A note on the Distributives Hilbert Algebras

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Abstract

In this paper the distributive Hilbert algebras are presented as algebras $\langle A, \to, \wedge, \vee, 0, 1\rangle$ of type $(2, 2, 2, 0, 0)$, where the reduct $\langle A, \to, 1\rangle$ is a Hilbert algebra, the reduct $\langle A, \wedge, \vee, 0, 1\rangle$ is a distributive lattice with first element 0 and last element 1, and the identities $x \wedge (x \to y) = x \wedge y$, $(x \to (y \wedge z)) \to ((x \wedge z) \to (x \wedge y)) = 1$, $(x \vee y) \to z = (x \to z) \wedge (y \to z)$ are verified. It is proved that the class of distributive Hilbert algebras is a variety of algebras that strictly include the variety of Heyting algebras. Besides, it is verified that the congruences of a distributive Hilbert algebra can be obtained by means of absorbent deductive systems of A, that is to say those subset of A which fulfil the properties: (D1) $1 \in D$, (D2) if $x, x \to y \in D$ then $y \in D$, (D3) if $d \in D$ then $x \to (x \wedge d) \in D$ for all $x \in A$. Finally, a family of subdirectly irreducible distributive Hilbert algebras is determined.

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1 Introduction and Preliminaries

In his paper on Hertz algebras of fractions, D. Busneag first considered the Hilbert algebras and gave the name Hertz algebra to every Hilbert algebra whose underlying ordered structure is a meet semi-lattice. Later, basing his work on an article written by H. Porta [11] and another produced by A. Monteiro [8], he stated that the Hertz algebras coincide with the variety of algebras that W. Nemitz called implicative semi-lattices [10]. With respect to what has been said above, we can make de following assertions:

(i) In [8] the Hertz algebras do not appear.

(ii) In [11], H. Porta used A. Diego's definition [5] to name the Hilbert algebras and, like D. Busneag, gave the name of Hertz algebras to the Hilbert algebras such that their underlying ordered set is a meet semi-lattice. He also asserted that the Hertz algebras coincide with the implicative semi-lattices.

(iii) In [12], H. Porta called the Nemitz implicative semi-lattices Hertz algebras.

(iv) The assertion about the equivalences betwen the two definitions of the Hertz algebras above mentioned is not true, as we shall show in Example 1.1.

(v) The fact mentioned in (iii) suggests the study of at least three classes of algebras that may be of interest on account of their relation to the intuitionistic propositional calculus. They are:

  (a) The Hilbert algebras with underlying structure of join semi-lattice.

  (b) The Hilbert algebras with underlying structure of meet semi-lattice.

  (c) The Hilbert algebras with underlying structure of lattice, and in particular those algebras whose ordered structure is a distributive lattice with a first and a last element.

We have obtained some results about particular cases of the class indicated in (a) (ver [6]). In addition, we are studying the classes indicated in (c), some results about which we shall communicate in this work. The layout of this work is as follows. In Section 1 we recall some basic definitions. In Section 2 we introduce the variety of distributive Hilbert algebras (or $dH-$algebras, to abbreviate), which are the Hilbert algebras with underlying structure of distributive lattices, but in which we have considered as primitive operations the lattice operations yielded by the underlying order. Then, the $dH-$algebras constitute a generalization of the Heyting algebras. In Section 3 we obtain the congruences of any $dH-$algebra and, in Section 4, we determine two classes of subdirectly irreducible $dH-$algebras.

The results on the Hilbert algebras may be consulted in [1, 2, 4, 5, 9, 13] and those on the distributive lattices in [3]. Throughout this paper we shall be including the definitions and properties necessary for the understanding of the remaing part.

Definition 1.1 ([5]) *A Hilbert algebra (or $H-$algebra) is an algebra* $\langle A, \rightarrow, 1\rangle$ *of the type* $(2,0)$ *which satisfies these identities:*

(H1) $x \rightarrow x = 1$,

(H2) $1 \rightarrow x = x$,

(H3) $x \to (y \to z) = (x \to y) \to (x \to z)$,

(H4) $(x \to y) \to ((y \to x) \to x) = (x \to y) \to ((y \to x) \to y)$.

We shall denote by $\boldsymbol{H}$ the variety of H−algebras. Then we state the following lemma:

Lemma 1.1 *For each $A \in \boldsymbol{H}$ the following properties are verified:*

(H5) $x \to 1 = 1$,

(H6) *if $x \to y = y \to x = 1$, then $x = y$,*

(H7) *the relation $\leq$ defined by $x \leq y$ if and only if $x \to y = 1$ is a partial order on A,*

(H8) $y \leq x \to y$,

(H9) $x \to (y \to z) = y \to (x \to z)$,

(H10) *if $x \leq y \to z$, then $y \leq x \to z$,*

(H11) *if $x \leq y$, then $z \to x \leq z \to y$,*

(H12) *if $x \leq y$, then $y \to z \leq x \to z$.*

In [4] and [11] a definition of the Hertz algebra equivalent to the following definition is used:

Definition 1.2 *A Hertz algebra (or implicative semi-lattice, according to Nemitz [10]) is an algebra $\langle A, \to, \wedge, 1\rangle$ of the type $(2, 2, 0)$ which satisfies these identities:*

(He1) $x \to x = 1$,

(He2) $(x \to y) \wedge y = y$,

(He3) $x \wedge (x \to y) = x \wedge y$,

(He4) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

Definition 1.3 ([3, 7]) *A Heyting algebra is an algebra $\langle A, \to, \wedge, \vee, 0, 1\rangle$ of the type $(2, 2, 2, 0, 0)$ such that the reduct $\langle A, \to, \wedge, 1\rangle$ is a Hertz algebra and the following identities are satisfied:*

(He5) $x \wedge 0 = 0$,

(He6) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$.

In a subsequent step, Example 1.1, we shall show that the identities (He1) to (He4) do no characterize the Hilbert algebras in which the order given by (H7) is of meet semi-lattice.

Example 1.1 *It is well known (see [4, 5]) that, if in an ordered set $(A, \leq)$ with a last element 1 the operation $\rightarrow$ is defined by the prescription*

$$x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise,} \end{cases}$$

then $\langle A, \rightarrow, 1\rangle$ is a Hilbert algebra. Then, considering the set $A = \{0, a, b, 1\}$ with the order indicated in the following Hasse diagram,
it follows that the Hilbert algebra $\langle A, \rightarrow, 1\rangle$, whose implication is given by this table

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	b	1
b	0	a	1	1
1	0	a	b	1

is such that every pair of elements has infimum, but it does not verify property (He4) *since* $a \rightarrow (a \wedge b) = a \rightarrow 0 = 0, \quad (a \rightarrow a) \wedge (a \rightarrow b) = 1 \wedge b = b$.

2 Distributive Hilbert Algebras

In this section we shall consider the Hilbert algebras with a natural ordered structure of distributive lattice with a first and a last element.

Let $A \in \boldsymbol{H}$ and $a, b \in A$, we shall denote by $a \wedge b$ and $a \vee b$ the infimum and the supremum of a and b respectively.

Lemma 2.1 *For every $A \in \boldsymbol{H}$ the following properties are verified:*

(H13) *if $a, b \in A$ are such that there exists $a \wedge b$, then $a = a \wedge b$ if and only if $a \rightarrow b = 1$,*

(H14) *if $a, b \in A$ are such that there exists $a \vee b$, then for every $c \in A$ there exists $(a \rightarrow c) \wedge (b \rightarrow c)$ and the following identity is verified:*
$(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$

(H15) *If* $a, b, c \in A$ *are such that there exist* $b \wedge c$ *and* $(a \rightarrow c) \wedge (b \rightarrow c)$, *then the condition* $a \rightarrow (b \wedge c) \leq (a \rightarrow c) \wedge (a \rightarrow b)$ *is verified,*

(H16) *If* $a, b \in A$ *are such that there exists* $a \wedge b$ *then, for every* $c \in A$ *the condition* $a \rightarrow (b \rightarrow c) \leq (a \wedge b) \rightarrow c$ *is verified,*

(H17) *If* $a, b \in A$ *are such that exist* $a \wedge b$ *and* $a \wedge (a \rightarrow b)$, *then the identity* $a \wedge (a \rightarrow b) = a \wedge b$ *is verified.*

Proof.

(H13) It derives directly from (H7).

(H14) (1) $a \leq a \vee b$,

(2) $b \leq a \vee b$,

(3) $(a \vee b) \rightarrow c \leq a \rightarrow c$, [(1), (H12)]

(4) $(a \vee b) \rightarrow c \leq b \rightarrow c$, [(2), (H12)]

(5) $(a \vee b) \rightarrow c \leq (a \rightarrow c) \wedge (b \rightarrow c)$. [(3), (4)]

(6) Let $z \in A$ such that $z \leq a \rightarrow c$ and $z \leq b \rightarrow c$. Then

(7) $a \leq z \rightarrow c$, [(6), (H10)]

(8) $b \leq z \rightarrow c$, [(6), (H10)]

(9) $a \vee b \leq z \rightarrow c$, [(7), (8)]

(10) $z \leq (a \vee b) \rightarrow c$, [(9),(H10)]

(11) $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$. [(5), (6), (10)]

(H15) (1) $b \wedge c \leq c$,

(2) $b \wedge c \leq b$,

(3) $a \rightarrow (b \wedge c) \leq (a \rightarrow c)$ [(1), (H11)]

(4) $a \rightarrow (b \wedge c) \leq (a \rightarrow b)$ [(2), (H11)]

(5) $a \rightarrow (b \wedge c) \leq (a \rightarrow c) \wedge (a \rightarrow b)$. [(3), (4)]

(H16) (1) $a \wedge b \leq b$,

(2) $b \to c \leq (a \wedge b) \to c,$ [(1), (H12)]

(3) $a \to (b \to c) \leq a \to ((a \wedge b) \to c),$ [(2), (H11)]

(4) $a \to (b \to c) \leq (a \wedge b) \to (a \to c),$ [(3), (H10)]

(5) $(a \wedge b) \to (a \to c) = ((a \wedge b) \to a) \to ((a \wedge b) \to c)$ [(H3)]

$= 1 \to ((a \wedge b) \to c)$ [(1), (H7)]

$= (a \wedge b) \to c,$ [(H2)]

(6) $a \to (b \to c) \leq (a \wedge b) \to c.$ [(4), (5), (H7)]

(H17) (1) $a \to ((a \to b) \to b) \leq (a \wedge (a \to b)) \to b,$ [(H16)]

(2) $1 \leq (a \wedge (a \to b)) \to b,$ [(1), (H9), (H1)]

(3) $a \wedge (a \to b) \leq b,$ [(2), (H7)]

(4) $a \wedge (a \to b) \leq a,$

(5) $a \wedge (a \to b) \leq a \wedge b,$ [(3), (4)]

(6) $b \leq a \to b,$ [(H8)]

(7) $a \wedge b \leq a \wedge (a \to b),$ [(6)]

(8) $a \wedge (a \to b) = a \wedge b.$ [(5), (7)]

■

Definition 2.1 *Let $A \in \boldsymbol{H}$. We shall say that A is distributive if its ordered structure $(A, \leq)$ is a distributive lattice set with a first element, where $\leq$ is the relation given by* (H7).

In Theorem 2.1 we shall show the equivalence between Definition 2.1 and Definition 2.2, which is given below.

Definition 2.2 *A dH-algebra is an algebra $\langle A, \to, \vee, \wedge, 0, 1\rangle$ of the type $(2, 2, 2, 0, 0)$ which satisfies the following properties:*

(i) *the reduct $\langle A, \to, 1\rangle$ is a Hilbert algebra ,*

(ii) *the reduct $\langle A, \vee, \wedge, 0, 1\rangle$ is a distributive lattice with first element 0 and last element 1,*

(iii) *the following identities are verified:*

(dH1) $x \wedge (x \to y) = x \wedge y$,

(dH2) $(x \to (y \wedge z)) \to ((x \to z) \wedge (x \to y)) = 1$,

(dH3) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

We shall denote the variety of dH−algebras by $\boldsymbol{dH}$.

Remark 2.1 *Taking account Example* 1.1, *we can state that in* $\boldsymbol{dH}$ *the identity* (He4) *does not hold. In the same example it can be shown that the following identity is not true either.*

(He7) $x \to (y \to z) = (x \wedge y) \to z$.

Nevertheless, it can be proved that in $\boldsymbol{dH}$ *weaker properties are valid:*

(dH4) $x \to (y \wedge z) \leq (x \to y) \wedge (x \to z)$,

(dH5) $x \to (y \to z) \leq (x \wedge y) \to z$.

Lemma 2.2 *For each* $A \in \boldsymbol{dH}$ *the following properties are verified:*

(dH6) $x \wedge 1 = x$,

(dH7) *for each* $x, y \in A$, *the property* (a) $x \to y = 1$ *is equivalent to the property* (b) $x = x \wedge y$.

Proof.

(dH6) It is a direct consequence of (dH1) and (H1).

(dH7) (a) ⇒ (b):

(1) $x \to y = 1$, [(a)]

(2) $x = x \wedge 1$ [(dH6)]

$= x \wedge (x \to y)$, [(1)]

$= x \wedge y$. [(dH1)]

(b) ⇒ (a):

(1) $x = x \wedge y$, [(b)]

(2) $1 = x \to (x \wedge y)$, [(1), (H1)]

(3) $(x \to (x \wedge y)) \to ((x \to y) \wedge (x \to x)) = 1$, [(dH2)]

(4) $x \to y = 1$, [(3), (2), (H2), (H1), (dH6)]

■

Is is clear that from Lemma 2.2 it follows that in a $dH-$algebra the order determined by the structure of the distributive lattice coincides with the order determined by the structure of an $H-$algebra. On the other hand, from definitions 2.1, 2.2 and Lemmas 2.1, 2.2 the following theorem results:

Theorem 2.1 *Let $\langle A, \to, \vee, \wedge, 0, 1\rangle$ be an algebra of the type $(2,2,2,0,0)$. Then, the following conditions are equivalent:*

(i) $\langle A, \to, \vee, \wedge, 0, 1\rangle \in \boldsymbol{dH}$,

(ii) *$\langle A, \to, 1\rangle$ is an $H-$algebra with first element 0 and for each $\{x, y\} \subseteq A$ the following identities are satisfied: $\inf\{x,y\} = x \wedge y$, $\sup\{x,y\} = x \vee y$.*

3 Congruences

Now we shall consider the way to determine the congruences of a $dH-$algebra. Let's recall that:

Definition 3.1 ([5]) *Let $A \in \boldsymbol{H}$. $D \subseteq A$ is a deductive system if the following conditions are verified:*

(D1) $1 \in D$,

(D2) *$x, x \to y \in D$ imply $y \in D$.*

We shall denote the family of the deductive systems of A by $\mathcal{D}(A)$.

Remark 3.1 *It is well known [5] that:*

(i) *If $A \in \boldsymbol{H}$, $D \in \mathcal{D}(A)$ and $R(D) = \{(x,y) \in A^2 : x \to y \in D, y \to x \in D\}$, then $R(D) \in Con_{\boldsymbol{H}}(A)$, where $Con_{\boldsymbol{H}}(A)$ is the set of all the $H-$congruences on A. If $R \in Con_{\boldsymbol{H}}(A)$ and x_R denotes the equivalence class of x, $x \in A$, then $1_{R(D)} = D$.*

(ii) *If $R \in Con_{\boldsymbol{H}}(A)$, then there exists a unique $D \in \mathcal{D}(A)$ such that $R = R(D)$ and $D = 1_R$.*

By means of Example 3.1 given below we shall show that the deductive systems do not determine the congruences of a $dH-$algebra.

Example 3.1 Let's consider the $dH-$algebra whose Hasse diagram and operation $\rightarrow$ are those indicated below :

Then, $D = \{b, c, d, 1\}$ is a deductive system of A. But the $H-$congruence $\theta(D)$ is not a $dH-$congruence because $(b, c), (c, c) \in \theta(D)$ but $(b \wedge c, c \wedge c) = (a, c) \notin \theta(D)$.

The previous result has led us to determine which are the deductive systems D such that $R(D)$ are $dH-$congruences.

Definition 3.2 *Let $A \in \mathbf{dH}$. We shall say that the deductive system $D \subseteq A$ is absorbent if it verifies this property :*

(D3) *If $x \in D$, then $z \rightarrow (z \wedge x) \in D$ for every $z \in A$.*

We shall denote by $\mathcal{D}_{ab}(A)$ the set of the absorbent deductive systems of A.

Lemma 3.1, which is given below, can be easily proved.

Lemma 3.1 *Let $A \in \mathbf{dH}$ and $D \in \mathcal{D}_{ab}(A)$, then D is a filter of A.*

Remark 3.2 *Let A be the algebra introduced in Example 3.1. Then $D = F(b) = \{b, d, 1\}$ is a filter which is not an absorbent deductive system. In effect, $b \in D$, and $c \rightarrow (c \wedge b) = c \rightarrow a \notin D$.*

Next we shall determine the $dH-$congruences of A and establish an isomorphism betwen the lattices $Con_{\mathbf{dH}}(A)$ and $\mathcal{D}_{ab}(A)$.

Lemma 3.2 *Let $A \in \mathbf{dH}$ and $D \in \mathcal{D}_{ab}(A)$. Then the following properties are verified:*

(i) $R(D) \in Con_{\mathbf{dH}}(A)$.

(ii) $1_{R(D)} = D$.

Proof. We shall only prove that, if $(x, y) \in R(D)$, then $(x \wedge z, y \wedge z) \in R(D)$ for every $z \in A$.

(1) Let $(x, y) \in R(D)$ and $z \in A$,

then these properties are verified:

(2) $x \rightarrow y \in D$,

(3) $y \rightarrow x \in D$,

(4) $(z \wedge x) \rightarrow ((z \wedge x) \wedge (x \rightarrow y)) \in D$ [(2), (D3)]

(5) $(z \wedge x) \rightarrow (z \wedge x \wedge y) \in D$ [(4), (dH1)]

(6) $z \wedge y \wedge x \leq z \wedge y$,

(7) $(z \wedge x) \rightarrow (z \wedge x \wedge y) \leq (z \wedge x) \rightarrow (z \wedge y)$, [(6),(H11)]

(9) $(z \wedge x) \rightarrow (z \wedge y) \in D$. [(5), (7)]

From (3), we can prove:

(10) $(z \wedge y) \rightarrow (z \wedge x) \in D$,

then

(11) $(z \wedge x, z \wedge y) \in R(D)$. [(9), (10)]

■

Lemma 3.3 *Let $A \in \boldsymbol{dH}$, $R \in Con_{\boldsymbol{dH}}(A)$ y $D = 1_R$. Then the following are verified:*

(i) $D \in \mathcal{D}_{ab}(A)$.

(ii) $R(D) = R$.

(iii) *The lattices $Con_{\boldsymbol{dH}}(A)$ and $\mathcal{D}_{ab}(A)$ are isomorphic if we consider the correspondences $R \longmapsto 1_R$ and $D \longmapsto R(D)$, one being the inverse of the other.*

As a direct consequence of Lemmas 3.2 and 3.3 we can formulate the following theorem:

Theorem 3.1 $Con_{\boldsymbol{dH}}(A) = \{R(D) : D \in \mathcal{D}_{ab}(A)\}$.

4 Subdirectly Irreducible Algebras

In this section we shall show some results on the subdirectly irreducible $dH-$algebras .

Lemma 4.1 *Let $A \in \mathbf{dH}$ and $D = F(a)$ be a principal filter of A. Then, the following are equivalent:*

(i) $F(a) \in \mathcal{D}_{ab}(A)$,

(ii) *for all $z \in A \setminus D$, $a \rightarrow (z \wedge a) = a \rightarrow z$ is verified.*

Proof.

(i) $\Rightarrow$ (ii):

(1) Let $z \in A \setminus D$. [hip.]

Then

(2) $z \rightarrow (z \wedge a) \in D$, [(1), (D3)]

(3) $a \leq z \rightarrow (z \wedge a)$, [(2), hip.]

(4) $1 = a \rightarrow (z \rightarrow (z \wedge a))$ [(3), (H7)]

$= (a \rightarrow z) \rightarrow (a \rightarrow (z \wedge a))$, [(H3)]

(5) $a \rightarrow z \leq a \rightarrow (z \wedge a)$, [(4), (H7)]

(6) $z \wedge a \leq z$,

(7) $a \rightarrow (z \wedge a) \leq a \rightarrow z$, [(6), (H11)]

(8) $a \rightarrow (z \wedge a) = a \rightarrow z$. [(5), (7), (H7)]

(ii) $\Rightarrow$ (i):

(1) $D = F(a)$ [hip.]

(2) $a \rightarrow (z \wedge a) = a \rightarrow z$ for all $z \in A \setminus D$, [hip.]

(3) $1 = (a \rightarrow z) \rightarrow (a \rightarrow (z \wedge a))$

$= a \to (z \to (z \wedge a)),$ [(2), (H3)]

(4) $a \leq z \to (z \wedge a),$ [(3), (H7)]

(5) $z \to (z \wedge a) \in D.$ [(4), (1)]

(6) Let $b \in D$.

Then

(7) $a \leq b$ [(1), (6)]

(8) $z \wedge a \leq z \wedge b$, for all $z \in A$, [(7)]

(9) $z \to (z \wedge a) \leq z \to (z \wedge b)$, for all $z \in A$, [(8), (H11)]

(10) $z \to (z \wedge b) \in D$, for all $z \in A$, [(9), (1)]

(11) D verifies $(D3)$. [(10)]

■

Theorem 4.1 *Let $A \in \mathbf{dH}$. If A has a penultimate element, then A is subdirectly irreducible.*

Proof. By hipothesis there exists $p \in A$, $p < 1$ such that

(1) $x \leq p$ for all $x \in A$, $x \neq 1$.

Let $F(p) = \{p, 1\}$ the principal filter of A generated by p. Then, if we choose $z \in A$ such that

(2) $z \in A \setminus F(p)$,

from (1) and (2), the identity :

(3) $p \to (z \wedge p) = p \to z$ is verified.

¿From (3) and Lemma 4.1 we obtain as a result that $F(p)$ is an absorbent deductive system of A. Clearly, $F(p)$ is the only atom of $(\mathcal{D}_{ab}(A), \subseteq)$. Then A is subdirectly irreducible. ■

Example 4.1 *Let $\langle A, \wedge, \vee, \to, 0, 1\rangle$ be the $dH-$algebra determined by the ordered set indicated in Example* 1.1. *Then, it can be easily verified that A is subdirectly irreducible and does not have a penultimate element.*

Definition 4.1 *Let A be an ordered set and $D \subseteq A$, $D \neq A$, $D \neq \emptyset$. We shall say that D is a node if, for every $a \in A \setminus D$ and $d \in D$, the relation $a < d$ is verified.*

Theorem 4.2 *Let $\langle A, \vee, \wedge, 0, 1\rangle$ be a bounded distributive lattice and D a node of A with four elements, whose order is that of a Boolean algebra. If the implication $\rightarrow$ given by the lattice order is like that indicated in Example* 1.1, *it follows that $\langle A, \rightarrow, \vee, \wedge, 0, 1\rangle$ is a subdirectly irreducible $dH-$algebra.*

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