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AN N-DIMENSIONAL FORCED PENDULUM EQUATION WITH FRICTION

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ABSTRACT. We study the elliptic boundary value problem

$$\begin{cases} \Delta u + b.\nabla u + a \sin u = p(x) & \text{in } \Omega \\ u|_{\partial\Omega} = \text{constant}, & \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = 0 \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. We prove that for any forcing term p orthogonal to constants there exists a compact interval $I_p \subset \mathbb{R}$ such that the problem is solvable for $\tilde{p}(x) = p(x) + c$ if and only if $c \in I_p$.

1. Introduction

The periodic problem for the forced pendulum equation has been studied by many authors. In 1922 Hamel [6] proved that the equation

$$u'' + a \sin u = \beta \sin t$$

for constant a and β admits a 2π-periodic solution that can be obtained as a minimum of the action functional

$$J(u) = \int_0^{2\pi} \frac{u'(t)^2}{2} + a \cos (u(t)) + u(t)\beta \sin t \, dt.$$

The same argument can be generalized for T-periodic solutions of the equation

$$u'' + a \sin u = p(t) \tag{1.1}$$

where p is T-periodic and orthogonal to constants, leading to the following result (see [7],[8]):

Theorem 1.1. *1.1 has at least one T-periodic solution for any $p \in L^1(\mathbb{R}/T\mathbb{Z})$ such that*

$$\int_0^T p(t) \, dt = 0.$$

If we allow the presence of friction, namely the equation

$$u'' + bu' + a \sin u = p(t) \tag{1.2}$$

(where b is a positive constant) then variational methods are not applicable to the periodic problem. The question of whether or not it was possible to extend Theorem 1.1 to (1.2) remained open until 1987, when Ortega [9] gave a negative answer for a and b large enough. Ten years later Alonso [1] obtained a nonexistence result for arbitrary a and b assuming that T is large. In a more recent work Ortega, Serra and Tarallo [10] have constructed a rather general class of counterexamples that are valid for arbitrary a, b and T.

However, existence results for (1.2) can be obtained by various methods: for example, Fournier and Mawhin proved in [5] the existence of a T-periodic solution of (1.2) using the method of upper and lower solutions and degree arguments. More precisely, they assume the condition

$$0 < \frac{a}{\omega\sqrt{\omega^2+b^2}} \leq \delta(p) \tag{1.3}$$

with $\omega = \frac{2\pi}{T}$, and

$$\delta(p) = \frac{1}{T}\left[\left(\int_0^T \sin\,(P(t))\,dt\right)^2 + \left(\int_0^T \cos\,(P(t))\,dt\right)^2\right]^{1/2} \leq 1$$

where P is the unique T-periodic function satisfying

$$P'' + bP' = p(t) - \frac{1}{T}\int_0^T p(t)\,dt, \qquad \int_0^T P(t)\,dt = 0.$$

If (1.3) holds, there exist $\alpha_1 \leq 0 \leq \alpha_2$ with $|\alpha_i| \geq \delta(p) - \frac{a}{\omega\sqrt{\omega^2+b^2}}$ such that (1.2) admits a T-periodic solution if and only if $a\alpha_1 \leq \frac{1}{T}\int_0^T p(t)\,dt \leq a\alpha_2$.

In this work we consider a generalization of the periodic problem for equation (1.2) to higher dimensions: with this aim, note that the periodic boundary condition can be written as

$$u(0) = u(T) = c, \qquad \int_0^T u''(t)\,dt = 0$$

where c is a non-fixed constant. Thus, by the divergence Theorem the problem may be generalized to a boundary value problem for an elliptic PDE in the following way:

$$\begin{cases} \Delta u + b.\nabla u + a\sin u = p(x) & \text{in } \Omega \\ u|_{\partial\Omega} = \text{constant}, \quad \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = 0 & \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,1}$ domain, $b \in \mathbb{R}^n$ and $p \in L^2(\Omega)$.

This kind of boundary conditions have been considered for example in [3], where the authors study a model describing the equilibrium of a plasma confined in a toroidal cavity.

The paper is organized as follows. First we show that Theorem 1.1 can be extended for the n-dimensional problem (1.4) with $b = 0$. Then we prove by topological methods that for a given p there exists a nonempty closed and bounded interval I_p such that problem (1.4) is solvable for $\tilde{p} = p + c$ if and only if $c \in I_p$. A similar result for the one-dimensional case has been proved by Castro [4] using variational methods, and by Fournier and Mawhin [5], using topological methods.

2. The n-dimensional problem

2.1. Existence by variational methods.

Theorem 2.1. *If $b = 0$ then (1.4) has at least one solution for any $p \in L^2(\Omega)$ such that*

$$\int_\Omega p(x)\, dx = 0.$$

Proof. Consider the functional $\mathcal{I} : \mathbb{R} + H_0^1(\Omega) \to \mathbb{R}$ given by

$$\mathcal{I}(u) = \int_\Omega \frac{|\nabla u(x)|^2}{2} + a \cos\,(u(x)) + p(x)u(x)\, dx.$$

By standard results, $\mathcal{I}$ is weakly lower semicontinuous in $\mathbb{R} + H_0^1(\Omega)$. Moreover, if u is a critical point of $\mathcal{I}$ then

$$\int_\Omega \nabla u(x)\nabla\varphi(x) - a \sin\,(u(x))\varphi(x) + p(x)\varphi(x)\, dx = 0$$

for any $\varphi(x) \in \mathbb{R} + H_0^1(\Omega)$. It follows that $\Delta u + a \sin u = p(x)$, and taking $\varphi \equiv 1$ we deduce that

$$a \int_\Omega \sin\,(u(t))\, dt = \int_\Omega p(t)\, dt = 0.$$

Hence, $\int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_\Omega \Delta u\, dx = 0$ and u is a solution of (1.4).

Furthermore, the functional verifies that

$$\mathcal{I}(u + 2\pi) = \mathcal{I}(u)$$

for any u. Thus, if $\{u_n\} \subset \mathbb{R} + H_0^1(\Omega)$ is a minimizing sequence of $\mathcal{I}$, we may assume that $c_n = u_n|_{\partial\Omega} \in [0, 2\pi]$. By Poincaré's inequality,

$$\|u_n - c_n\|_{H^1}^2 \le c\|\nabla u_n\|_{L^2}^2 \le 2c\Big(\mathcal{I}(u_n) + |a|.|\Omega| + \|p\|_{L^2}\|u\|_{L^2}\Big)$$

It follows that $\{u_n\}$ is bounded, and hence $\mathcal{I}$ has a minimum on $\mathbb{R} + H_0^1(\Omega)$. □

2.2. The maximal interval I_p.

For fixed $p \in L^2(\Omega)$ such that $\int_\Omega p(x)\, dx = 0$, let us consider the problem

$$\text{(2.1)} \qquad \begin{cases} \Delta u + b.\nabla u + a \sin u = p(x) + c & \text{in } \Omega \\ u|_{\partial\Omega} = \text{constant} \qquad \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = 0 & \end{cases}$$

with $c \in \mathbb{R}$. Integrating the equation it follows that necessarily $-|a| \le c \le |a|$. In the next theorem we establish also a sufficient condition. More precisely, if we define

$$I_p = \{c \in \mathbb{R} :\ (2.1) \text{ admits a solution in } H^2(\Omega)\},$$

we shall prove that I_p is a nonempty compact interval. In the particular case $b = 0$, from Theorem 2.1, it follows that $I_p = [A_p, B_p]$, where

$$-|a| \le A_p \le 0 \le B_p \le |a|.$$

In order to prove our assertion, let $P \in H^2 \cap H_0^1(\Omega)$ be the unique function verifying

$$\Delta P + b.\nabla P = p.$$

It follows that $\int_{\partial\Omega} \frac{\partial P}{\partial \nu} = 0$; thus, for $v := u - P$ problem (2.1) becomes

$$\begin{cases} \Delta v + b.\nabla v + a \sin(v + P) = c & \text{in } \Omega \\ v|_{\partial\Omega} = \text{constant} \quad \int_{\partial\Omega} \frac{\partial v}{\partial \nu} = 0 \end{cases} \tag{2.2}$$

For each $v \in L^2(\Omega)$ define

$$c_v = \frac{a}{|\Omega|} \int_\Omega \sin(v(x) + P(x))\, dx$$

We shall need the following lemmas:

Lemma 2.2. *For each $s \in \mathbb{R}$ the integro-differential boundary value problem*

$$\begin{cases} \Delta v + b.\nabla v + a \sin(v + P) = c_v & \text{in } \Omega \\ v|_{\partial\Omega} = s \quad \int_{\partial\Omega} \frac{\partial v}{\partial \nu} = 0 \end{cases} \tag{2.3}$$

admits at least one solution $v \in H^2(\Omega)$.

Proof. For $\tilde{v} \in H^1(\Omega)$ define $v = T\tilde{v}$ as the unique solution of the linear Dirichlet problem

$$\begin{cases} \Delta v + b.\nabla v = c_v - a \sin(v + P) & \text{in } \Omega \\ v|_{\partial\Omega} = s \end{cases}$$

By standard a priori estimates, a straightforward application of Schauder Theorem shows that T has a fixed point v, and integrating the equation it is immediate that $\int_{\partial\Omega} \frac{\partial v}{\partial \nu} = 0$. □

Lemma 2.3. *Let*

$$E = \{v \in H^2(\Omega) : v \text{ is a solution of } (2.3) \text{ for some } s \in \mathbb{R}\}$$

Then the (nonempty) set

$$c(E) := \{c_v : v \in E\} \subset \mathbb{R}$$

is compact. Furthermore, $I_p = c(E)$.

Proof. It is clear that $c \in I_p$ if and only if $c = c_v$ for some $v \in E$. On the other hand, if $v_n \in E$ and $c_n := v_n|_{\partial\Omega}$, by standard a priori estimates we have that $\|v_n - c_n\|_{H^2} \le C$ for some constant C. As $E = E + 2\pi$ and $c_v = c_{v+2\pi}$, from the compact imbedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$ it is immediate that $\{c_{v_n}\}$ has a convergent subsequence.

□

Then we have:

Theorem 2.4. *I_p is a nonempty compact interval.*

Proof. From Lemma 2.3, it suffices to show that I_p is connected. Let $c_1, c_2 \in I_p$, $c_1 < c_2$, and take $v_1, v_2 \in E$ such that $c_{v_i} = c_i$. As $v_i \in C(\overline{\Omega})$, adding $2k\pi$ for some integer k if necessary, we may assume that $v_2 \leq v_1$. For $c \in [c_1, c_2]$ we have that

$$\Delta v_1 + b.\nabla v_1 + a \sin (v_1 + P) = c_1 \leq c \leq c_2 = \Delta v_2 + b.\nabla v_2 + a \sin (v_2 + P)$$

It follows that (v_2, v_1) is an ordered couple of a lower and an upper solution and the proof follows as in [2].

□

Remark 2.1. From Lemma 2.3, E is infinite. In particular, if $I_p = \{c\}$ then (1.4) admits a continuum of solutions. More precisely, if $I_p = \{c\}$ then every solution of

$$\Delta u + b.\nabla u + a \sin u = p(x) + c, \qquad u|_{\partial\Omega} = \text{constant}$$

is a solution of (2.1).

The problem of finding p with $I_p = \{c\}$ or proving that such a p does not exist is still open. In the one-dimensional case

$$u'' + bu' + a \sin u = p(t) \tag{2.4}$$

Ortega and Tarallo have proved in [11] that the following statements are equivalent:

i) $I_p = \{c\}$.

ii) For any $s \in \mathbb{R}$ there exists a unique T-periodic solution u_s of (2.4) such that $u_s(0) = s$.

iii) There exists a continuous path $s \to u_s$ which satisfies

$$\lim_{s\to\pm\infty} u_s(t) = \pm\infty$$

uniformly in t.

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