ACTAS DEL VI CONGRESO
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(2001),Páginas 51 – 68

TILINGS ASSOCIATED WITH NUMBER SYSTEMS AND THE GEOMETRY OF SETS DERIVED FROM THE BASES –n+i

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ABSTRACT. The number system with base $b=-n+i$, $n\in\{1,2,...\}$, and set of ciphers (digits) $D=\{0,1,...,n^2\}$ gives rise to a congruent tiling of the plane in which each tile touches exactly six different tiles whenever $n\neq 2$. Instead, if $n=2$, each tile touches ten different ones but it is in contact with four of them in only a finite set of points. The cardinalities of these sets are given and their elements are determined. It holds that

$$1 < s = \dim_H(\partial F(n)) = \dim_B(\partial F(n)) = \log\lambda / \log|b| < 2$$

where $F(n)$ is the central tile and λ is the spectral radius of a nonnegative matrix associated with the number system. Moreover, $\partial F(n)$ is an s-set. The preceding equalities and inequalities hold for more general tilings associated with number systems, (cf. Ths. 2 and 4). •

1. INTRODUCTION. A) The number system (b,D) with base $b=b(n)=-n+i$, n a positive integer and set of ciphers $D=D(n)=\{0,1,2,...,n^2\}$ was studied by Kátai and Szabó in [KS]. The complex numbers are representable in each of these bases. $F=F(n)$ is the set of complex numbers that have a representation with integer part zero in the number system $(-n+i,D(n))$. It defines the central tile of a tiling $\tau=\tau(n)$ of the plane derived from the base $-n+i$. $L=[1,i]$ denotes the set of Gaussian integers. Observe that $bL\subset L\supset D$ and that D is a complete set of residues modulus b, that is, for $y\in L$ there exist $x\in L$ and $c\in D$ such that $y=bx+c$ where x and c are uniquely determined.
The family of traslations of $F(n)$ by numbers in L is in fact a *tessellation* $\tau(n)$ of the plane, $\tau=\{F_t=F+t: t\in L\}$. This means that it is a covering of R^2 (**C** is representable) with tiles such that two different traslations of F have an intersection of Lebesgue measure zero. (This is a consequence of the facts that D is a complete set of residues modulus b and $|b|^2=n^2+1=card\,D$, cf. [K] or [B] Th. 2 and Prop. 5).
$F(1)$ is the so called *twin dragon*. We shall study in this paper **only the bases $b(n)$, $n>1$**, since Davies and Knuth's space filling twin dragon curve is well known. For $n=1$ and in the context of number representation the reader can consult, among several other references, our paper [BP], where this case is studied in great detail. There it is shown that the conclusions to which we arrive here for $n>2$ hold for $n=1$.
Our purpose is to prove that the compact sets $F(n)$, $n>2$, like $F(1)$, have a common frontier with exactly six different tiles and that the tile $F(2)$ has points in common with exactly ten different ones.

B) W. J. Gilbert proved in [G] the formula

(1) $$\dim_H(\partial F(n)) = \log\lambda / \log|b|$$

where $\lambda = \lambda\,(n)$ is the spectral radius of a primitive (nonnegative) matrix and the greatest root of the polynomial

(2) $$r(x)\,..\,x^3 - (2n-1)x^2 - (n-1)^2 x - (n^2+1)\,.$$

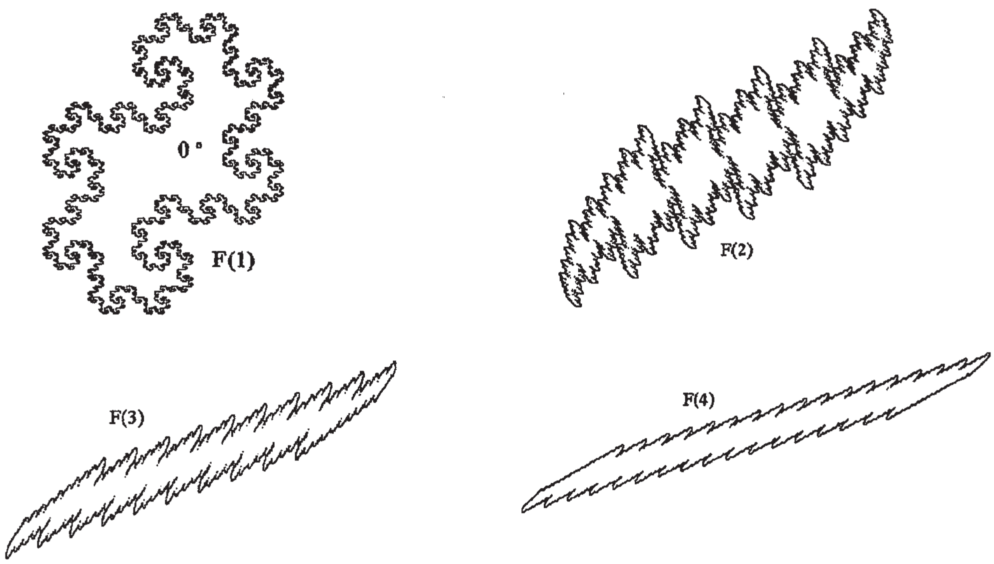
This is proved by Gilbert making use of the δ -Hausdorff measure of $\partial F(n)$ and showing that $H^\delta(\partial F(n))$ is finite for $\delta = s := \log\lambda/\log|b|$ and infinite for $\delta < s$. It also follows from the Corollary (1.14) of [V] where it is shown that $\log\lambda/\log|b| < 2$. Moreover, it can be obtained from [DKV], Th. 2.

$F(n)$ is the invariant (compact) set of the iterated function system given by $\{\Phi_c(z) = b^{-1}z + c : c \in D(n)\}$. It satisfies the open set condition with the open (non void) set $\mathrm{int}(F)$. Since ∂F is a sub-self-similar set, according to [F2], Th. 3.5, we have $H^s(\partial F) > 0$ where $s = \dim_H(\partial F) = \dim_B(\partial F)$.

Collecting results,

(3) $$0 < H^s(\partial F(n)) < \infty, \quad s = \dim_H(\partial F) = \dim_B(\partial F) = \log\lambda/\log|b| < 2\,.$$

C) Formulae like (2) and (3) hold for tiles that appear in several tilings. In this relation the reader could consult [DKV] and [V] where very general results are proved. We show in the following sections of part I, precisely in Ths. 2 and 4, that the formulae hold for tessellations derived from quite general number systems.



I. GENERAL THEORY.

2. BASIC RESULT ON THE HAUSDORFF DIMENSION. The next Theorem 1 can be proved repeating almost *verbatim* the proof given in Theorem 3.1 of Falconer's book [F] only replacing the functions g_i^{-1} that appear there by new functions f_i.

THEOREM 1. Let E be a non trivial compact set and a and r_0 two positive numbers, $r_0 <1$, such that for any set $U \subset E, 0<|U|<r_0$, there exist $V=V(U) \subset E$ and a map f from V onto U that verifies

(4) $$v, w \in V \Rightarrow |f(v)-f(w)| \leq \frac{|U|}{a}|v-w|.$$

Then, the box dimension $\dim_B(E)$ exists and if $s=\dim_H(E)$ then i) and ii) hold:
i) $H^s(E) \geq a^s$
ii) $s=\dim_B(E)$.•

We shall assume the next two hypothesis.
H) Let $b \in \mathbf{C}$ $(\equiv R^N, N=2)$, $|b|>1$, be the base of the number system $\{b,D\}$ with $D=\{0,a_1,\ldots,a_n\} \subset R^N$ its set of ciphers (digits) such that there exists a point lattice $L=[1,g]:=\{m+ng: m,n \in Z\} \subset R^N$ verifying $bL \cup D \subset L$ with D a complete set of residues modulo b, (*i.e.*, each point y of L can be written in a unique way as $y=bx+c$, $x \in L$, $c \in D$).

DEFINITION 0. $F:=\{z: z=0.c_1c_2\ldots; c_i \in D\}$ and $F_t:=t+F$. •

H') $\{F_t: t \in L\}$ is a tessellation of R^N, (*i.e.*, $R^N=\bigcup F_t$, $m(F_u \cap F_v)=0$ for $u \neq v$).

THEOREM 2. If **H)** and **H')** hold then
a) the box dimension of $E:=\partial F$ exists,
b) $s=\dim_H E=\dim_B E$,
c) $H^s(E)>0$,
d) $1 \leq s<N$. •

PROOF. a), b) and c) will follow from Theorem 1. In fact, suppose $U \subset E$ has diameter $0<|U|<r_0:=\rho /|b|$ where $2\rho:=\min\{|\lambda|; 0 \neq \lambda \in L\}$. Let k be the positive integer verifying $\rho /|b| \leq |U||b|^k<\rho$.

We write $U=\bigcup_{j=1}^{M} U_j$ where each U_j is of the form $U \cap (F_{0.b_1\ldots b_k} \cap F_{\gamma.c_1\ldots c_k})$ (cf. Def. 3), $\gamma \in S^0:=\{t \in L: t \neq 0, F \cap F_t \neq \varnothing\}$ (Def. 1) and $b_i, c_i \in D$ depend on j. Let $g_j(z):=b^k z+t_j$ where $t_j:=-\sum_{i=1}^{k} b_i b^{k-i}$ is a point of the lattice L (this because of $bL \cup D \subset L$). Each similitude g_j maps U_j into E and $|\ g_j(z)-g_h(z)\ |$ is either identically 0 or $\geq 2\rho$.

Therefore, if the maps are not identical then

(5) $$dist(g_j(U), g_h(U)) \geq 2\rho-|U||b|^k>\rho .$$

Let $V=\bigcup_{j=1}^{M} V_j$ where $V_j := g_j(U_j)$ and define $f:V\to U$ by $f(z)=g_j^{-1}(z)$ if $z\in V_j$.
Observe that if $V_j \cap V_h \neq \varnothing$ then, by (5), g_j and g_h must be identical. Therefore, *f* is well defined and onto *U*. We claim that if $z, w \in V$ then

(6) $$|f(z)-f(w)| \le \frac{|U|}{a}|z-w|, \text{ where } a=\rho/|b|.$$

This will show that the hypothesis of theorem 1 are fullfilled, so a), b) and c) are true.
Let $z\in V_j$, $w\in V_h$. There are two possibilities:

i) g_j and g_h are identical. Then, $|f(z)-f(w)| = |z-w||b|^{-k} \le \frac{|U|}{\rho/|b|}|z-w|$.

ii) g_j and g_h are not identical. Then, using (5), one gets $|z-w| \ge dist\{V_j, V_h\} > \rho$ and $|f(z)-f(w)| \le |U| = \frac{\rho}{\rho}|U| < \frac{|U|}{\rho}|z-w|$.

Thus, in any case (6) is true with $a=\rho/|b|$.
Let us prove d). $s<N$ is a consequence of c) and the definition of tessellation. On the other hand, *F* is a compact set with non void interior and *E* is compact. Any compact set with Hausdorff dimension less than 1 is totally disconnected. Then, if $s<1$, the complement *E'* of *E* in R^N, $N>1$, is a connected set. A polygonal path in *E'* from one point in int(*F*) to a point in ext(*F*) contains necessarily a point in *F* with two representations. That is, a point in *E*, a contradiction, QED.
Note that Theorems 1 and 2 were proved in [Z].

3. BASIC RESULTS ON THE BOX DIMENSION. Since we wish to make more precise the statement a) of Theorem 2 we introduce a method, that we borrow from [BA] pgs. 5-9, for estimating the box dimension of certain sets. It is a slight variant of the method presented in [K], pgs. 11-12. We restrict ourselves to plane sets. However, the preceding results and those that follow can be extended without much difficulty to dimension $N > 2$ and generalized number systems, (about these systems cf. for example [BO]).
Assume that the hypothesis **H)** of Theorem 2 holds. Observe that this hypothesis implies the important

PROPOSITION 1. The integers of the system, $W:=\left\{(c_m...c_0)_b : m\ge 0, c_j\in D\right\} \equiv \left\{\sum_0^m c_j b^j\right\} \subset L$, have a unique positional representation as integers of the number system $\{b,D\}$.•

Let S_r be a *net of cubes* in R^d with sides of length *r* parallel to the axes, $S_r=\{q+\tau : \tau\in rZ^d\}$, *q* a cube of side of length *r*. The *upper box dimension* of the bounded set $E\subset R^d$ is defined by $\overline{\dim}_B(E) := \overline{lim}\,\frac{\log N(r,E)}{\log 1/r}$ for $r\to 0$ where $N(r,E)$=card {cubes $\in S_r$ that intersect *E*}. Several other families can play the role of the S_r's. We wish to use families of sets *T* of positive measure for which there exists a fixed constant *K* verifying $|T|^d/m(T) < K$ *where* $|T|$=diam(*T*). We use the notation $B_1=B(0;1)$ for the *unit ball* with center the origin.

LEMMA 1. Suppose that $\{T+\gamma:\gamma\in\Lambda\}$ is a *covering* of the bounded set E, that is, $E\subset\bigcup\{T+\gamma:\gamma\in\Lambda\}$ and $(T+\gamma)\cap E\neq\varnothing$ for $\gamma\in\Lambda$. Assume that a) and b) hold:

a) $q\in S_r$, $m(T)=m(q)$, $\dfrac{(|T|+|q|)^d}{m(T)}\leq K<\infty$,

b) $E\subset\bigcup\{T+\gamma:\gamma\in\Lambda\}$ a covering such that $m((T+\gamma)\cap(T+\gamma'))=0$ if $\gamma\neq\gamma'$.

If $M(r,E):=\operatorname{card}\Lambda$ then

$$M(r,E)\leq Km(B_1).N(r,E)\ ,\ N(r,E)\leq Km(B_1).M(r,E).\ \bullet$$

PROOF. Let $E\subset\bigcup\{q+\tau:(q+\tau)\cap E\neq\varnothing\}$ and for $\gamma\in\Lambda$, $\Xi=\Xi(\gamma)$ the family of τ's for which $(T+\gamma)\cap(q+\tau)\cap E\neq\varnothing$.

If $y=\operatorname{card}\Xi$ then $ym(q)=ym(T)\leq m(B(0,|T|+|q|)=(|T|+|q|)^d m(B_1)$ .Thus, $y\leq Km(B_1)$ and

$$N(r,E)\leq(\sup y)\ M(r,E)\leq K\,m(B_1)\ M(r,E)$$

and the lemma follows, QED.

LEMMA 2. Let k be a positive integer and $r_j\downarrow 0$ a decreasing sequence such that $kr_{j+1}\geq r_j$. Then, the following expressions have the same limits for $j\to\infty$ and $r\to 0$, respectively,

$$\frac{\log N(r_j,E)}{\log 1/r_j}\ ,\ \frac{\log N(r,E)}{\log 1/r}.\ \bullet$$

PROOF. A cube $q\in S_{r_j}$ intersects at most $(k+1)^d$ cubes of $S_{r_{j+1}}$. Therefore,

$$N(r_{j+1},E)\leq(k+1)^d N(r_j,E)\,.$$

We used only that $kr_{j+1}\geq r_j$. However, if $r<r_j$ it holds that

$$N(r_j,E)\leq 2^d N(r,E)\,.$$

Then, for $r_{j+1}<r<r_j\leq kr_{j+1}$, we obtain,

$$N(r_{j+1},E)\leq(k+1)^d N(r_j,E)\leq(2(k+1))^d N(r,E)\leq 4^d(k+1)^d N(r_{j+1},E)\,.$$

Besides, $\log 1/r_{j+1}\geq\log 1/r\geq\log 1/r_j\geq\log 1/r_{j+1}-\log k$. Thus,

$$\text{B.}\quad \frac{\log N(r_{j+1},E)}{\log 1/r_{j+1}}\leq\frac{\log N(r,E)+c}{\log 1/r}\leq\frac{\log N(r_{j+1},E)+\tilde c}{\log 1/r_{j+1}-\log k}.$$

From (7) we obtain, for example, $\overline{\lim}_{j\to\infty}\dfrac{\log N(r_j,E)}{\log 1/r_j}=\overline{\lim}_{r\to 0}\dfrac{\log N(r,E)}{\log 1/r}$, QED.

LEMMA 3. Let $r_j\downarrow 0$ with $kr_{j+1}\geq r_j$, $q\in S_{r_j}$, k a positive integer. For each j let $T=T(j)$ be such that $m(T)=m(q)$, $(|T|+|q|)^d/m(T)\leq K<\infty$ with K independent of j.

Assume that $\{T+\gamma:\gamma\in\Lambda\}$, $\Lambda=\Lambda(j)$, is a covering of E verifying

$$\gamma\neq\gamma'\Rightarrow m((T+\gamma)\cap(T+\gamma'))=0.$$

If $M(r_j,E)=\operatorname{card}\Lambda$ then

$$(8)\qquad \overline{\dim}_B(E)=\overline{\lim}_{j\to\infty}\frac{\log M(r_j,E)}{\log 1/r_j}\ ,\ \underline{\dim}_B(E)=\underline{\lim}_{j\to\infty}\frac{\log M(r_j,E)}{\log 1/r_j}.\ \bullet$$

PROOF. Because of Lemma 1, $\overline{\lim}\dfrac{\log M(r_j,E)}{\log 1/r_j}=\overline{\lim}\dfrac{\log N(r_j,E)}{\log 1/r_j}$. It follows from Lemma 2 that both are equal to $\overline{\lim}_{r\to 0}\dfrac{\log N(r,E)}{\log 1/r}$, QED.

DEFINITION 1. $S^\circ := \{\gamma \in L, \gamma \neq 0 : \exists z = 0.\varepsilon_1\varepsilon_2... = \gamma.\tilde{\varepsilon}_1\tilde{\varepsilon}_2...\}$. *In other words,* $S^\circ = \{\gamma \in L, \gamma \neq 0 : F \cap F_\gamma \neq \varnothing\}$. *If* $\gamma \in S^\circ$ *then* $V_\gamma := F \cap F_\gamma$. $\Gamma :=$card S°.•

DEFINITION 2. *$G(S^\circ)$ will denote the graph with set of nodes S° and arrows* $\gamma \xrightarrow{\binom{\varepsilon}{\tilde{\varepsilon}}} \gamma' = \gamma b + \tilde{\varepsilon} - \varepsilon$ *whenever* $\gamma b + d \in S^\circ$, $d = \tilde{\varepsilon} - \varepsilon$ *and* $\varepsilon, \tilde{\varepsilon} \in D$.•

We know from the definition of S° that given $\gamma \in S^\circ$ there exists a point z such that $z = 0.\varepsilon\varepsilon_2... = \gamma + 0.\tilde{\varepsilon}\tilde{\varepsilon}_2... \in V_\gamma$. For this point, $(\gamma b + \tilde{\varepsilon} - \varepsilon) + 0.\tilde{\varepsilon}_2... = 0.\varepsilon_2...$. Since D is a complete set of residues modulo b, $\gamma b + \tilde{\varepsilon} - \varepsilon \neq 0$. Then $\gamma b + \tilde{\varepsilon} - \varepsilon \in S^\circ$. Hence, from γ starts an infinite path in $G(S^\circ)$ that defines the point $z = 0.\varepsilon\varepsilon_2... \in V_\gamma$. It is not difficult to see that *any* infinite path in $G(S^\circ)$ starting from γ determines a point in V_γ. We have then the

PROPOSITION 2. From each $\gamma \in S^\circ$ starts, at least, one arrow which is the beginning of an infinite path in $G(S^\circ)$ that determines, as shown above, a point $z \in V_\gamma$. •

DEFINITION 3. $F_{\sigma+0.abc...m} := \{z : z = \sigma + 0.abc...mc_1c_2...; c_j \in D\}, \sigma \in L,\ a, b, ..., m \in D$. *We write also $\sigma.abc...m$ instead of $\sigma + 0.abc...m$ mainly when $\sigma \in W$, the set of integers of (b,D).*

DEFINITION 4. *$M(n,\gamma)$ is the number of paths in $G(S^\circ)$ of length n starting from $\gamma \in S^\circ$. That is,*

$$M(n,\gamma) = \text{card}\left\{(\varepsilon_1,...,\varepsilon_n;\tilde{\varepsilon}_1,...,\tilde{\varepsilon}_n) : \exists\varepsilon_{n+1},...;\exists\tilde{\varepsilon}_{n+1},... : \sum_1^\infty \varepsilon_j b^{-j} = \gamma + \sum_1^\infty \tilde{\varepsilon}_j b^{-j}\right\};$$

$$m(n,\gamma) := \text{card}\{(\varepsilon_1,...,\varepsilon_n) : \exists\varepsilon_{n+1},\varepsilon_{n+2},...;\exists\tilde{\varepsilon}_1,\tilde{\varepsilon}_2,... : 0.\varepsilon_1...\varepsilon_n\varepsilon_{n+1}... = \gamma + 0.\tilde{\varepsilon}_1\tilde{\varepsilon}_2...\}.\bullet$$

The family $\{F_{t+0.a_1...a_n} : t \in L, a_i \in D\}$ is a tessellation of the plane like $\{F_t : t \in L\}$ but with smaller tiles in a ratio $|b|^{-n}$ and $m(n,\gamma)$ *is the number of such tiles contained in F and in contact with* V_γ. They form a covering of V_γ by essentially disjoint sets of diameter $|F||b|^{-n}$ and measure $m(F)|b|^{-2n}$. Obviously, $m(n,\gamma) \le M(n,\gamma)$.

On the other hand, $M(n,\gamma)$ counts the number of pairs of tiles in $\{F_{t+0.a_1...a_n} : t \in L, a_i \in D\}$ with non void intersection such that one of the tiles is in F and the other one in F_γ. For fixed $\varepsilon_1,...,\varepsilon_n$ the number of possible sets $\tilde{\varepsilon}_1,...,\tilde{\varepsilon}_n$ is not greater than Γ. In fact, $\{\gamma b^n + (\tilde{\varepsilon}_1...\tilde{\varepsilon}_n)_b - (\varepsilon_1...\varepsilon_n)_b\} + 0.\tilde{\varepsilon}_{n+1}... = 0.\varepsilon_{n+1}...$. If τ is the number inside the brackets then $\tau \in S^\circ$ and $(\tilde{\varepsilon}_1...\tilde{\varepsilon}_n)_b = \tau - \gamma b^n + (\varepsilon_1...\varepsilon_n)_b \in L$. Because of Proposition 1, the $\tilde{\varepsilon}_j$'s are uniquely determined by τ. Thus

(9) $$m(n,\gamma) \le M(n,\gamma) \le \Gamma\,.m(n,\gamma).$$

Observe that $M(1,\gamma_j)$ is the number of arrows in $G(S^\circ)$ starting from γ_j.

DEFINITION 5. p_{jk} *(or also* $P_{\gamma_j\gamma_k}$*) denotes the number of* 1*-paths from* γ_j *to* γ_k*.•*

Then, for each γ_j, $M(1,\gamma_j)=\sum_k p_{jk}\geq 1$. If $M(0,\gamma_j):=1$, we have, for $n\geq 0$, that

$$M(n+1,\gamma_j)=\sum_{k=1}^{s} p_{jk}M(n,\gamma_k). \quad (10)$$

If $\vec{Y}^{(n)}:=\begin{pmatrix} M(n,\gamma_1)\\ \vdots \\ M(n,\gamma_\Gamma)\end{pmatrix}$, $P=\begin{pmatrix} p_{11} & \cdots & p_{1\Gamma}\\ & \cdots & \\ p_{\Gamma 1} & \cdots & p_{\Gamma\Gamma}\end{pmatrix}$ then $\vec{Y}^{(n+1)}=P\vec{Y}^{(n)}=P^n\vec{Y}^{(1)}=P^{n+1}\vec{Y}^{(0)}$.

P is a *nonnegative* matrix that verifies $P\vec{Y}^{(0)}=\vec{Y}^{(1)}\geq 1.\vec{Y}^{(0)}>0$. Therefore, its spectral radius λ *is an eigenvalue not less than* 1. Besides, *there exists a non null* Γ *-dimensional vector* $\vec{v}\geq 0$ *such that* $P\vec{v}=\lambda\vec{v}$.

There also exists $\mu>0$ such that $\vec{Y}^{(1)}\geq\mu\vec{v}$. In consequence,

$$\mu\lambda^n\vec{v}=P^n\mu\vec{v}\leq P^n\vec{Y}^{(1)}=\vec{Y}^{(n+1)}.$$

Then, if γ is such that v_γ (the γ^{th} element of $\vec{v}$) is positive then $M(n+1,\gamma)\geq\mu\lambda^n v_\gamma>0$. Therefore, for an adequate constant B, we have,

$n\log\lambda+B\leq\log M(n+1,\gamma)$ and $\dfrac{n\log\lambda}{\log|b^n|}+o(1)\leq\dfrac{\log M(n,\gamma)}{\log|b^n|}$. On the other hand we obtain from (9) that $\dfrac{\log M(n,\gamma)}{\log|b^n|}$ and $\dfrac{\log m(n,\gamma)}{\log(|b^n|)}$ *have the same limits.*

We call $r_j=\sqrt{m(F)}|b|^{-j}$, $K'=\dfrac{|F_{0.\varepsilon_1\ldots\varepsilon_j}|^2}{m(F_{0.\varepsilon_1\ldots\varepsilon_j})}=\dfrac{|F|^2}{m(F)}$ and apply Lemma 3 with K=2K'+4 and $M(r_j,V_\gamma)=m(j,\gamma)$. It follows that

$$\frac{\log\lambda}{\log|b|}\leq\underline{\lim}\frac{\log M(j,\gamma)}{\log 1/r_j}=\underline{\lim}\frac{\log m(j,\gamma)}{\log 1/r_j}=\underline{\lim}\frac{\log M(r_j,V_\gamma)}{|\log r_j|}. \quad (11)$$

Since the last limit is equal to $\underline{\dim}_B V_\gamma$ we have,

$$\log\lambda/\log|b|\leq\underline{\dim}_B V_\gamma. \quad (12)$$

THEOREM 3. Assume the hypothesis **H)** and **H')**. Let λ be the spectral radius of P.
i) If $V_\gamma=F\cap F_\gamma$ then $\overline{\dim}_B V_\gamma\leq\log\lambda/\log|b|$.
ii) If γ is such that $v_\gamma>0$ (v_γ = the γ^{th} element of $\vec{v}$, $\vec{v}$ an eigenvector corresponding to λ) then the box dimension of V_γ exists and

$$\log\lambda/\log|b|=\dim_B V_\gamma.\ \bullet \quad (13)$$

PROOF. i) $\|\vec{Y}^{(n)}\|_\infty \le \|P^n\| \|\vec{Y}^{(0)}\|_\infty$ implies that $M(n,\gamma) \le C\|P^n\|$ and therefore $\frac{\log M(n,\gamma)}{n} \le \log(\|P^n\|^{1/n}) + o(1)$. Thus, $\overline{lim}\frac{\log M(n,\gamma)}{\log|b^n|} \le \frac{\log\lambda}{\log|b|}$. As above, applying Lemma 3, we get,

$$\overline{\dim}_B V_\gamma = \overline{lim}\frac{\log m(n,\gamma)}{\log|b^n|} = \overline{lim}\frac{\log M(n,\gamma)}{\log|b^n|} \le \frac{\log\lambda}{\log|b|}.$$

ii) follows from i) and (12), QED.

4. THE MATRICES *P* AND *Q*. One can find $\lambda = \rho(P)$ =the spectral radius of P, from a nonnegative matrix Q whose order is the half of the order of P and verifies $\rho(P) = \rho(Q)$. Observe that $S°$ and the graph $G(S°)$ are *symmetric*. In fact, given $\gamma \in S°$, the equality $0.\varepsilon ... = \gamma + 0.\tilde{\varepsilon}...$ implies $-\gamma + 0.\varepsilon ... = 0.\tilde{\varepsilon}...$, so $-\gamma \in S°$. Besides $\gamma \xrightarrow{\binom{\varepsilon}{\tilde{\varepsilon}}} \gamma' = \gamma b + \tilde{\varepsilon} - \varepsilon$ and $-\gamma \xrightarrow{\binom{\tilde{\varepsilon}}{\varepsilon}} -\gamma' = b(-\gamma) + \varepsilon - \tilde{\varepsilon}$. Therefore,

$$S° = -S° \;,\; M(n,\gamma) = M(n,-\gamma). \tag{14}$$

DEFINITION 6. *Let S'' be a subset of $S°$ not containing opposite elements and such that $S° = S'' \cup (-S'')$. Let $\Delta := card\ S'' = \Gamma/2$.* •

We get from (10) and (14) and for $\gamma \in S''$, that

$$M(n+1,\gamma) = \sum_{\delta \in S°} P_{\gamma\delta} M(n,\delta) = \sum_{\delta \in S''} (P_{\gamma\delta} + P_{\gamma -\delta}) M(n,\delta) = \sum_{\delta \in S''} Q_{\gamma\delta} M(n,\delta). \tag{15}$$

This defines the matrix $Q = [Q_{\gamma\delta}]$, $\gamma, \delta \in S''$. Using the set of indices $S'' \cup (-S'')$ we see that $P = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ and using only the indices in S'' we have $Q = A + B$. Then,

$$\det(P - xI) = \det\begin{pmatrix} A + B - xI & B \\ A + B - xI & A - xI \end{pmatrix} = c(x)\det(A + B - xI) = c(x)\det(Q - xI).$$

It follows from this that *the spectrum of Q is contained in the spectrum of P*. For certain nonnegative vectors $\vec{v}$, $\vec{w}$ with $\vec{v} + \vec{w} \neq 0$, it holds that $\begin{pmatrix} A & B \\ B & A \end{pmatrix}\begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix} = \lambda \begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix}$. Thus,

$$(A + B)(\vec{v} + \vec{w}) = \lambda(\vec{v} + \vec{w}). \tag{16}$$

Therefore, λ *is in the spectrum of Q* and the two matrices have the same spectral radius. Thus, we proved c) of the next theorem.

THEOREM 4. Assume that the hypothesis **H)** and **H')** in Theorem 2 hold:

H+H') Let $b \in C$ be the base of the number system $\{b, D\}$ with $|b| > 1$ and $D = \{0, a_1, ... a_n\} \subset C$ $(..R^N, N = 2)$ its set of ciphers. Assume that there exists a point lattice $L = [1,g] := \{m + ng : m,n \in Z\} \subset R^N$ such that $bL \cup D \subset L$ and that D is a complete set of residues modulo b. Suppose that the family $\{F_t : t \in L\}$ is a tessellation of R^N where $F := \{z : z = 0.c_1c_2...; c_i \in D\}$ and $F_t := t + F$.

Then,

a) For any $\gamma \in S^{\circ}$, $\overline{\dim}_B V_\gamma \le \log\lambda / \log|b|$ but $\log\lambda/\log|b| = \dim_B V_\gamma$ for each $V_\gamma = F \cap F_\gamma$, $\gamma \in S^{\circ}$, such that the element v_γ of an eigenvector $\vec{v}$ corresponding to the eigenvalue $\lambda = \rho(P)$ of the nonnegative matrix $P = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ is positive.

b) The box dimension of $E = \partial F$ exists and $\dim_B E = \dfrac{\log\lambda}{\log|b|}$.

c) λ is equal to the spectral radius of the nonnegative matrix Q=A+B.

d) λ may have a geometric dimension greater than one.

e) Let us denote with $\vec{y}^{(0)}$ the column vector of dimension Δ such that $(\vec{y}^{(0)})_\omega = 1$, $\omega \in S''$. Assume that 1 is in the spectrum of Q, that $\vec{A}$ is an eigenvector corresponding to the eigenvalue 1 and that $\vec{W} \ge 0$ is an eigenvector corresponding to the eigenvalue λ. Suppose that $\vec{y}^{(0)} = (1,\ldots,1)' \le \vec{W} + \vec{A}$. If $\gamma \in S''$ is such that $(\vec{W})_\gamma$=0 then

$$card(V_\gamma) = card(V_{-\gamma}) < \infty.$$

f) Assume that there exists a family of eigenvectors $\vec{A}_k$, k=1,...,r, corresponding to eigenvalues e_k of Q such that $\vec{y}^{(0)} \le \sum_1^r \vec{A}_k$. Then, for any $\gamma \in S$, $H^s(V_\gamma) < \infty$ and $0 < H^{(s)}(E) < \infty$, *i.e.*, E is an s-set. •

PROOF. a) is the content of Theorem 3 and c) was already proved. From a) and the finite stability of the upper box dimension we obtain

$$\frac{\log\lambda}{\log|b|} = \overline{\dim}_B E \ge \underline{\dim}_B E \ge \sup_\gamma(\underline{\dim}_B V_\gamma) = \frac{\log\lambda}{\log|b|}$$

and b) follows. In part II we give an example that demonstrates that d) is true.

e) Recall (16) and that Δ=$card$ S''. We have $Q^n\vec{y}^{(0)} \le Q^n\vec{W} + \vec{A} = \lambda^n\vec{W} + \vec{A}$. Then, using (15), we arrive at the inequality

(17) $$\vec{y}^{(n)} = (M(n,\gamma_1),\ldots,M(n,\gamma_\Delta))' \le \lambda^n\vec{W} \quad ..$$

If the γ-element of $\vec{W}$ is null, $(\vec{W})_\gamma = 0$, then $M(n,\gamma) \le (\vec{A})_\gamma$. The positive integer $M(m,\gamma)$ is the number of m-paths in $G(S^{\circ})$ starting from γ. We know from Proposition 2 that $M(m,\gamma) \le M(m+1,\gamma)$. Thus, we have from the last inequality and for some p that $M(p+j,\gamma) = c$ if $j > 0$. Therefore, it follows that there are c ∞-paths in $G(S^{\circ})$ starting from γ. Since to each such path there corresponds a z=$0.\varepsilon_1 \ldots \varepsilon_p \ldots = \gamma.\tilde{\varepsilon}_1 \ldots \tilde{\varepsilon}_p \ldots \in V_\gamma$ and this correspondence is onto, there are at most c points in V_γ. (It might not be 1:1 since the representation may not be unique). Thus, $card(V_\gamma) = c' \le c < \infty$. Observe that, by symmetry, $card(V_{-\gamma}) = card(V_\gamma)$.

f) We have $H^s_\delta(V_\gamma) \le m(n,\gamma)\,\delta^s \le M(n,\gamma)\,\delta^s$ for $\gamma \in S^{\circ}$ and $\delta = |F||b|^{-n}$. Thus, $H^s_\delta(V_\gamma) \le |F|^s M(n,\gamma)|b|^{-ns}$. By Theorems 2 and 4 we have $\lambda = |b|^s$. Since Q and $\vec{y}^{(0)}$ are non negative, instead of (17) we obtain,

(17') $$\vec{y}^{(n)} = (M(n,\gamma_1),\ldots,M(n,\gamma_\Delta))' \le \sum_{k=1}^r (e_k)^n \vec{A}_k.$$

But $|e_k| \leq \lambda$ implies that $0 < \vec{y}^{(n)} \leq \lambda^n \vec{Z}$ with $\vec{Z}$ a positive vector. Therefore, $M(n,\gamma) \leq K\lambda^n$ for every $\gamma \in S^{\circ}$. Thus, $H^s_\delta(V_\gamma) \leq C(|b|^s)^n.|b|^{-ns} = C$. In consequence, $H^s(V_\gamma) < \infty$. Now, it follows that $H^s(E) < \infty$, QED.

REMARK 1. i) If $\vec{v} > 0$ there is a constant K such that $\vec{Y}^{(0)} \leq K\vec{v}$ and by f) E is an s-set. ii) Observe that (17) is a particular case of (17'). iii) Since $0 < \vec{y}^{(0)} \leq \vec{y}^{(m)}$, in the hypothesis of e) and f), $\vec{y}^{(0)}$ could be replaced by $\vec{y}^{(m)}$.

II. APPLICATIONS.

5. THE NUMBER SYSTEM (b(n),D(n)), n>1. PRELIMINARY RESULTS. The diagrams in Figs. 1 and 2 show the central tile of the tessellations derived from the bases −3+i and −2+i, respectively. In the first case the tiles F_j, j = 0, ± 1, ± (3+i), ± (2+i) can be seen and in Fig. 2, the tiles F_j, j = 0, ± 1, ± i, ± (2+i), ± (2+2i), ± (1+i) are shown. If b=−n+i, n≥2, we have $b^2 = n^2 - 1 - 2ni$, $b^3 = -n^3 + 3n + (3n^2 - 1)i$. Thus, −1, b, $-b^2$ and b^3 are in the second quadrant of R^2.

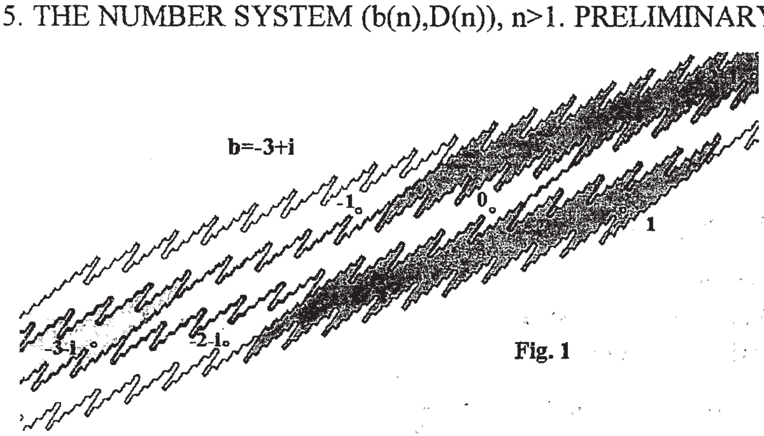


Fig. 1

LEMMA 4. The diameter of $F(n)$, $|F(n)|$, verifies the inequalities

$$|b-1| = |n+1+i| > |F(n)| \geq \frac{n^2}{|b+1|} > |b|. \blacklozenge$$

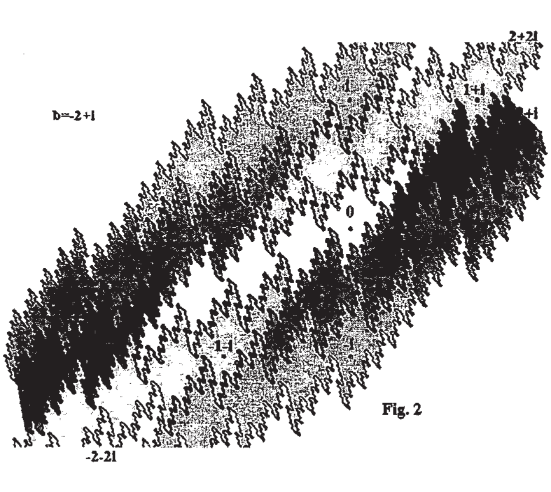


Fig. 2

PROOF. Let $d_j \in \mathbf{R}$ and $z = d_0 + d_1 b + d_2 b^2 + d_3 b^3$, $w = -|d_0| + b|d_1| - b^2|d_2| + b^3|d_3|$.
Then, $|\mathrm{Re}(z)| \le |\mathrm{Re}(w)|$, $|\mathrm{Im}(z)| \le |\mathrm{Im}(w)|$, $|z| \le |w|$. Besides, if $|d_j|$ increases then $|w|$ increases. Therefore, for $d_j \in D-D$, $D=\{0,1,\dots,n^2\}$, we have

$$|z| \le n^2|-1+b-b^2+b^3| = n^2|(b^2+1)(b-1)| = n^3\sqrt{n^2+4}\sqrt{n^2+2n+2}.$$

If $\alpha = \sum_1^\infty c_j b^{-j}, \beta = \sum_1^\infty \tilde{c}_j b^{-j}$, $c_j, \tilde{c}_j \in D$, then $|\alpha - \beta| = \left|\sum_1^\infty d_j b^{-j}\right| =$

$$= \left|\sum_1^\infty b^{-4j}(d_{4j} + d_{4j-1}b + d_{4j-2}b^2 + d_{4j-3}b^3)\right| \le n^3\sqrt{(n^2+4)(n^2+2n+2)}\frac{1}{|b|^4-1}.$$ Thus,

(18) $$|\alpha - \beta| < \sqrt{n^2+2n+2} = |n+1+i| < n + \frac{3}{2}.$$

If $c_{2j} = \tilde{c}_{2j+1} = 0$, $c_{2j+1} = \tilde{c}_{2j} = n^2$ then $\alpha - \beta = n^2/(b+1)$ and the Lemma follows, QED.

6. THE SET *S*. To find the tiles in contact with $F=F(n)$ we have to study a finite set of gaussian integers, S^o, that was introduced in §3. Recall that $L = [1, i]$, (cf. §1).

DEFINITION 7. $S = S(n) := \{\gamma \in L : \exists z = 0.c_{-1}c_{-2}\dots = \gamma.\tilde{c}_{-1}\tilde{c}_{-2}\dots\}$.♦

Thus, $S^o = S\setminus\{0\} = \{\gamma \in L, \gamma \ne 0 : \exists z = 0.c_{-1}\dots = \gamma.\tilde{c}_{-1}\dots\} = \{\gamma = \alpha - \beta \in L : \gamma \ne 0, \alpha, \beta \in F\}$.

It follows from formulae (18) that $S=S(n) \subset B=B(0, |n+1+i|)=$ *the ball* of center 0 and radius $\sqrt{n^2+2n+2}$.

In the next proofs one should keep in mind that if $\gamma = x+iy \in S\ (S^o)$ then

(19) $$\gamma' = \xi + i\eta := b\gamma + \tilde{c}_{-1} - c_{-1} = b\gamma + d \in S\ (S^o),$$
$$\xi = -nx - y + d,\ \eta = x - ny,\qquad d = \tilde{c}_{-1} - c_{-1} \in D - D.$$

PROPOSITION 3. If $x + iy \in S(n)$ then

i) if $|y| \ge 1$ then $|x| \le n$,

ii) $|y| \le 2$.♦

PROOF. If $|y| \ge 1$ then $x^2 + y^2 < n^2 + 2n + 2$ implies that $x^2 < n^2 + 2n + 1 = (n+1)^2$ and then $|x| \le n$. If $|y| > 2$ then $|x - ny| \ge n(|y| - 1) \ge 2n$. That is, $2n \le |\xi + i\eta| < n + 3/2$, a contradiction, QED.

PROPOSITION 4. If $n>2$ then $S(n) \subset B\cap\{\mathbf{Z}\cup(\mathbf{Z}\pm i)\}$.
If $n=2$ then $S \subset B\cap\{\mathbf{Z}\cup(\mathbf{Z}\pm i)\cup\{\pm(2+2i)\}\}$.♦

PROOF. $n>2$: assume that $|y| = 2$. Because of Proposition 3, $|\eta| = |x - ny| \ge n \ge 3$, a contradiction. $n=2$: $|y| = 2 \Rightarrow 2 \ge |\eta| = |x - 2y| \ge 2$. Thus, $2=|x-2y|$. If $y=2$ then $x=2$ and if $y=-2$ then $x=-2$, QED.

PROPOSITION 5. Let $n>2$. Then $S \subset \{0, \pm 1, \pm(n+i), \pm(n-1+i)\}$.
Let $n=2$. Then $S \subset \{0, \pm 1, \pm(2+i), \pm(1+i), \pm i, \pm(2+2i)\}$.♦

PROOF. If $0 \ne x+0i \in S$ then $x \in \mathbf{Z}\setminus\{0\}$ and $\eta=x$. If $n>2$ then, because of Proposition 4, $\xi + i\eta \in S$ implies that $|\eta| \le 1$. So, $|x| \le 1$ and $x=\pm 1$. If $n=2$ then $|\eta| = |x| \le 2$. However, if $\eta=2$ then $\xi=2$. Thus, $2=-2x+d=-4+d$. But this equality cannot be satisfied by any d in $D-D$. If

$\eta=-2$ then $-2=4+d$ and again this is impossible. In consequence, even for $n=2$, $0\neq x+0i \in S$ implies $x=\pm1$.

Let $x+i \in S$. Then, $\eta=x-n$. Using what has been proved and Proposition 4, we have,

a) $x=n \Rightarrow \eta=0 \Rightarrow |\xi|\leq 1 \Rightarrow |-n^2-1+d|\leq 1$ and this can be accomplished with $d=n^2$. Thus, $n+i$ is a possible gaussian integer in S. Since $S=-S$, $-n-i$ is in the same situation,

b) $x=n-1 \Rightarrow \eta=-1 \Rightarrow |\xi|\leq n \Rightarrow |-n^2+n-1+d|\leq n$, thus, $n-1+i$ and its opposite cannot be excluded as possible candidates in S,

c) $x=n-2 \Rightarrow \eta=-2 \Rightarrow n=2 \Rightarrow x=0$, thus, $\pm i$ are, in principle, gaussian integers in $S(2)$ and the proposition follows, QED.

7. A GRAPH ASSOCIATED TO $S^o(n)$. As we already know the gaussian integers of the set S^o identify the tiles in contact with the central tile.

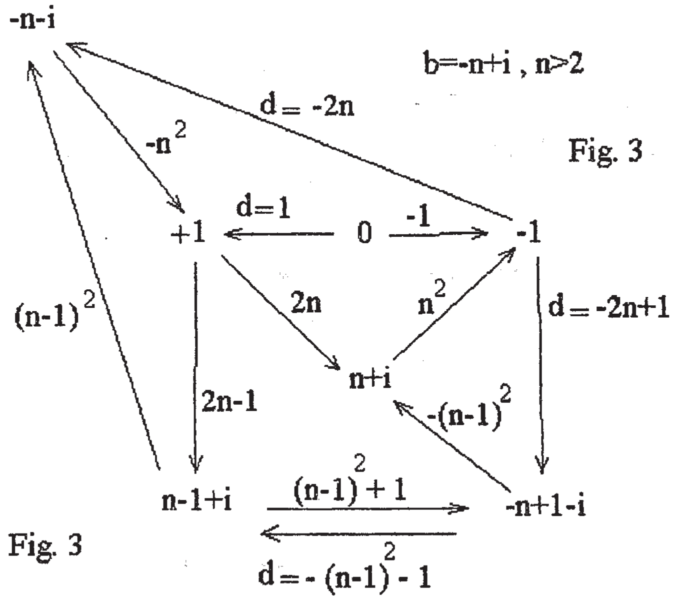


Fig. 3

THEOREM 5. Let $n>2$. Then $S^o(n)=\{\pm 1, \pm(n+i), \pm(n-1+i)\}$.

Let $n=2$. Then $S^o(2)=\{\pm 1, \pm i, \pm(2+i), \pm(1+i), \pm(2+2i)\}$.♦

PROOF. Let us call $T=T(n)$ the set of points inside the brackets. We proved that $S^o(n)\subset T(n)$ for $n\geq 2$. The directed graphs in Figs. 3 and 4 show all the arrows from points $\gamma\in T$ to points $\gamma b+d\in T$ with the real integer $d\in\{-n^2,\ldots,n^2\}$ beside the corresponding arrow. d is obtained using formulae (19) and Proposition 5. From every γ in T starts at least one such arrow. This is sufficient to ensure that $S^o=T$ (cf. [K] or [B] p. 31), QED.

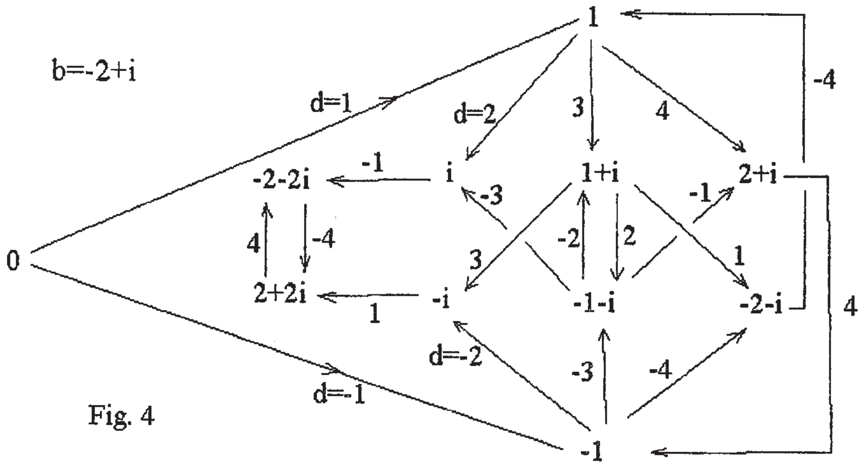


Fig. 4

8. A MATRIX ASSOCIATED TO $S°(n)$. Let d be the integer beside an arrow of the graphs above. If $d \geq 0$, it can be written in $n^2 - d + 1$ ways as a difference of ciphers in D, $d = n^2 - (n^2 - d) = \ldots = d - 0$. In general, $d = \tilde{c}_{-1} - c_{-1}$ can be written in $n^2 - |d| + 1$ different ways as a difference of two ciphers. Thus, an arrow in Figs. 3 and 4 from

$P(n)$ $n>2$	$n-1+i$	1	$n+i$	$-n+1-i$	-1	$-n-i$
$n-1+i$	0	0	0	$2n-1$	0	$2n$
1	n^2-2n+2	0	n^2-2n+1	0	0	0
$n+i$	0	0	0	0	1	0
$-n+1-i$	$2n-1$	0	$2n$	0	0	0
-1	0	0	0	n^2-2n+2	0	n^2-2n+1
$-n-i$	0	1	0	0	0	0

$P(2)$	1	i	$1+i$	$2+i$	$2+2i$	-1		$-1-i$	$-2-i$	$-2-2i$
1	0	3	2	1	0	0	0	0	0	0
i	0	0	0	0	0	0	0	0	0	4
$1+i$	0	0	0	0	0	0	2	3	4	0
$2+i$	0	0	0	0	0	1	0	0	0	0
$2+2i$	0	0	0	0	0	0	0	0	0	1
-1	0	0	0	0	0	0	3	2	1	0
$-i$	0	0	0	0	4	0	0	0	0	0
$-1-i$	0	2	3	4	0	0	0	0	0	0
$-2-i$	1	0	0	0	0	0	0	0	0	0
$-2-2i$	0	0	0	0	1	0	0	0	0	0

$\gamma = x + iy$ to $\delta = b\gamma + d = (-nx - y + d) + i(x - ny) = \xi + i\eta$ in fact represents $n^2 - |d| + 1$ arrows corresponding to the several ways in which d can be written. This information is contained in the matrix $P=P(n)$ shown above where $P_{\gamma\delta} = n^2 - |d| + 1$. Since $P_{-\gamma\ -\delta} = P_{\gamma\delta}$, the matrix P is of the form $P(n)=\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ where $A=A(n)$ and $B=B(n)$ are square matrices of equal order (=3 if $n>2$, =5 if $n=2$). The matrix P was defined in §3.

We deal next with the matrix $Q(n)$, $Q:=A+B$, (see § 4). We have for $n>2$,

$$\begin{array}{cccc} & n-1+i & 1 & n+i & \\ & & & & \\ Q= & \left(\begin{array}{c} 2n-1 \\ n^2-2n+2 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ 0 \\ 1 \end{array} & \left.\begin{array}{c} 2n \\ (n-1)^2 \\ 0 \end{array}\right) & \begin{array}{c} n-1+i \\ 1 \\ n+i \end{array} \end{array}$$

(20) $$\det(Q-xI)=-x^3+(2n-1)x^2+(n^2-2n+1)x+(n^2+1).$$

LEMMA 5. If $n>2$ then $Q(n)$is a nonnegative primitive matrix. ◆
In fact, $Q^j>0$ for $j>2$. Thus, the Perron-Frobenius theory applies to it.

For n=2 we have,

$$Q(2)=\begin{array}{c} \begin{array}{ccccc} 1 & i & 1+i & 2+i & 2+2i \end{array} \\ \begin{pmatrix} 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 2 & 3 & 4 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ i \\ 1+i \\ 2+i \\ 2+2i \end{array}$$

(21) $$\det(Q(2)-xI)=x(x-1)(-x^3+3x^2+x+5).$$

Observe that the cubic factor in the formula (21) coincides with the polynomial (20) if n=2. It is the characteristic polynomial of the matrix $Q_0=\begin{pmatrix} 0 & 2 & 1 \\ 0 & 3 & 4 \\ 1 & 0 & 0 \end{pmatrix}$. Q_0 is a primitive matrix since $Q_0^j>0$ whenever $j>2$.

Let $\lambda=\lambda(n)$ be the (unique positive) eigenvalue of $Q(n)$, n>2, of maximum modulus. It is easy to check that $\lambda>\sqrt{n^2+1}$. $\lambda(2)$, the greatest root of (21), is also an eigenvalue of Q_0 and verifies the preceding inequality. Thus, for $n\geq 2$, $\lambda(n)$ is not only the unique eigenvalue of maximum modulus of Q but it is also equal to the spectral radius of Q or Q_0. Besides, we have the

PROPOSITION 6. $|b|=(n^2+1)^{1/2}<\lambda(n)<n^2+1=|b|^2$. ◆

PROOF. The first inequality was already proved. $\lambda<n^2+1$ is a consequence of the last inequality in the following Theorem 6, QED.

9. THE HAUSDORFF AND BOX DIMENSIONS OF $\partial F(n)$, $n\geq 2$. We introduce next an auxiliary set $S'\subset S°$. Observe that if n>2, $S'=S°$.

DEFINITION 8. *For* $n\geq 2$, $S':=S°\backslash\{\pm i, \pm(2+2i)\}$.◆

THEOREM 6. For $n\geq 2$, $E=\partial F(n)$ and $\lambda=\lambda(n)=\rho(Q(n))$, the following equalities and inequalities hold:

(22) $$1<s=\dim_H(E)=\dim_B(E)=\log\lambda(n)/\log|b|<2.◆$$

PROOF. Assume $n>2$. Let $\vec{v}$ be the normalized positive eigenvector of the matrix $Q=A+B$ correponding to the eigenvalue λ. Then $P=\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ verifies $P\begin{pmatrix} \vec{v} \\ \vec{v} \end{pmatrix}=\lambda\begin{pmatrix} \vec{v} \\ \vec{v} \end{pmatrix}$. Therefore, according to Theorem 4 we have $\log\lambda(n)/\log|b|=\dim_B V_\gamma$ for any γ in S°. Now (22) follows from Theorem 2 and the first inequality in Proposition 6.

Assume $n=2$. If $\vec{v}=(v_1,v_{1+i},v_{2+i})'$ is the normalized positive eigenvector of the matrix Q_0 corresponding to λ then

(23) $$\vec{W}:=(v_1,0,v_{1+i},v_{2+i},0)'$$

is a nonnegative eigenvector of the matrix Q for the eigenvalue $\lambda=\lambda(2)$ and $\begin{pmatrix} A & B \\ B & A \end{pmatrix}\begin{pmatrix} \vec{W} \\ \vec{W} \end{pmatrix}=\lambda\begin{pmatrix} \vec{W} \\ \vec{W} \end{pmatrix}$. Thus, $\log\lambda/\log|b|=\dim_B V_\gamma$ for all γ in S'. To finish the proof of (22) in the case $n=2$, it is sufficient to observe that for $\gamma\in S^\circ\backslash S'=\{\pm i,\ \pm(2+2i)\}$, by next Theorem 7, $card(V_\gamma)<\infty$, (cf. e) Th. 4) and therefore $\dim_B(V_\gamma)=0$, QED.

THEOREM 7. If $n=2$ then card V_i = card V_{-i}= 4; card V_{2+2i}= card V_{-2-2i}= 1.♦

PROOF. It will suffice to prove the theorem for i and $2+2i$ because of the symmetry of the tilings (see Fig. 2 and recall that $S^\circ=-S^\circ$).

We have $M(1,\gamma)>0$. We also have,

(24) $$M(n+1,\gamma_j)=\sum p_{jk}M(n,\gamma_k)$$

where $\gamma_1=1, \gamma_2=i, \gamma_3=1+i, \gamma_4=2+i, \gamma_5=2+2i$ and $\gamma_j=-\gamma_{j-5}$ for $j=6,7,8,9,10$.

If $\vec{Y}^{(n)}=\begin{pmatrix} M(n,\gamma_1) \\ \vdots \\ M(n,\gamma_{10}) \end{pmatrix}$ then $\vec{Y}^{(n+1)}=P\vec{Y}^{(n)}=P^n\vec{Y}^{(1)}$. If $\vec{y}^{(n)}=\begin{pmatrix} M(n,\gamma_1) \\ \vdots \\ M(n,\gamma_5) \end{pmatrix}$ then

$\vec{y}^{(n+1)}=Q\,\vec{y}^{(n)}=Q^n\vec{y}^{(1)}$. Besides, $Q(2)\vec{W}=Q\begin{pmatrix} v_1 \\ 0 \\ v_{1+i} \\ v_{2+i} \\ 0 \end{pmatrix}=\lambda\begin{pmatrix} v_1 \\ 0 \\ \\ v_{2+i} \\ 0 \end{pmatrix}$ where the v_j's are positive. Now we take advantage of the existence of the eigenvalue 1:

(25) $$\vec{y}^{(1)}=\begin{pmatrix} 6 \\ 4 \\ 9 \\ 1 \\ 1 \end{pmatrix}=\begin{pmatrix} 5 \\ 0 \\ 15 \\ 0 \\ 0 \end{pmatrix}+\begin{pmatrix} 1 \\ 4 \\ -6 \\ 1 \\ 1 \end{pmatrix}=\vec{W}'+\vec{V} \text{ where } \vec{V} \text{ verifies } Q\,\vec{V}=\vec{V}.$$

On the other hand observe that there are positive constants c, C, such that the vector $\vec{W}$ defined in (23) verifies: $c\vec{W}\le Q\vec{W}'\le C\vec{W}$. Then, $Q^{n+1}\vec{y}^{(1)}=Q^nQ\vec{W}'+\vec{V}$ implies that

(26) $$\lambda^n c\vec{W}+\vec{V}\le Q^{n+1}\vec{y}^{(1)}\le\lambda^n C\vec{W}+\vec{V}=C\lambda^n\vec{W}+(1,4,-6,1,1)'.$$

Thus, for $n>0$, $c\lambda^{n-1}\begin{pmatrix} v_1 \\ 0 \\ v_{1+i} \\ v_{2+i} \\ 0 \end{pmatrix} + \vec{V} \le \begin{pmatrix} M(n,\gamma_1) \\ \cdot \\ \cdot \\ \cdot \\ M(n,\gamma_5) \end{pmatrix} \le C\lambda^{n-1}\begin{pmatrix} v_1 \\ 0 \\ v_{1+i} \\ v_{2+i} \\ 0 \end{pmatrix} + \vec{V}$. We get from these inequalities: $M(n,i)$=4, $M(n,2+2i)$=1 and $M(n,\gamma) \approx \lambda^{n-1}$ for $\gamma = 1, 1+i, 2+i$.

To prove the theorem it only remains to show that $4 = M(n,i) = card(V_i)$, (cf. e) Theorem 4 and Remark 1). From Fig. 4 we obtain

(27) $\gamma = i \xrightarrow{d=-1} \gamma' = \gamma b + (-1) = -2-2i \xrightarrow{d=-4} 2+2i \xrightarrow{4} -2-2i$.

Now, $d=-1=0-1=1-2=2-3=3-4$, $d=-4=0-4$ and $d=4=4-0$. Thus, we have the eight representations:

(28) $0.(f+1)\overline{40} = i.f\overline{04}$ where f= 0,1,2,3.

They correspond to four infinite strings in the graph $G(S^o(2))$ that represent four different points, (see Fig. 2). Precisely,

(29) $0.(f+1)\overline{40} = \dfrac{3i-(2+i)f}{5}$, $V_i = \left\{\dfrac{3i}{5}, \dfrac{-2+2i}{5}, \dfrac{-4+i}{5}, \dfrac{-6}{5}\right\}$.

Finally, we obtain from (27) that

(30) $V_{2+2i} = \{0.\overline{04}\}$, $\dfrac{2+4i}{5} = 0.\overline{04} = (2+2i).\overline{40}$, QED.

THEOREM 8. If $s = \dim_H \partial F(n)$ then $E = \partial F(n)$ is an s-set. •

PROOF. This is the content of f) Theorem 4, QED.

REMARK 2. E is not always a simple closed curve. In fact, $\partial F(1)$ is a Jordan curve (cf. [BP]) but $\partial F(2)$ is not (see [M]). This may be conjectured after looking at Fig. 2.

10. THE NUMBER SYSTEM (2,{0,1,*i*,1+*i*}). We show with a simple example that the eigenvalue λ in Theorem 4, i.e., the spectral radius of P and Q, is not necessarily of geometric dimension 1. In other words, the eigenvector $\vec{v}$ is not uniquely determined.

Let b=2, $D = \{0,1,i,1+i\}$, $L = [1,i]$. Then, D is a complete set of residues modulus 2 and $bL \cup D \subset L$. Thus, $D-D = \{0, \pm 1, \pm i, \pm(1+i), \pm(1-i)\}$.

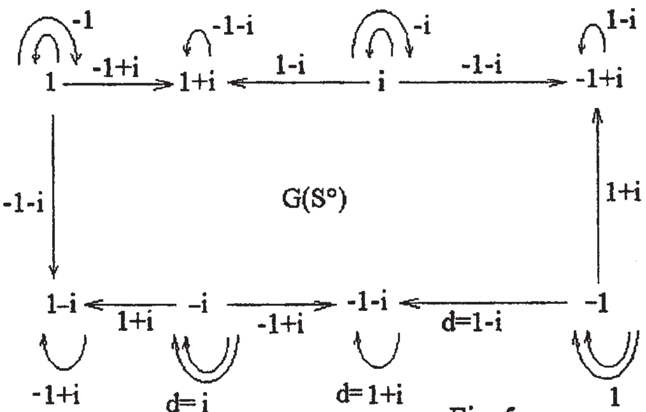


Fig. 5

$\{b,D\}$ satisfies **H**) and **H'**) with $F=\{x+iy: 0\le x\le 1, 0\le y\le 1\}$. Besides, $D-D=$ $S=\{\gamma\in L:\gamma=\alpha-\beta,\ \alpha,\beta\in F\}$. The family of representable numbers is the set of complex numbers with nonnegative real and imaginary parts. The integers of the number system $\{b,D\}$ are the gaussian integers with nonnegative integral real and imaginary parts. In this number system there exist numbers with four representations: $(1+i).\bar{0}=0.\overline{(1+i)}=i.\bar{1}=1.\bar{i}$.
Define $m(d)$ as the number of ways $d\in D-D$ can be written as a difference of two ciphers. So, $m(0)=4$, $m(\pm 1)=2$, $m(\pm i)=2$, $m(\pm(1+i))=1$, $m(\pm(1-i))=1$. The graph $G(S^{o})$, (cf. Def. 2), where $S^{o}=\{\pm 1,\pm i,\pm(1+i),\pm(1-i)\}$, is seen in Fig. 5 with the correct number of arrows for each d. That is, each arrow is repeated $m(d)$ times. The matrix Q is now (cf. §4):

(31)

	1	i	$1+i$	$1-i$
1	2	0	1	1
i	0	2	1	1
$1+i$	0	0	1	0
$1-i$	0	0	0	1

Its spectrum is $\{1,2\}$ and both eigenvalues are of geometric dimension 2. The eigenvectors are $(-1,-1,1,0)'$, $(-1,-1,0,1)'$ and $(1,0,0,0)'$, $(0,1,0,0)'$.

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