A note about $U-$ operators on $(n+1)-$bounded Wajsberg Algebras

Marina Beatriz Lattanzi

Abstract

The $UW_{n+1}-$algebras have been defined in [10] as algebras $\langle A, \rightarrow, \sim, \forall, 1\rangle$, where $\langle A, \rightarrow, \sim, 1\rangle$ is a $(n+1)-$bounded Wajsberg algebra (see [13]) and $\forall$ is a $U-$operator on A, i.e., a unary operation defined on A which satisfies the identities: $\forall x \rightarrow x = 1$, $\forall(x \rightarrow y) \rightarrow (\forall x \rightarrow \forall y) = 1$, $\forall(\forall x \rightarrow \forall y) = \forall x \rightarrow \forall y$.

Moisil's modal operators $\sigma_1, \sigma_2, \ldots, \sigma_n$ that can be defined on a Wajsberg algebra have been studied in [14].

In this note conditions are given on the $\forall$, $\rightarrow$ and $\sim$ operations, so that in a $UW_{n+1}-$algebra $\forall\sigma_j = \sigma_j\forall$, $j = 1, 2 \ldots, n$ is verified.

It is also shown that if $n \geq 1$ is even or n is odd and ≤ 7, then the Moisil's modal operators can be defined on any $(n+1)-$valued Wajsberg algebra on the basis of the unary operations of power and product by a natural number.

The results on Wajsberg algebras and Łukasiewicz algebras can be found in [13, 8] and in [4, 5, 2], respectively. The definitions and properties needed to understand the rest of the text will be described throughout this note.

Let A_1 and A_2 be two algebras with the same universe A; algebra A_1 is said to be a *reduct* of A_2 if every fundamental operation of A_1 is a term in the language of A_2.

Let us remember that (see [13]) an algebra $\mathcal{A} = \langle A, \rightarrow, \sim, 1\rangle$ of type $(2, 1, 0)$ is a Wajsberg algebra (or $W-$algebra) if the following identities are satisfied:

(W1) $1 \rightarrow x = x$,

(W2) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,

(W3) $(x \to y) \to y = (y \to x) \to x$,

(W4) $(\sim y \to \sim x) \to (x \to y) = 1$.

We will denote with $\mathbf{W}$ the variety of $W-$algebras.

We will indicate with C_{n+1} the $W-$algebra with universe $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$ and the operations defined by:

$x \to y := min\ \{1, 1 - x + y\}$ and $\sim x := 1 - x$.

On $A \in \mathbf{W}$ the following binary operations can be defined:

(D1) $x \vee y = (x \to y) \to y$,

(D2) $x \wedge y = \sim (\sim x \vee \sim y)$,

(D3) $x * y = \sim (x \to \sim y)$,

(D4) $x + y = \sim y \to x$.

For any $A \in \mathbf{W}$ the following properties are verified (see [8, 13]):

(W5) $x \to x = 1$,

(W6) $(A, \vee, \wedge, \sim, 0, 1)$ is a Kleene algebra where $0 = \sim 1$ and $x \leq y$ if and only if $x \to y = 1$,

(W7) $x \to 0 = \sim x$,

(W8) $x \leq y$, $z \leq t$ implies $x + z \leq y + t$,

(W9) $x \leq y$, $z \leq t$ implies $x * z \leq y * t$.

For each $A \in \mathbf{W}$, $x \in A$ and n integer, $n \geq 0$, the power x^n and the product nx can be defined as follows:

(D5) $x^n = \begin{cases} 1 & \text{if } n = 0 \\ x^{n-1} * x & \text{if } n \geq 1 \end{cases}$

(D6) $n \cdot x = \begin{cases} 0 & \text{if } n = 0 \\ (n-1)\cdot x + x & \text{if } n \geq 1 \end{cases}$

Then in **W** the following property is verified

(W10) $n \cdot (\sim x) = \sim (x^n)$.

The classes of $(n+1)-$bounded and $(n+1)-$valued Wajsberg algebras are the subvarieties of **W** generated by chains of length less or equal to $n+1$ and by the chain of length $n+1$, respectively.

Definition 1 [10] *Let $\langle A, \rightarrow, \sim, 1\rangle$ be a $W-$algebra. A $U-$operator on A is an application $\forall : A \longrightarrow A$ which satisfies the following identities:*

(U1) $\forall x \rightarrow x = 1$,

(U2) $\forall(x \rightarrow y) \rightarrow (\forall x \rightarrow \forall y) = 1$,

(U3) $\forall(\forall x \rightarrow \forall y) = \forall x \rightarrow \forall y$.

An algebra $\langle A, \rightarrow, \sim, \forall, 1\rangle$ is a $UW-$algebra if the reduct $\langle A, \rightarrow, \sim, 1\rangle$ is a Wajsberg algebra and $\forall$ is a $U-$operator on A.
An algebra $\langle A, \rightarrow, \sim, \forall, 1\rangle$ is a $(n+1)-$bounded $UW-$algebra (or $UW_{n+1}-$algebra) if the reduct $\langle A, \rightarrow, \sim, 1\rangle$ is a $(n+1)-$bounded Wajsberg algebra and $\forall$ is a $U-$operator on A.

Lemma *Let A be a $UW-$algebra. For all $x, y \in A$ and every positive integer m, the following properties are verified:*

(U4) $\forall 0 = 0$,

(U5) $\forall 1 = 1$,

(U6) $x \leq y$ *implies* $\forall x \leq \forall y$,

(U7) $\forall \sim \forall x = \sim \forall x$,

(U8) $\forall(\forall x + \forall y) = \forall x + \forall y$, (U9) $\forall x + \forall y \leq \forall(x+y)$,

(U10) $m \cdot \forall x \leq \forall(m \cdot x)$,

(U11) $\forall(\forall x * \forall y) = \forall x * \forall y$,

(U12) $\forall x * \forall y \leq \forall(x * y)$,

(U13) $(\forall x)^m \leq \forall(x^m)$.

Proof

(U4) Is an immediate result of (W6) and (U1).

(U5) $\forall 1 = \forall(\forall x \to \forall x) = \forall x \to \forall x = 1$. [W5,U3]

(U6) $1 = \forall 1 = \forall(x \to y) \leq \forall x \to \forall y$. [U5,W6,U2]

(U7) $\forall \sim \forall x = \forall(\forall x \to 0) = \forall(\forall x \to \forall 0) = \forall x \to 0 = \sim \forall x$. [W7,U4,U3]

(U8) Results of (D4), (U7) and (U3).

(U9) Results of (W6), (U1), (W8), (U6) and (U8).

(U10) Results of (U9) and (D6) applying induction on m.

(U11) Results of (D3), (U7) and (U3).

(U12) Results of (W6), (U1), (W9), (U6) and (U11).

(U13) Results of (U12) and (D5) applying induction on m. ■

An algebra $\langle A, \vee, \wedge, \sim, \sigma_1^{n+1}, \ldots, \sigma_n^{n+1}, 0, 1\rangle$, $n \geq 1$, is a $(n+1)-$ valued Łukasiewicz algebra (or $L_{n+1}-$algebra) if the reduct $\langle A, \vee, \wedge, \sim, 0, 1\rangle$ is a De Morgan algebra and $\sigma_1^{n+1}, \sigma_2^{n+1}, \ldots, \sigma_n^{n+1}$ are unary operators, called Moisil's modal operators, which satisfy the identities (see [5]):

(L1) $\sigma_j^{n+1}(x \vee y) = \sigma_j^{n+1}x \vee \sigma_j^{n+1}y$, $1 \leq j \leq n$,

(L2) $\sigma_j^{n+1}x \vee \sigma_{j+1}^{n+1}x = \sigma_{j+1}^{n+1}x$, $1 \leq j \leq n-1$, (L3) $\sigma_j^{n+1}x \vee \sim \sigma_j^{n+1}x = 1,\ 1 \leq j \leq n,$

(L4) $\sigma_i^{n+1} \sim x = \sim \sigma_{n-i+1}^{n+1}x,\ 1 \leq i \leq n,$

(L5) $\sigma_i^{n+1}\sigma_j^{n+1}x = \sigma_j^{n+1}x,\ 1 \leq i,\ j \leq n,$

(L6) $x \vee \sigma_n^{n+1}x = \sigma_n^{n+1}x,$

(L7) $(x \wedge \sim \sigma_j^{n+1}x \wedge \sigma_{j+1}^{n+1}y) \vee y = y,\ 1 \leq j \leq n-1.$

Example 1 Let $n = 2$. It is easy to see that $\langle C_3, \rightarrow, \sim, \forall, 1\rangle$ is a $UW_{2+1}-$algebra, where $\forall 0 = \forall\frac{1}{2} = 0$ and $\forall 1 = 1$. If we consider the structure of three-valued Łukasiewicz algebra of C_3, we have that $\sigma_2^3\left(\forall\frac{1}{2}\right) = 0$ whereas $\forall\sigma_2^3\left(\frac{1}{2}\right) = 1$. Therefore, if we have $\exists x = \sim \forall \sim x$, it results that the class of $UW_{2+1}-$algebras is not equivalent to the class of monadic three-valued Łukasiewicz algebras introduced by L. Monteiro in [12].

In [14, Theorem 15, pg.16] it is stated that every $(n+1)-$bounded Wajsberg algebra has a $L_{n+1}-$algebra reduct. So we have analysed, in the first place, how to define such operators on a $(n+1)-$bounded Wajsberg algebra with the purpose of determining the conditions on $\forall$, $\rightarrow$ y $\sim$ operations so that a $U-$operator commutes with Moisil's modal operators.

A family of unary terms $p(n,m)(x)$, $n, m \geq 0$ of language of the $W-$algebras is defined by induction in [14] as follows:

(P1) for any $n \geq 0$, $p(n,0)(x) = 1$,

(P2) for any $m \geq 1$, $p(0,m)(x) = 0$,

(P3) if $p(r,m)(x)$ is defined for $r \leq n$ and $m \geq 0$, then

$$p(n+1,m)(x) = \sim ((\sim p(n,m)(x) \rightarrow x) \rightarrow \sim p(n,m-1)(x)).$$

The following properties hold (see [14]):

(P4) $p(n,1)(x) = n \cdot x$, for every $n \geq 1$, (P5) $p(n, n+m)(x) = 0$, for every $n \geq 0$ and $m \geq 1$,

(P6) $p(n, n)(x) = x^n$, for every $n \geq 0$.

Besides, in [14, pg. 13] it is stated that if $\langle A, \to, \sim, 1\rangle$ is a $(n+1)-$bounded Wajsberg algebra, then $\langle A, \vee, \wedge, \sim, \sigma_1^{n+1}, \ldots, \sigma_n^{n+1}, 0, 1\rangle$, is a $(n+1)-$valued Łukasiewicz algebra, where for every $1 \leq i \leq n$, the unary operators σ_i^{n+1} are the interpretation of terms $p(n, n+1-i)$ on A, i.e., $(\sigma_i^{n+1})^A = (p(n, n+1-i))^A$.

As follows in Example 2 we shall show that the previous affirmation is false.

Example 2 Let $A = C_{2+1} \times C_{3+1}$; it is clear that A is a $(3+1)-$bounded but not a $(3+1)-$valued Wajsberg algebra. For $n = 3$, the following polynomials are obtained:

$$\begin{aligned} \sigma_1^{3+1}(x) &= p(3,3)(x) = x^3, \\ \sigma_2^{3+1}(x) &= p(3,2)(x) = (x + x^2) * (2 \cdot x), \\ \sigma_3^{3+1}(x) &= p(3,1)(x) = 3 \cdot x. \end{aligned}$$

For $x = (\frac{1}{2}, 0) \in A$ we have that $\sigma_2^{3+1}((\frac{1}{2}, 0)) = (\frac{1}{2}, 0)$, then $\sigma_2^{3+1}((\frac{1}{2}, 0)) \vee \sim \sigma_2^{3+1}((\frac{1}{2}, 0)) = (\frac{1}{2}, 1)$ and the property (L3) is not verified.

Remark We cannot always define the operators σ_i^{n+1}, $1 \leq i \leq n$ on a $(n+1)-$bounded $W-$algebra, so that it can be endowed with a structure of $(n+1)-$valued Łukasiewicz algebra. Indeed, every $(3+1)-$valued Łukasiewicz finite algebra is direct product of chains with 2 and/or 4 elements (see [3, pg. 120]) then its cardinal must be a power of 2. Therefore the $(3+1)-$bounded Wajsberg algebra $C_{2+1} \times C_{3+1}$ is not a $(3+1)-$valued Łukasiewicz algebra.

In [14, Theorem 15, pg. 16] it is stated that if A is a $W-$algebra then the following conditions are equivalent (part (ii) has been omitted):

(i) *A admits a $(n+1)-$valued Łukasiewicz algebra reduct, for some $n > 1$,*

(iii) *A is $n+1$ bounded for some $n < \omega$.*

where n is the same in (i) *and* (iii).

But the remark above contradicts the previous affirmation.

On the other hand, the following result is verified:

Theorem 1 *If A is a $(n+1)-$ bounded $W-$algebra, then A admits a $(m+1)-$valued Łukasiewicz algebra reduct, where m is the least common multiple of the integers r, $1 \leq r \leq n$. That is to say, if $\langle A, \rightarrow, \sim, 1\rangle$ is a $(n+1)-$ bounded Wajsberg algebra, then $\langle A, \vee, \wedge, \sim, \sigma_1^{m+1}, \ldots, \sigma_m^{m+1}, 0, 1\rangle$ is a $(m+1)-$valued Łukasiewicz algebra where $0 = \sim 1$, m is the least common multiple of the integers r, $1 \leq r \leq n$ and for every $a, b \in A$:*

$$x \vee y = (x \rightarrow y) \rightarrow y,$$

$$x \wedge y = \sim (\sim x \vee \sim y),$$

$$\sigma_i^{m+1}(x) = p(m, m+1-i)(x), \ 1 \leq i \leq m.$$

Proof It is simple to check that the interpretation of the unary terms $p(m, m+1-i)(x)$, $1 \leq i \leq m$, on C_{r+1} for any $1 \leq r \leq n$ satisfies the equations (L1) to (L7). ∎

¿From this analysis we can conclude that in order to study the relation between modal operators and the $U-$operator, it is necessary to restrict ourselves to $(n+1)-$valued Wajsberg algebras.

Theorem 2 *Let $\langle A, \rightarrow, \sim, 1\rangle$ be a $(n+1)-$valued Wajsberg algebra and $\forall : A \longrightarrow A$ a $U-$operator. If for every $x, y \in A$ the two following properties are satisfied:*

(U14) $\forall(x+y) \leq \forall x + \forall y$,

(U15) $\forall(x * y) \leq \forall x * \forall y$,

then, it is verified that:

(U16) $\forall \sigma_i^{n+1}(x) = \sigma_i^{n+1}(\forall x)$, *for all i, $1 \leq i \leq n$, whereas for every $x \in A$*

(1) $\sigma_i^{n+1}(x) = p(n, n+1-i)(x)$.

Proof From (1), it is enough to prove

(2) $p(n, n+1-i)(\forall x) = \forall p(n, n+1-i)(x)$, for all $x \in A$ and $1 \leq i \leq n$.

We will prove (2) by induction on n. For $n = 1$, it is a direct consequence of (P4). Suppose now that (2) is true for every $r < n$, i.e., it is verified that:

(3) $p(r, r+1-i)(\forall x) = \forall p(r, r+1-i)(x)$, for every $x \in A$, $r < n$ and $1 \leq i \leq r$,

and we shall prove that it holds for n.

From (P3), (D3) and (D4) it follows

(4) $p(n, n+1-i)(\forall x) = (\forall x + p(n-1, n+1-i)(\forall x)) * p(n-1, n-i)(\forall x)$.

For $r = n-1$, (5) results from (3):

(5) $$\begin{aligned} p(n-1, n-i)(\forall x) &= p(r, r+1-i)(\forall x) \\ &= \forall p(r, r+1-i)(x) \\ &= \forall p(n-1, n-i)(x). \end{aligned}$$

On the other hand, considering $r = n-1$ and $j = i-1$ it is possible to write

(6) $p(n-1, n+1-i)(\forall x) = p(r, r+2-i)(\forall x) = \forall p(r, r+1-j)(\forall x)$.

If $i > 1$, $1 \leq j < i \leq r$ then, considering (3) the following identities are verified:

(7) $\forall p(r, r+1-j)(\forall x) = \forall p(r, r+1-j)(x) = \forall p(n-1, n+1-i)(x)$.

For $i = 1$, (8) results from (P5):

(8) $p(n-1, n)(\forall x) = 0 = \forall p(n-1, n)(x)$,

therefore, from (6), (7) and (8) we obtain

(9) $p(n-1, n+1-i)(\forall x) = \forall p(n-1, n+1-i)(x)$.

Equality (2) is a consequence of (4), (5), (9), (U9), (U14), (U12), (U15), (P3), (D3) and (D4). Indeed:

$$\begin{aligned}
p(n, n+1-i)(\forall x) &= (\forall x + \forall p(n-1, n+1-i)(x)) * \forall p(n-1, n-i)(x) \\
&= \forall(x + p(n-1, n+1-i)(x)) * \forall p(n-1, n-i)(x) \\
&= \forall((x + p(n-1, n+1-i)(x)) * p(n-1, n-i)(x)) \\
&= \forall p(n, n+1-i)(x). \qquad \blacksquare
\end{aligned}$$

Theorem 2 takes us to the following definition:

Definition 2 *Let $\forall$ be a $U-$operator on a Wajsberg algebra A. We shall say that $\forall$ is a universal quantifier if the two following properties are satisfied:*

(i) $\forall(\sim x \rightarrow y) \rightarrow (\sim \forall x \rightarrow \forall y) = 1$,

(ii) $\forall \sim (x \rightarrow \sim y) \rightarrow (\sim (\forall x \rightarrow \sim \forall y)) = 1$.

It is easy to verify that if $\forall$ is a universal quantifier on a Wajsberg algebra A, then for every $x, y \in A$ and every positive integer m, the following properties are satisfied:

(U17) $\forall(m \cdot x) \leq m \cdot \forall x$,

(U18) $\forall(x^m) \leq (\forall x)^m$.

The monadic $MV_{n+1}-$algebras are defined as pairs $\langle A, \exists \rangle$, where A is a $MV_{n+1}-$algebra (i.e., A is polynomially equivalent to a $(n+1)-$valued Wajsberg algebra) and $\exists : A \longrightarrow A$ is an application that verifies the following axioms (see [15, 16, 9]):

(M0) $\exists 0 = 0$,

(M1) $x \leq \exists x$,

(M2) $\exists(x * \exists y) = \exists x * \exists y$,

(M3) $\exists(x + \exists y) = \exists x + \exists y$,

(M4) $\exists(x * x) = \exists x * \exists x$, (M5) $\exists(x+x) = \exists x + \exists x$.

It is proved in [9] that the monadic MV_{n+1}–algebras are polynomially equivalent to the $(n+1)$–valued Łukasiewicz algebras for $n = 2$ and $n = 3$. Particulary, it is shown that every monadic MV_{3+1}–algebra has a structure of monadic L_{3+1}–algebra (see [1]) with respect to

$$\begin{aligned} \sigma_1^{3+1}(x) &= x^3, \\ \sigma_2^{3+1}(x) &= (2\cdot x)^3 = 3\cdot x^2, \\ \sigma_3^{3+1}(x) &= 3\cdot x. \end{aligned}$$

When Mosil's modal operators on a $(n+1)$–valued Wajsberg algebra are defined as in Theorem 2, i.e., $\sigma_k^{n+1}(x) = p(n, n+1-k)(x)$, for each $1 \leq k \leq n$, it results

$$\begin{aligned} \sigma_1^{3+1}(x) &= x^3, \\ \sigma_2^{3+1}(x) &= (x + x^2) * (2\cdot x), \\ \sigma_3^{3+1}(x) &= 3\cdot x. \end{aligned}$$

If A is a $(3+1)$–valued Wajsberg algebra, then $(x+x^2)*(2\cdot x) = 3\cdot x^2$, for every $x \in A$.

It is well known that the chain C_{n+1} with the lattice operations defined on the basis of their natural order, the negation defined by $\sim x := 1 - x$ and the operators

$$\sigma_k^{n+1}\left(\tfrac{j}{n}\right) := \begin{cases} 0 & \text{if } k+j \leq n \\ 1 & \text{otherwise} \end{cases}, \text{ for every } k, j,\ 1 \leq k \leq n \text{ and } 0 \leq j \leq n,$$

is a L_{n+1}–algebra.

On the other hand, if we consider the W–algebra C_{n+1}, from (P4) and (P6) we have that $\sigma_1^{n+1}(x) = p(n,n)(x) = x^n$ and $\sigma_n^{n+1}(x) = p(n,1)(x) = n\cdot x$, for every $x \in C_{n+1}$ and $n \geq 1$.

In the Theorem 3, which is developed below, we shall prove that every Moisil's modal operator on the chain of length $n+1$, and therefore on any $(n+1)$–valued Wajsberg algebra, can be defined in terms of the power and product unary operations by a natural number (see (D5) and (D6)).

Theorem 3 *Let n be an integer, $n \geq 3$, $C_{n+1} \in \mathbf{W}$ and, for every k, j, $1 < k < n$ and $0 \leq j \leq n$, $\sigma_k^{n+1}\left(\frac{j}{n}\right) = \begin{cases} 0 & \text{if } k+j \leq n \\ 1 & \text{otherwise} \end{cases}$. Then, for any $x \in C_{n+1}$ it is verified that:*

(a) *If n is even:*

(i) $k \leq \frac{n}{2}$ *implies* $\sigma_k^{n+1}(x) = n \cdot x^{\frac{n}{2}-k+2}$,

(ii) $k > \frac{n}{2}$ *implies* $\sigma_k^{n+1}(x) = ((k+1-\frac{n}{2}) \cdot x)^n$.

(b) *If n is odd and $n \leq 7$:*

(i) $k < \frac{n+1}{2}$ *implies* $\sigma_k^{n+1}(x) = n \cdot x^{\frac{n+1}{2}-k+2}$,

(ii) $k \geq \frac{n+1}{2}$ *implies* $\sigma_k^{n+1}(x) = ((k+1-\frac{n-1}{2}) \cdot x)^n$.

Proof

(a) Let n be even.

(i) Let $2 \leq k \leq \frac{n}{2}$. If $\sigma_k^{n+1}\left(\frac{j}{n}\right) = 0$, then $k+j \leq n$, hence resulting:

(1) $(\frac{n}{2} - k + 2)(n-j) \geq (\frac{n}{2} - k + 2)k$.

On the other hand, conditions (2), (3) and (4) below are equivalent:

(2) $(\frac{n}{2} - k + 2)k \geq n$,

(3) $-k^2 + (\frac{n}{2} + 2)k - n \geq 0$,

(4) $2 \leq k \leq \frac{n}{2}$.

Since (4) is a hypothesis, from (1) and (2) it follows that $(\frac{n}{2}-k+2)(n-j) \geq n$, and due to (W10) this condition is equivalent to $\left(\frac{j}{n}\right)^{\frac{n}{2}-k+2} = 0$, therefore $n \cdot \left(\frac{j}{n}\right)^{\frac{n}{2}-k+2} = 0$.

Reciprocally, suppose now $n \cdot \left(\frac{j}{n}\right)^{\frac{n}{2}-k+2} = 0$; from this and (W10) it follows $(\frac{n}{2} - k + 2)(n-j) \geq n$, which is equivalent to

(5) $j + k \leq \dfrac{kn - 2n - n^2 + 2k^2 - 4k}{2k - 4 - n}$.

On the other hand, $\frac{kn-2n-n^2+2k^2-4k}{2k-4-n} \leq n$ if and only if $2 \leq k \leq \frac{n}{2}$, which is true by hypothesis. Therefore, from (5) it follows $j + k \leq n$.

(ii) If $k > \frac{n}{2}$ then $n+1-k \leq \frac{n}{2}$; therefore from (L4), (i) and (W10) we have that $\sigma_k^{n+1}(x) = \sim \sigma_{n+1-k}^{n+1}(\sim x) = ((k+1-\frac{n}{2}) \cdot x)^n$.

(b) For $n=3$, $n=5$ and $n=7$ it is easy to do the verification. ■

As consequence of this result, we have that for n even or n odd and $n \leq 7$, Theorem 2 is true if hypotheses (U14) and (U15) are replaced by (U17) and (U18).

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Marina Beatriz Lattanzi,
Facultad de Ciencias Exactas y Naturales,
Universidad Nacional de La Pampa,
Av. Uruguay 151 - 6300 Santa Rosa - Argentina.
e-mail: mblatt@exactas.unlpam.edu.ar