Current research in many-valued logic: a short survey

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Introduction

The infinite-valued propositional calculus of Łukasiewicz and its associated algebras, Chang's MV-algebras, today find applications to the treatment of uncertain information, and also have deep relationship with various mathematical areas, such as desingularization of toric varieties, abelian lattice-ordered groups, and the C*-algebras of quantum spin systems. The increased relevance of this subject is witnessed by the growing number of projects (notably, the European COST Action 15 on Many-valued logic for computer science applications) special issues of journals in logic and computer science (including Studia Logica, Journal of Logic, Language and Information, Mathware and Soft Computing, Journal of Applied Nonclassical Logic) along with many textbooks, (e.g., the expanded version of Gottwald's book [32], Hájek's book [35], the monographs [17], [18], as well as the handbook chapter by Panti [59]) and several Ph. D. theses (notably, [29], [38]). The two papers [15], [16] give a self-contained introduction to the completeness theorem and to the categorical equivalence between MV-algebras and abelian lattice-ordered groups with strong unit. See [14] for a compact technical survey of MV-algebras and their neighbours.

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Introduced in the late fifties by Chang (see [11]), after some years of quiescency, today MV-algebras are thoroughly investigated by several groups of researchers. An *MV-algebra* is a set equipped with an associative-commutative operation $\oplus$, with a neutral element 0, and with an involutive operation $\neg$ such that $x \oplus \neg 0 = \neg 0$ and, characteristically,

(1) $\quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$

These equations express some properties of the real unit interval $[0,1]$ equipped with negation $\neg x = 1 - x$ and truncated addition $x \oplus y = \min(1, x + y)$. For instance, direct inspection shows that the left hand member of equation (1) coincides with the maximum of x and y, whenever $x, y \in [0,1]$, and hence the equation expresses the commutativity of the max operation. The basic theorem on MV-algebras, *Chang's completeness theorem*, [9], [10] states that a formula follows from the Łukasiewicz axioms iff it is valid in $[0,1]$. This is a generalization of the well known fact that the two element set $\{0,1\}$ equipped with the operations of involution and max generates the variety of boolean algebras. The literature contains many distinct proofs, notably [13], [58] and [65]—the first published proof of the completeness of the infinite-valued Łukasiewicz calculus. Nevertheless, each of these proofs requires substantial background prerequisites from such disparate areas as first order model theory, linear inequalities, free abelian lattice-ordered groups, toric varieties. To the best of our knowledge the first elementary proof is given in [15].

A second significant fact is that *MV-algebras are categorically equivalent to abelian lattice-ordered groups (for short, ℓ-groups) with a distinguished strong unit* [42]. The functor Γ realizing this equivalence takes any such a group G with strong unit u and equips its unit interval $[0, u]$ with the involution $u - x$ and with truncated addition $u \wedge (x + y)$, generalizing the transformation of the additive real line $\mathbf{R}$ into the MV-algebra $[0,1]$. Further, for any morphism $\psi\colon (G, u) \to (H, v)$ of ℓ-groups with strong unit, the functor Γ restricts ψ to the unit interval $[0, u] \subseteq G$.

A third major result is *McNaughton's representation theorem* stating that, up to logical equivalence, formulas in n variables coincide with the totality of continuous $[0,1]$-valued piecewise linear functions f over $[0,1]^n$, where each piece of f is a linear polynomial with integer coefficients. This is a generalization of the well known fact that, up to logical equivalence, formulas in two-valued logic are boolean functions. McNaughton proved this theorem in 1951 using a nonconstructive reductio ad absurdum argument [36]. In the 1994 paper [48] one can find a direct geometric proof, stressing the role of desingularization, an important concept in the theory of toric varieties. The main ideas of this new proof are sketched in the next section.

Normal forms and toric desingularizations

In many-valued logic, as well as in boolean logic, disjunctive normal forms (DNF) play a fundamental role, both for automated deduction, and for a deeper understanding of the algebra of formulas [46], [55]. Let $\psi = \psi(x_1, \ldots, x_n)$ be a formula in the infinite-valued calculus.

Since by Chang's completeness theorem, the variety of MV-algebras is generated by the unit interval $[0,1]$, a routine construction shows that, as an element of the free n-generated MV-algebra, the equivalence class of ψ can be identified with a piecewise linear (continuous) function p, each piece having integer coefficients. This is the easy part of McNaughton's theorem. To get the converse direction, assuming the function $p: [0,1]^n \to [0,1]$ to be piecewise linear with integer coefficients, one proceeds in several steps as follows.

FIRST STEP. We can partition the n-cube $[0,1]^n$ into a complex $\mathcal{C}$ of convex polyhedra with rational vertices such that, over any such polyhedron, the function p is linear. Generalizing the familiar construction for 2-dimensional polyhedra (where one adds a maximal set of diagonals) we can subdivide $\mathcal{C}$ into a simplicial complex $\mathcal{S}$ without adding new vertices. Using Minkowski's convex body theorem, we can further subdivide $\mathcal{S}$ into a *unimodular* simplicial complex $\mathcal{T}$: in other words, for each n-simplex Δ in $\mathcal{T}$ writing in homogeneous integer coordinates the vertices of Δ, we obtain an $(n+1) \times (n+1)$ integer valued matrix M_Δ whose determinant is equal to ± 1.

SECOND STEP. One then constructs the family $H(\mathcal{T})$ of Schauder hats of $\mathcal{T}$. By the *Schauder hat* of $\mathcal{T}$ at vertex w we mean the uniquely determined piecewise linear continuous function h such that $h(w) = 1/d$ (where d is the least common denominator of the homogeneous integer coordinates of w), $h(v) = 0$ for every vertex v of $\mathcal{T}$ other than w, and h coincides with a linear function h_Δ over each n-simplex Δ of $\mathcal{T}$. As an equivalent reformulation of the unimodularity property, the coefficients of the linear polynomial representing h_Δ are integers. An easy lemma, first proved by McNaughton, and then simplified by Rose and Rosser, yields a formula ϕ_Δ representing h_Δ over Δ. A routine min-max argument now shows that all Schauder hat functions are representable by formulas, and therefore p can be expressed as a disjunction (sum) of Schauder hat formulas. We naturally regard the formulas representing the functions in $H(\mathcal{T})$ as the basic constituents of a DNF reduction of p.

THIRD STEP. To grasp the connection with toric varieties, upon writing in homogeneous integer coordinates the vertices of each simplex Δ in $\mathcal{T}$, we obtain a complex Ψ of simplicial cones, also known as a *fan*. As one more equivalent reformulation of the unimodularity of $\mathcal{T}$ we can say that Ψ is "regular". By the well known vocabulary of toric geometry [24, p. 329-330], (smooth) toric varieties correspond to (regular) fans, and hence to (unimodular) simplicial complexes of the above kind. It follows [50] that desingularizing a toric variety amounts to subdividing a complex into a unimodular simplicial complex, precisely as is done to compute DNF reductions of McNaughton functions. Our desingularization algorithms, arising from DNF reduction algorithms in infinite-valued logic, yield tight estimates of the Euler characteristic of desingularizations of low-dimensional toric varieties [4]. Further, Panti [58] gives a geometric proof of Chang's completeness theorem using the De Concini-Procesi theorem on elimination of points of indeterminacy [24, p. 252].

Product, Probability, Partitions

Several results on "generalized conjunction connectives" (alias, T-norms) [8], [35], [64] show that a significant portion of the expressive power needed for applications in fuzzy control and many-valued probability theory would be provided by a logic incorporating a *product* connective jointly with Łukasiewicz disjunction and negation. In general, Łukasiewicz's conjunction $x \odot y = \neg(\neg x \oplus \neg y)$ does not yield such connective. Several people are actively involved in this line of research (see, e.g., [41] [23] and [61]). A different approach is taken in [54], using semisimple tensor products—the latter perhaps being the bare minimum needed for if-then-else approximations of continuous (control) functions.

Introduced in [49] and [51], *states* are the MV-algebraic generalization of *finitely additive* probability measures on boolean algebras. On the other hand, countably infinitary operations are needed for the development of MV-algebraic measure theory. Accordingly, σ-complete MV-algebras and σ-additive states are systematically used in the book by Riečan and Neubrunn [64]. As shown by Riečan and his School, many important results of classical probability theory based on σ-complete boolean algebras and σ-fields of sets have interesting MV-algebraic generalizations. One more example can be found in [54].

While the theory of σ-additive MV-algebraic states is fairly well understood, random variables (alias, observables) still lack a definitive systematization in the context of MV-algebras. A number of technical problems, also involving product and infinite distributive laws are posed by the theory of continuous functions of several (joint) MV-algebraic observables. (See [62] and [63] for interesting positive results). A useful tool for understanding such observables is given by the MV-algebraic generalization of the notion of boolean partition [51], [52]. An *MV-partition* in A is a multiset of linearly independent elements of A whose sum equals 1. This definition makes sense, by referring to the underlying $\mathbf{Z}$-module structure of the unique ℓ-group G with unit 1 given by $\Gamma(G, 1) = A$. The joint refinability of any two MV-algebraic partitions on an MV-algebra A depends on the *ultrasimplicial* property of its associated ℓ-group G, in the sense that every finite set in G^+ is contained in the monoid generated by some *basis* $B \subseteq G^+$, i.e., a set B of positive elements that are independent in the $\mathbf{Z}$-module G. When G is countable, an equivalent reformulation of this property is that G is the limit of an ascending sequence of free abelian groups of finite rank with product ordering, and one-one monotone homomorphisms. After some partial results of Elliott, Handelman and others (see [44], [43], [53], [57] and references therein), recently Marra [37] has proved that every abelian ℓ-group is ultrasimplicial.

MV-algebras and C*-algebraic infinite quantum systems

See [31] for background on AF (approximately finite-dimensional) C*-algebras, the operator algebras currently used for the mathematical description of infinite quantum spin systems [47]. The same techniques giving DNF reductions for sets of formulas in the infinite-valued calculus, together with Elliott's classification theory [21], yield a one-one correspondence between countable MV algebras and AF C*-algebras whose (Grothendieck) K_0-group is lattice-ordered. Specifically, using the above mentioned categorical equivalence Γ between ℓ-groups with strong unit and MV-algebras, the composite functor ΓK_0 yields the required correspondence.

To see how an AF C*-algebra is approximable by its finite-dimensional subalgebras one can work in an MV-algebraic (equivalently, in an ℓ-group-theoretical) framework, using the Schauder hat machinery developed for DNF reductions. The properties of the composite functor ΓK_0 ensure that the free MV-algebra F over countably many generators corresponds to a "free" ℓ-group G with strong unit, and also to a "free" AF C*-algebra U, inheriting from F all universal properties. In particular, every AF C*-algebra with lattice-ordered K_0-group is a quotient of U [42].

As one more application of the composite functor ΓK_0, every AF C*-algebra E with lattice-ordered $K_0(E)$ can be presented as a sequence of strings of symbols—the Lindenbaum algebra of some theory Θ in the infinite-valued calculus. From Θ one can uniquely recover E. The complexity of the decision problem of Θ measures the complexity of E. While, as proved in [43], the tautology problem for the infinite-valued calculus in co-NP complete, many AF C*-algebras in the literature have polynomial time complexity.

As a final application, MV-algebraic states were introduced in [49], [51] as an averaging tool for the truth-value of formulas in Łukasiewicz logic. They have already been mentioned in connection with MV-algebraic probability theory. In the functional analytical context, states turn out to be the logical counterpart of AF C*-algebraic *tracial states.* States are also used in [26] for a probabilistic approach to Ulam game.

Ulam game and coding with feedback

In Ulam game [68, p. 78] with L errors/lies two players, named Questioner and Respondent, fix a search space $S_n = \{0, 1, \ldots, 2^n - 1\}$. The Respondent chooses a secret number $x \in S_n$, and the Questioner must find x by asking the smallest possible number of yes-no questions, knowing that *the Respondent can lie at most L times.* As explained in [45], the following logical counterpart of Ulam's game provides a natural interpretation of formulas in infinite-valued logic: *An equation holds for arbitrary n and L, and for all possible conjunctions of answers* *if and only if it is derivable in the infinite-valued calculus of Łukasiewicz.* This follows from Chang's completeness theorem.

An equivalent description of Ulam's game arises when the Respondent is not aware of giving erroneous answers—that is, he is not lying, but his answers may be distorted during transmission. There is no substantial difference between such kind of Respondent and an artificial satellite that is transmitting bits b_j (the yes-no answers), some of which may be received as $1 - b_j$ instead of b_j, as a result of distortion. In the present cooperative model, where decoding strategies are also known to the satellite, the effect of a question is the same as sending back to the satellite the actually received bit via a noiseless channel. Thus Ulam game becomes an interesting chapter of Berlekamp's theory of communication with feedback [7].

In the *nonadaptive* case of Ulam game all questions are asked at the outset, before receiving any answer. A moment's reflection shows that in this case, finding an optimal searching strategy amounts to finding an optimal L-error correcting code—a very difficult task in general. At the opposite extreme, let us consider the case when questions are *adaptively* asked using the information of the previous answers. Then for small L, optimal search strategies were discovered already in the eighties (see e.g., [60], [20]). Later on, these results were extended to all L in [66].

A promising new line of research is given by the investigation of searching strategies with *minimum adaptiveness.* Joint work of Cicalese and the present author [12] shows that, by asking all questions in *two* nonadaptive batches, one at the very beginning of the game, and the other almost at the end, the questioner can still guess the unknown number with as many questions as in the fully adaptive case.

References

[1] AGUZZOLI S. (1998) The complexity of McNaughton functions of one variable, *Advances in Applied Mathematics*, **21**, 58-77.

[2] AGUZZOLI, S. (1998) A note on the representation of McNaughton functions by basic literals. *Soft Computing*, **2**, p. 111-115.

[3] AGUZZOLI, S., CIABATTONI, A. (1999) Finiteness in infinite-valued Łukasiewicz logic. *Journal of Logic, Language and Information.* Special issue on Logics of Uncertainty (D. Mundici, Ed.). To appear.

[4] AGUZZOLI, S., MUNDICI, D. (1994) An algorithmic desingularization of three-dimensional toric varieties. *Tohoku Mathematical Journal*, **46**, p. 557-572.

[5] AIGNER, M. (1996) Searching with Lies. *Journal of Combinatorial Theory*, Series A, **74**, p. 43-56.

[6] BAAZ, M., HÁJEK, P., KRAIJCEK, J., SVEIDA, D. (1998) Embedding logics into product logic. *Studia Logica.* Special issue on Many-valued logics, (D. Mundici, Ed.,) **61**, p. 35-47.

[7] BERLEKAMP, E.R. (1968) Block coding for the binary symmetric channel with noiseless, delayless feedback. In: **Error-correcting Codes**. (Mann, H. B., Ed.) Wiley, New York, p. 330-335.

[8] BUTNARIU, D., KLEMENT, E.P.K. (1995) **Triangular norm-based measures and games with fuzzy coalitions**. Kluwer, Dordrecht.

[9] CHANG, C.C. (1958) Algebraic analysis of many-valued logics. *Transactions of the American Mathematical Society*, **88**, p. 467-490.

[10] CHANG, C.C. (1959) A new proof of the completeness of the Łukasiewicz axioms. *Transactions of the American Mathematical Society*, **93**, p. 74-90.

[11] CHANG C.C. (1998) The writing of the MV-algebras, *Studia Logica*, special issue on Many-valued Logics, (D. Mundici, Ed.,) **61**, 3-6.

[12] CICALESE, F., MUNDICI, D. (199?) Optimal binary search with two unreliable tests and minimum adaptiveness, In: **Proceeding of the European Symposium on Algorithms, ESA'99**. *Lecture Notes in Computer Science*. To appear.

[13] CIGNOLI, R. (1993) Free lattice-ordered abelian groups and varieties of MV-algebras. In: **IX Latin American Symposium on Mathematical Logic**, Bahía Blanca, p. 113-118. (*Notas de Lógica Matemática*, v. 38, part I)

[14] CIGNOLI, R., MUNDICI, D. (1998) An invitation to Chang's MV-algebras. In: **Advances in Algebra and Model-Theory**, (M.Droste and R.Göbel, Eds.,) Gordon and Breach Publishing Group, Readings UK, p. 171-197.

[15] CIGNOLI, R., MUNDICI, D. (1997) An elementary proof of Chang's completeness theorem for the infinite-valued calculus of Łukasiewicz. *Studia Logica*, **58**, p. 79-97.

[16] CIGNOLI, R., MUNDICI, D. (1998a) An elementary presentation of the equivalence between MV-algebras and ℓ-groups with strong unit. *Studia Logica*, special issue on Many-valued logics, (D. Mundici, Ed.,) **61**, p. 49-64.

[17] CIGNOLI, R. D'OTTAVIANO I.M.L., MUNDICI, D. (1995) **Algebras of Łukasiewicz Logics**, (in Portuguese). Second Edition. Editions CLE, State University of Campinas, Campinas, S.P., Brazil.

[18] CIGNOLI, R. D'OTTAVIANO I.M.L., MUNDICI, D. (199?) **Algebraic Foundations of Many-valued Reasoning**, Kluwer, Dordrecht. To appear.

[19] CIGNOLI, R., ELLIOTT, G.A., MUNDICI, D. (1993) Reconstructing C^*-algebras from their Murray von Neumann orders. *Advances in Mathematics*, **101**, p. 166-179.

[20] CZYZOWICZ, J., MUNDICI, D., PELC, A. (1989) Ulam's searching game with lies. *Journal of Combinatorial Theory*, Series A, **52**, p. 62-76.

[21] ELLIOTT, G.A. (1976) On the classification of inductive limits of sequences of semisimple finite-dimensional algebras. *Journal of Algebra*, **38**, p. 29-44.

[22] ELLIOTT, G.A., MUNDICI, D. (1993) A characterization of lattice-ordered abelian groups. *Mathematische Zeitschrift*, **213**, p. 179-185.

[23] ESTEVA, F., GODO, L., MONTAGNA, F. (199?) The LP and LP1/2 logics: two complete fuzzy systems joining Łukasiewicz and product logic. *Archive for Math. Logic*. To appear.

[24] EWALD G. (1996) **Combinatorial convexity and algebraic geometry.** Springer-Verlag, Berlin, Heidelberg, New York. (*Graduate Texts in Mathematics, vol. 168*)

[25] GEORGESCU G., LEUSTEAN I., (1998) Probabilities on Łukasiewicz-Moisil algebras, *International Journal of Approximate Reasoning*, **18** 201-215.

[26] GERLA B. (1999) A probabilistic approach to Ulam games. *Theoretical Computer Science*. To appear.

[27] GISPERT, J., TORRENS, A. (1998) Quasivarieties generated by simple MV-algebras. *Studia Logica*, special issue on Many-valued logics, (D. Mundici, Ed.,) **61**, p. 79-99.

[28] GISPERT, J., MUNDICI, D. TORRENS, A. (1999) Ultraproducts of **Z** with an application to many-valued logics. *Journal of Algebra*. To appear.

[29] GLUSCHANKOF, D. (1989) *Objetos inyectivos en álgebra de la lógica*. Universidad de Buenos Aires. (Ph.D. Thesis)

[30] GLUSCHANKOF, D. (1992) Prime deductive systems and injective objects in the algebras of Łukasiewicz infinite-valued calculi. *Algebra Universalis*, **29**, p. 354-377.

[31] GOODEARL, K. (1982) **Notes on Real and Complex C*-algebras.** Birkhäuser, Boston. (*Shiva Mathematics Series, vol. 5*)

[32] GOTTWALD, S. (1989) **Mehrwertige Logik**. Akademie-Verlag, Berlin. Expanded edition in English, in preparation.

[33] HÄHNLE, R. (1993) **Automated Deduction in multiple-valued Logics**. Clarendon Press, Oxford.

[34] HÄHNLE, R., ESCALADA-IMAZ, G. (1997) Deduction in many-valued logics: a Survey. *Mathware and Soft Computing*, **4**, p.69-97.

[35] HÁJEK, P. (1998) **Metamathematics of fuzzy logic**. Kluwer, Dordrecht. (*Trends in Logic, Studia Logica Library*)

[36] MCNAUGHTON, R. (1951) A theorem about infinite-valued sentential logic. *The Journal of Symbolic Logic*, **16**, p. 1-13.

[37] MARRA, V. (199?) Every abelian ℓ-group is ultrasimplicial. *Journal of Algebra*. To appear.

[38] MARTÍNEZ, N.G. (1990) *Una dualidad topológica para estructuras algebraicas ordenadas*. Universidad de Buenos Aires. (Ph.D. Thesis)

[39] MARTÍNEZ, N.G. (1990a) The Priestley duality for Wajsberg algebras. *Studia Logica*, **49**, p. 31-46.

[40] MARTÍNEZ, N.G., PRIESTLEY, H.A.P. (1995) Uniqueness of MV-algebra implication and De Morgan negation. *Mathware and Soft Computing*, **2**, p. 229-245.

[41] MONTAGNA F. (1999) An algebraic approach to propositional fuzzy logic. *Journal of Logic, Language and Information*. Special issue on Logics of Uncertainty (D. Mundici, Ed.) To appear.

[42] MUNDICI, D. (1986) Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus. *Journal of Functional Analysis*, **65**, p. 15-63.

[43] MUNDICI, D. (1987) Satisfiability in many-valued sentential logic is NP-complete. *Theoretical Computer Science*, **52**, p. 145-153.

[44] MUNDICI, D. (1988) Farey stellar subdivisions, ultrasimplicial groups and K_o of AF C^*-algebras. *Advances in Mathematics*, **68**, p. 23-39.

[45] MUNDICI, D. (1992) The logic of Ulam's game with lies. In: **Knowledge, Belief and Strategic Interaction**. (Bicchieri, C., Dalla Chiara, M.L., Eds.) Cambridge University Press, p. 275-284. (*Cambridge Studies in Probability, Induction and Decision Theory*)

[46] MUNDICI, D. (1992a) Normal forms in infinite-valued logic: the case of one variable. *Lecture Notes in Computer Science*, **626**, p. 272-277.

[47] MUNDICI, D. (1993) Logic of infinite quantum systems. *International Journal of Theoretical Physics*, **32**, p. 1941-1955.

[48] MUNDICI, D. (1994) A constructive proof of McNaughton's Theorem in infinite-valued logics. *The Journal of Symbolic Logic*, **59**, p. 596-602.

[49] MUNDICI, D. (1995) Averaging the truth value in Łukasiewicz sentential logic. *Studia Logica*, Special issue in honor of Helena Rasiowa. **55**, p.113-127.

[50] MUNDICI, D. (1996) Łukasiewicz normal forms and toric desingularizations. In: **Proceedings of Logic Colloquium 1993**, Keele, England. (W. Hodges et al., Eds.) Oxford University Press, p. 401–423.

[51] MUNDICI, D. (1996a) Uncertainty measures in MV-algebras and states of AF C^*-algebras. *Notas de la Sociedad de Matematica de Chile*, Special issue in memoriam Rolando Chuaqui, **15**, p. 42-54.

[52] MUNDICI, D. (1998) Nonboolean partitions and their logic, In: **First Springer-Verlag Forum on Soft Computing**. Prague, August 1997, *Soft Computing* 2, p.18-22.

[53] MUNDICI, D. (1999) Classes of ultrasimplicial lattice-ordered abelian groups. *Journal of Algebra,* **213**, p.596-603.

[54] MUNDICI, D. (1999a) Tensor products and the Loomis-Sikorski theorem for MV-algebras. *Advances in Applied Mathematics,* **22**, p. 227-248.

[55] MUNDICI, D., OLIVETTI, N. (1998) Resolution and model building in the infinite-valued calculus of Łukasiewicz. *Theoretical Computer Science,* **200**, p. 335-366.

[56] MUNDICI, D., PANTI, G. (1993) Extending addition in Elliott's local semigroup. *Journal of Functional Analysis*, **171**, p. 461-472.

[57] MUNDICI, D., PANTI, G. (1999) A constructive proof that every 3-generated ℓ-group is ultrasimplicial. In: **Logic, Algebra and Computer Science** Polish Academy of Sciences, Warszawa 1999, p.169-178. (Banach Center Publications, vol. 46)

[58] PANTI, G. (1995) A geometric proof of the completeness of the Łukasiewicz calculus. *Journal of Symbolic Logic*, **60**, p. 563-578.

[59] PANTI, G. (1998) Multi-valued Logics. In: **Quantified Representation of Uncertainty and Imprecision**, vol. 1, (P. Smets, Ed.), Kluwer, Dordrecht, p.25-74.

[60] PELC, A. (1987) Solution of Ulam's problem on searching with a lie. *Journal of Combinatorial Theory*, Series A, **44**, p. 129-140.

[61] RIEČAN,B. (1999) On the product MV-algebras. *Tatra Mountain Math. Publications*, **16**. To appear.

[62] RIEČAN, B. (199?) On the joint distribution of observables. *Soft Computing*. To appear.

[63] RIEČAN, B. (199?) On the probability theory on MV-algebras. *Soft Computing*. To appear.

[64] RIEČAN, B., NEUBRUNN, T. (1997) **Integral, Measure, and Ordering**. Kluwer, Dordrecht.

[65] ROSE, A., ROSSER, J.B. (1958) Fragments of many-valued statement calculi. *Transactions of the American Mathematical Society*, **87**, p. 1-53.

[66] SPENCER, J. (1992) Ulam's searching game with a fixed number of lies. *Theoretical Computer Science*, **95**, p. 307-321.

[67] TARSKI, A. (1956) **Logic, Semantics, Metamathematics**. Clarendon Press, Oxford. Reprinted (1983), Hackett, Indianapolis.

[68] ULAM, S. (1976) **Adventures of a Mathematician**. Scribner's, New York.