overline{01}$ are two solutions for e=1. However, if $|e| = \sqrt{3}$ then $e \in S'$ and (*a*) and (*b*) are determined: $e = b_1 - a_1$, $a_i = b_{i+1}$, $b_i = a_{i+1}$. Thus, we have proved the next theorem that we borrowed from [Z].

THEOREM II 1. i) The numbers in S\{0} can be written in a unique way as a difference of two ciphers.
ii) Let be $z=e.(b)=e.b_{-1}b_{-2}\ldots$, $e\in W$ and $z=0.(a)=0.a_{-1}a_{-2}\ldots$ Then $e\in$S. If $|e|=\sqrt{3}$ then $e\in$S', (a) and (b) are uniquely determined and $b_{-1}-a_{-1}=e$, $a_i=b_{i-1}$, $b_i=a_{i-1}$.
iii) $F\cap F_e\neq\varnothing\Rightarrow e\in S$ and $e\in S'\Rightarrow(e+F)\cap F$ contains only one point.•

The **state k of the p-expansion** of z, $z=\sum_{-\infty}^{L}p_jb^j$, is the number p(k) in W defined by $p(k):=\left(\sum_{k}^{L}p_jb^j\right)b^{-k}$. p(k) will also be called the **kth state of the p-representation** $p_L\cdots p_0.p_{-1}p_{-2}\cdots$. If z has also a q-expansion $z=\sum_{-\infty}^{L}q_jb^j$ then by Theorem II 1, ii), p(k)–q(k) belongs to S since $(p(k)-q(k)).p_{k-1}\ldots=0.q_{k-1}\cdots$.

LEMMA II 1. $z=p_L\cdots p_0.p_{-1}p_{-2}\cdots$ and $\zeta=q_L\cdots q_0.q_{-1}q_{-2}\cdots$ are equal if and only if $\forall k{:}\,p(k)-q(k)\in S$.•

PROOF. The if part follows from $|b^{-k}(z-\varsigma)|\leq|p(k)-q(k)|+2<4$ letting $k\to-\infty$, QED.

We have $p(k-1)=p(k)b+p_{k-1}$ and a similar expression for the q-expansion. Thus,

(II 2) $$(p(k)-q(k))b+(p_{k-1}-q_{k-1})=p(k-1)-q(k-1).$$

Since b=–2, this formula can be written as

(II 2) $$p_{k-1}-q_{k-1}=(p(k-1)-q(k-1))+2(p(k)-q(k)).$$

By the **state k of the p, q-representations** of z we mean the pair of states (p(k),q(k)) and will also refer to it as the **kth state** (p(k),q(k)). Most of the times it is not necessary to consider the kth state (p(k),q(k)) but only the difference $\Delta=p(k)-q(k)$. We call this number in S the **type** of the kth state (p(k),q(k)). That is,

DEFINITION II 1. Given a number z with two positional representations p,q, we say that the kth state, (p(k),q(k)), is of type <Δ> if $\Delta=p(k)-q(k)$.•

The formula (II.2) gives the transition from the type Δ of the state k to the type Δ_1 of the state $(k-1)$ in terms of the ciphers p_{k-1}, q_{k-1}. We shall represent it graphically as

(II 3) $$<\Delta>\xrightarrow[\binom{a}{c}]{}<\Delta_1>$$

where $\Delta_1=p(k-1)-q(k-1)$ and $a=p_{k-1}$, $c=q_{k-1}$. Thus, (II 3) stands for

(II 3) $$2\Delta+\Delta_1=a-c.$$

One readily sees that if the type Δ is not zero then neither the type Δ_1 nor $a-c$ can be zero. So, $a-c\in$S\{0} in this case. Since any number in S\{0} can be uniquely written as a difference of two numbers in D, a and c are uniquely determined. We shall construct a digraph Γ with nodes the types <Δ>, $\Delta\in$S, and arrows given by (II 3). To this end we examine the possible ciphers a and c that can occur in (II 3) in order that $\Delta_1\in$S assuming that $\Delta\in$S.

THEOREM II 2. i) If $a,c \in$ D and $\Delta = a-c \neq 0$ then $\langle a\text{-}c \rangle \xrightarrow[\binom{a}{c}]{} \langle c\text{-}a \rangle$.

ii) If $\Delta \in S'$ ($|\Delta| = \sqrt{3}$) then $\Delta_1 = -\Delta = c-a$.

iii) If $\Delta = \pm 1$ then Graph 1 and Graph -1 show all the possibilities for $\Delta_1 \in S$.

iv) $\langle 0 \rangle \xrightarrow[\binom{a}{c}]{} \langle a\text{-}c \rangle$ for any a, c belonging to D.

v) the state <0> can only be reached from <0>.•

PROOF. The proofs of all the statements follow from (II 3). For example $\Delta_1 = 0$ and $\Delta \neq 0$ implies $|a-c| \geq 2 >$ the modulus of any number in S. This contradiction proves v). If $\Delta = 1$ then $\Delta_1 = a-c-2$. So $\Delta_1 \in$ S only if $\mathrm{Re}(a-c) \geq \frac{1}{2}$. This occurs in five cases, yielding the five arrows in Graph 1. We leave the details to the reader, QED.

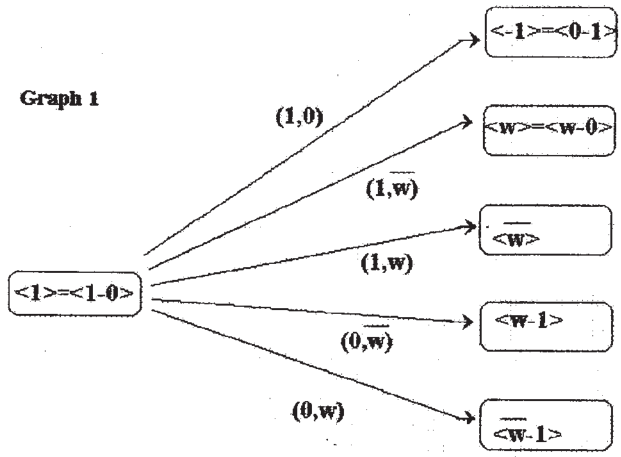
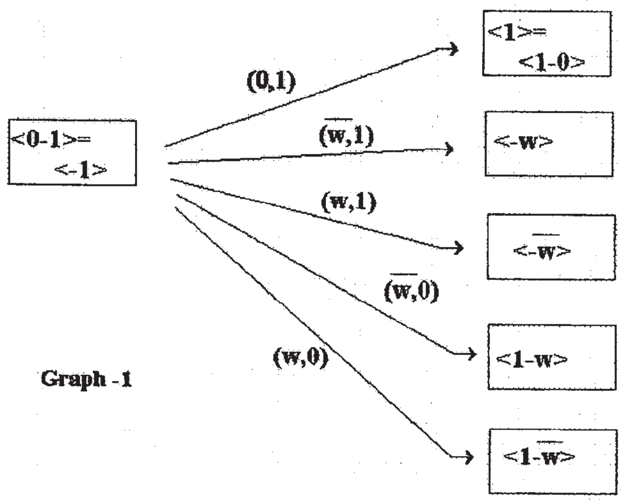


Fig. 1. Note that $(a,c) = \binom{a}{c}$.

The types reached from $\langle \pm d \rangle$, $d \in$ D\{0}, can be obtained from Graph 1 and Graph -1. In fact since D\{0} is a multiplicative group, $\langle \pm 1 \rangle \xrightarrow[(a,c)]{} \langle s \rangle$ and $\langle \pm d \rangle \xrightarrow[(da,dc)]{} \langle ds \rangle$ are equivalent for $d \in$ D\{0}. Hence multiplying Graphs 1 and -1 in Fig. 1 by w and x we obtain the Graphs $w, -w, x$ and $-x$ with the arrows starting at $w, -w, x$ and $-x$, respectively.



DEFINITION II 2. We call Γ the digraph with nodes the types in S and arrows from $\langle \Delta \rangle$ to $\langle \Delta_1 \rangle$ if $2\Delta + \Delta_1 = a-c$ with $a, c \in$ D. •

The arrows of Γ starting at <0> are shown in Fig.2 except for a loop at <0>, (Th.II 2 v)).

THEOREM II 3. The digraph Γ is obtained superposing the digraph with the arrows starting at <0> shown in Fig. 2, a loop at <0>, the three digraphs obtained from the one shown in Fig. 3 multiplied by 1, w and x and the three digraphs obtained from Fig. 5 multiplied by 1, w and x.•

ii) of Th. II 1 explains the oscillation in the semicircular arcs in Figs. 3 and 4. The dotted edges that appear in Figs. 3, 5 and 4 *bis* have arrows only in one direction. In Fig. 4 we show in full lines the edges of the graph Γ which have arrows in both directions. In Fig. 4 *bis* we have added, as dotted lines, the remaining edges that start at $\sigma\setminus\{0\}$. Therefore one obtains the **complete graph** Γ by superposing the **graph** $V\Gamma$ of Fig. 4 *bis* with the graph in Fig. 2 and a loop at <0>. Suppose z has **two** different positional representations. Then for some fixed k, the kth type is a node $<\Delta>$ of Γ different from <0>. The successive (k–1), (k–2), ...–th types of the representations are obtained following an infinite string starting at $<\Delta>$ in the digraph Γ. The ciphers (p_{k-1}, q_{k-1}), (p_{k-2}, q_{k-2}), ..., are completely determined by the arrows, (cfr. (II 2) or (II 3)).
Combining this with Lemma II 1 the next result is obtained.

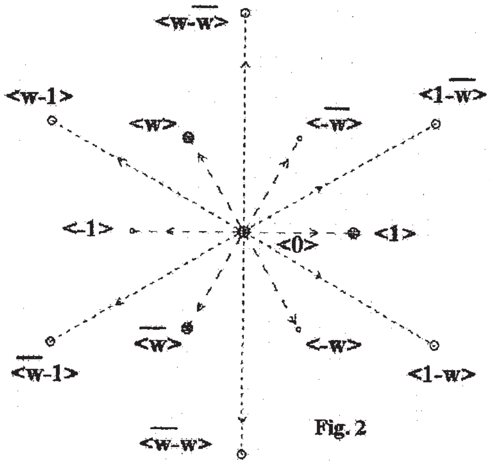


Fig. 2

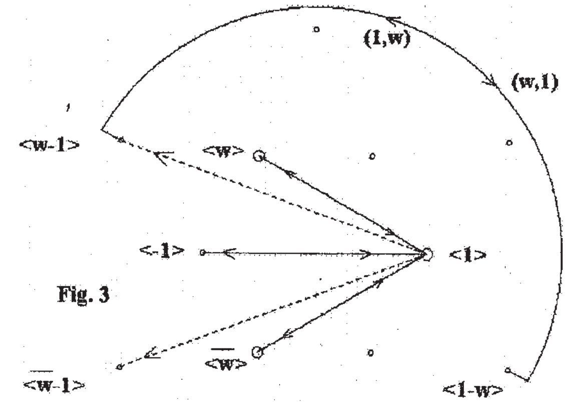
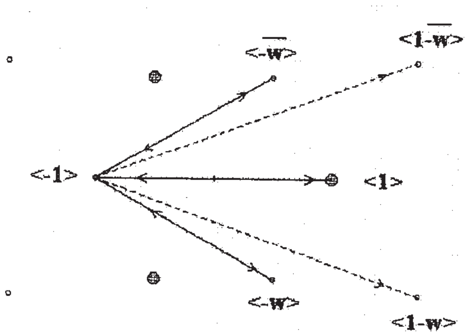


Fig. 3

THEOREM II 4. Given $k \in Z$ and a node $<\Delta> \neq <0>$ in the digraph Γ then two positional representations p,q of a number $z \in \mathbf{C}$ are obtained following an infinite string starting at $<\Delta>$ in such a way that $p(k)-q(k)=\Delta$. z is defined by any of these representations. Conversely, for a number with two positional representations p,q, the types of the successive states follow an infinite string in the graph Γ.•

Fig. 5



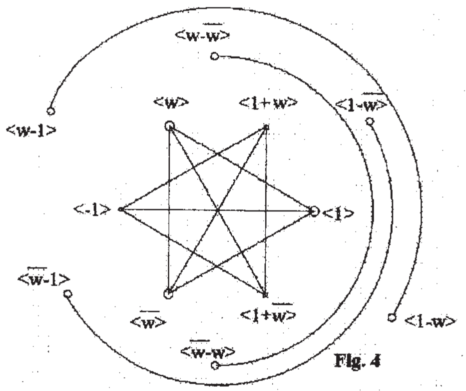


Fig. 4

We leave to the reader the remaining details of the proofs of Theorems II 3 and 4.
Next Fig. 5 reproduces Fig. 4 *bis* but with the ciphers beside the arrows that permit the passage from one state to the next one. Once a state different of <0> is reached, the states in Γ follow an infinite string in the digraph $V\Gamma$.

To say “two representations” means “at least two”. As a matter of fact, there are numbers with three representations that we shall characterize elsewhere. An example: $-1/3=0.\bar{1}=w.\overline{wx}=x.\overline{xw}$. However, the following result holds,

THEOREM II V. There is no number with four representations. •

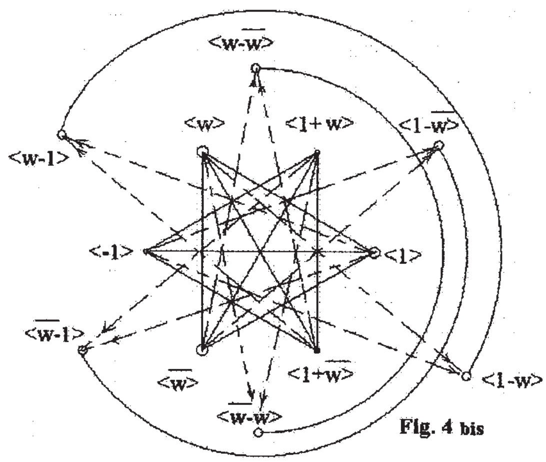


Fig. 4 bis

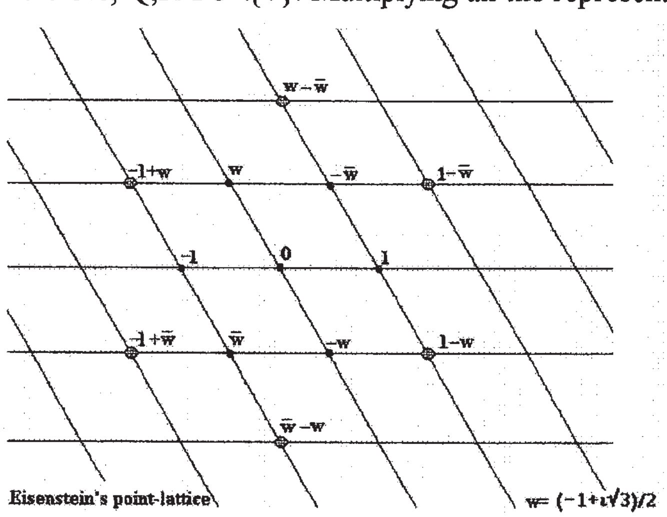
PROOF. Assume that a number y has a p-, q-, r- and z-expansion, pairwise different. This means that y = $P.p_{-1}p_{-2}\ldots$ = Q. ... = = R. ... = Z. ... . Multiplying by an adequate power of b and adding an integer we may assume without loss of generality that P, Q, R, Z are pairwise different, Z=0, and $|P|>1$. By theorem II 1 ii), P, Q, R∈S\0, so P∈S'. Then y is equal to

$$u-v.\overline{uv}=0.\overline{vu},\ u,v\in D\backslash\{0\}.$$

If Q∈S' then $y=m-n.\overline{mn}=0.\overline{nm}$, $m,n\in D$. It is easy to see that

$0.\overline{vu}=0.\overline{nm}\Rightarrow v=n,u=m$.

Therefore, $Q,R\in\sigma\backslash\{0\}$. Multiplying all the representations by a non null cipher we may assume that R=1. Then P=1−x, Q=−x or P=1−w, Q=−w. Let us consider the case where R=1, P=1−x, Q=−x. Then, y = $1-x.\overline{1x}=0.\overline{x1}=1.r_{-1}\ldots=-x.q_{-1}\ldots$ From the third equality we get that there is an infinite string starting at <1> such that the ciphers beside the arrows are $\begin{pmatrix} r_{-1}\\ x\end{pmatrix},\begin{pmatrix} r_{-2}\\ 1\end{pmatrix},\begin{pmatrix} r_{-3}\\ x\end{pmatrix},\ldots$. But digraph VΓ shows that such a string does not exist. Similarly, if



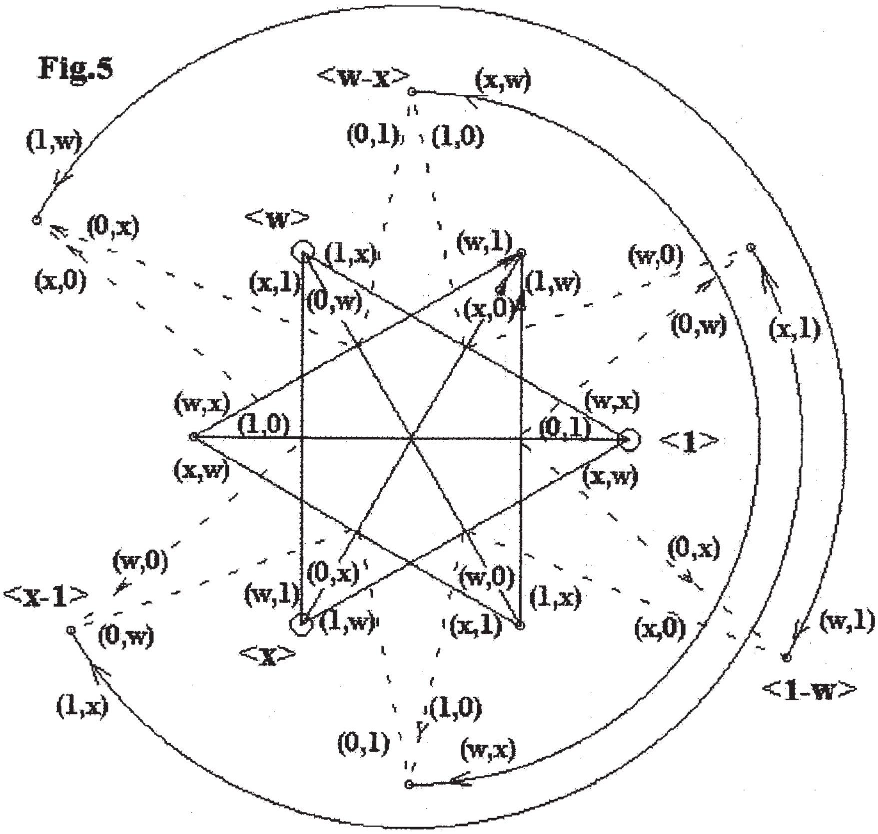
Eisenstein's point-lattice

P=1−w, Q=−w then $y=1-w.\overline{1w}$=$0.\overline{w1}$=$1.r_{-1}\ldots=-w.q_{-1}\ldots$. This is again impossible since there is no infinite string in VΓ starting at <1> such that the ciphers beside the arrows are $\begin{pmatrix} r_{-1}\\ w\end{pmatrix},\begin{pmatrix} r_{-2}\\ 1\end{pmatrix},\begin{pmatrix} r_{-3}\\ w\end{pmatrix},\ldots$ .QED.

Final remarks. W=E is also a consequence of the fact that the family of periodic points in (−2,D) is equal to {0}, (cf. [K], §2 and 9). On the other hand W=E implies that G=**C**, (cf. [KS] or [IKR], Th. 2). The behaviour of the ciphers corresponding to numbers that have three positional representations in the number system (−2,D) will be studied in [Q].

Digraph V Γ

Fig.5



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INMABB, (UNS-CONICET), Av. Alem 1253, (8000) Bahía Blanca, Argentina.