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A GENERALIZATION OF THE SPACES $\mathcal{H}_\mu$, $\mathcal{H}'_\mu$ AND THE SPACE OF MULTIPLIERS.

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ABSTRACT. The Hankel transformation is defined by A.H. Zemanian ([1]) as follows:

$$\hbar_\mu f = \int_0^\infty f(x)\sqrt{xy}J_\mu(xy)\,dx$$

where $0 < y < \infty$, $\mu \in I\!R$, $\mu \geq -\frac{1}{2}$ and J_μ designates the well-know Bessel function of first kind and order μ. This transformation has been studied in the Zemanian space $\mathcal{H}_\mu$. The testing-function space $\mathcal{H}_\mu$ is countably multinormed and a Fréchet space. Moreover, the Hankel transformation is an automorphism of $\mathcal{H}_\mu$ whenever $\mu \geq -\frac{1}{2}$ and so it allows to define the Hankel transformation in $\mathcal{H}'_\mu$ by the adjoint transformation. In this work, we obtain some characterizations and topological properties of an n-dimensional generalization of the spaces $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$. Certain properties are considered on $\mathcal{O}$, the space of multipliers of $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$.

1. NOTATIONS

Let $I\!R^n$ denote the real n-dimensional euclidean space, $I\!R^n_+$ the n-tuples of positive real numbers. $I\!N$ the set $\{1, 2, 3 \ldots\}$ and $I\!N_0 = I\!N \cup \{0\}$, $\mid x \mid = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$. If $x, y \in I\!R^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, the notations $x < y$ and $x \leq y$ mean, respectively, $x_i < y_i$ and $x_i \leq y_i$ for $i = 1, \ldots, n$. Moreover, $x = a$ for $x \in I\!R^n$, $a \in I\!R$ means $x_1 = x_2 = \cdots = x_n = a$, and e_j for $j = 1, \ldots, n$, denote the members of the canonical basis of $I\!R^n$. An element $k = (k_1, \ldots, k_n) = (k_j) \in I\!N_0^n = I\!N_0 \times I\!N_0 \times \cdots \times I\!N_0$ is called multiindex. For k, m multiindex we set

$$\mid k \mid = k_1 + \cdots + k_n,$$

$$k! = k_1!, \ldots, k_n!,$$

$$\binom{k}{m} = \binom{k_1}{m_1} \cdots \binom{k_n}{m_n},$$

$$\sum_{j=m}^{k} f(j) = \sum_{j_1=m_1}^{k_1} \sum_{j_2=m_2}^{k_2} \cdots \sum_{j_n=m_n}^{k_n} f(j_1, \ldots, j_n).$$

If $x \in I\!R^n$, $x = (x_1, \ldots, x_n)$, we set

$$x^m = x_1^{m_1} \ldots . x_n^{m_n}.$$

If $D_j = \frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$, then a differentiation partial respect to x is denoted by

$$D^k = D_1^{k_1} \ldots D_n^{k_n}$$

2. The spaces $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$.

Let us put $I\!R^n_+ = (0,\infty)\times(0,\infty)\times\cdots\times(0,\infty)$ and μ a n-tuple of real numbers $\mu = (\mu_1,\mu_2,\ldots,\mu_n)$. We define the operators

$$T_i = x_i^{-1}\frac{\partial}{\partial x_i}$$

for $i = 1,\ldots,n$. For multiindex k we shall write

$$T^k = T_n^{k_n}\circ T_{n-1}^{k_{n-1}}\circ\cdots\circ T_1^{k_1},$$

where $T_i^j = T_i\circ\cdots\circ T_i$ (j times), "$\circ$" denote the usual composition. In the following we shall write T_iT_j instead of $T_i\circ T_j$. We define the space $\mathcal{H}_\mu$ as follows

$$\mathcal{H}_\mu = \{\phi\in C^\infty(I\!R^n_+) : \sup_{x\in I\!R^n_+} \mid x^m T^k\Big\{x^{-\mu-\frac{1}{2}}\phi(x)\Big\}\mid < \infty,\ \forall m,k\in I\!N^n_0\} \tag{1}$$

where $\mu\in I\!R^n$ and $-\mu-\frac{1}{2} = (-\mu_1-\frac{1}{2},-\mu_2-\frac{1}{2},\ldots,-\mu_n-\frac{1}{2})$.

Observation 2.1.. *T_i, T, T^k are linear operators such that $T_iT_j = T_jT_i$ for $i,j = 1,\ldots,n$, and $T_i^nT_j^m = T_j^mT_i^n$ for $n,m\in I\!N_0$.*

Observation 2.2.. *$\mathcal{H}_\mu$ is a linear space with a countable collection of seminorms $\{\gamma^\mu_{m,k}\}_{m,k\in I\!N^n_0}$ defined by:*

$$\gamma^\mu_{m,k}(\phi) = \sup_{x\in I\!R^n_+} \mid x^m T^k\Big\{x^{-\mu-\frac{1}{2}}\phi(x)\Big\}\mid . \tag{2}$$

Moreover (2) *is a separating collection of seminorms because $\{\gamma^\mu_{m,0}\}_{m\in I\!N^n_0}$ are norms.*

Observation 2.3.. *Let k be a multiindex, the following equality is valid*

$$T^k\{\theta.\varphi\} = \sum_{j=0}^{k}\binom{k}{j}T^{k-j}\theta.T^j\varphi, \tag{3}$$

where "." denote the usual product of functions, $\binom{k}{j}$ and $\sum_{j=0}^k$ must be interpreted as in section 1 for $j = 0 = (0,\ldots,0)$.

The equality (3) can be derived from the following equation

$$T_i^k\{\theta.\varphi\} = \sum_{j=0}^{k}\binom{k}{j}T_i^{k-j}\theta.T_i^j\varphi \tag{4}$$

valid for $i = 1,\ldots,n$, $k\in I\!N$, which can be obtained by induction on k. Moreover, if $k\in I\!N^n$, $k = (k_1,\ldots,k_n)$ then, we have

$$T_2^{k_2}T_1^{k_1}\{\theta\varphi\} = T_2^{k_2}\Big\{T_1^{k_1}\{\theta\varphi\}\Big\} = T_2^{k_2}\left[\sum_{j_1=0}^{k_1}\binom{k_1}{j_1}T_1^{k_1-j_1}\theta.T_1^{j_1}\varphi\right] =$$

$$= \sum_{j_1=0}^{k_1} \binom{k_1}{j_1} T_2^{k_2}\left\{T_1^{k_1-j_1}\theta . T_1^{j_1}\varphi\right\} =$$

$$= \sum_{j_1=0}^{k_1} \binom{k_1}{j_1} \left[\sum_{j_2=0}^{k_2} \binom{k_2}{j_2} T_2^{k_2-j_2}\left\{T_1^{k_1-j_1}\{\theta\}\right\} . T_2^{j_2}\left\{T_1^{j_1}\{\varphi\}\right\}\right] =$$

$$= \sum_{j_1=0}^{k_1}\sum_{j_2=0}^{k_2} \binom{k_1}{j_1}\binom{k_2}{j_2} \left[T_2^{k_2-j_2}T_1^{k_1-j_1}\{\theta\}\right] . \left[T_2^{j_2}T_1^{j_1}\{\varphi\}\right].$$

Repeating this process we obtain that

$$T^k\{\theta.\varphi\} = T_n^{k_n}\dots T_1^{k_1}\{\theta.\varphi\} =$$

$$= \sum_{j_1=0}^{k_1}\cdots\sum_{j_n=0}^{k_n} \binom{k_1}{j_1}\cdots\binom{k_n}{j_n} \left[T_n^{k_n-j_n}\dots T_1^{k_1-j_1}\{\theta\}\right] . \left[T_n^{j_n}\dots T_1^{j_1}\{\varphi\}\right] =$$

$$\sum_{j=0}^{k} \binom{k}{j} T^{k-j}\theta . T^j\varphi.$$

Lemma 2.1.. *If $\phi \in \mathcal{H}_\mu$, for each multiindex k, $D^k\phi(x)$ is rapid descent as $| x | \to \infty$, (i.e, for each pair of multiindex m, k then $x^m D^k\phi = 0(1)$ as $| x | \to \infty$).*

Proof: Let be $\phi \in \mathcal{H}_\mu$, $k, m \in I\!N_0^n$ we shall prove that there exists $C_{m,k} \in I\!R^+$ such that

$$| x^m D^k\phi(x) | < C_{m,k} \tag{5}$$

for all $x \in B$ where $B = I\!R_+^n - Q$ and $Q = (0,1] \times \cdots \times (0,1]$. From (5) we deduce that $| x^m D^k(\phi) | = 0(1)$ whenever $| x | \to \infty$. We shall use induction on $| k | = k_1 + k_2 + \cdots + k_n$ to prove (5). To do this, we write B as a finite union of disjoint subsets. Considering for each $1 \le s \le n$ the collection $\mathcal{P}_s = \{\mathcal{A}_{j_1\dots j_s}\}_{\substack{j_1,\dots,j_s=1\\ j_1<\dots<j_s}}^n$ such that

$$\mathcal{A}_{j_1\dots j_s} = \left\{x \in I\!R_+^n : x_{j_r} \in (1,\infty), r = 1,\dots,s, \quad \text{y} \quad x_j \le 1 \quad \text{si} \quad j \ne j_r\right\} \tag{6}$$

Note that $\mathcal{P}_0 = \{Q\}$ and $\mathcal{P}_n = (1,\infty)^n$. Let us put

$$\mathcal{P} = \bigcup_{i=1}^n \mathcal{P}_i \ ,$$

then $B = I\!R_+^n - Q = \bigcup_{\mathcal{A}\in\mathcal{P}} \mathcal{A}$.

Now, we are going to consider that $| k | = 0$, $(k = (0,\dots,0))$ and $m \in I\!N_0^n$. We choose $m' \in I\!N_0^n$ such that $m < m' - \mu - \frac{1}{2}$. Since $\phi \in \mathcal{H}_\mu$, there exists a constant $C_{m',0} \in I\!R_+$ which verifies

$$\sup_{x\in I\!R_+^n} | x^{m'-\mu-\frac{1}{2}}\phi(x) | < C_{m',0} \ . \tag{7}$$

Then

$$\sup_{x\in B} | x^m\phi | = \max_{\mathcal{A}\in\mathcal{P}} \{\sup_{x\in\mathcal{A}} | x^m\phi |\}. \tag{8}$$

If the maximal in (8) is attained by some $\mathcal{A}'\in\mathcal{P}$, then there is an integer number s $(1\leq s\leq n)$ such that $\mathcal{A}'\in\mathcal{P}_s$. Let $\mathcal{A}'=\mathcal{A}_{j_1,\ldots,j_s}$ and

$$\mathcal{C}=\{x\in\mathcal{A}' : \ x_i=1 \quad \text{para} \quad i\neq j_1\ldots j_s\},$$

then for $x\in\mathcal{A}'$:

$$| x^m\phi | \leq | x_{j_1}^{m_{j_1}}\ldots x_{j_s}^{m_{j_s}}\phi | \leq | x_{j_1}^{m'_{j_1}-\mu_{j_1}-\frac{1}{2}}\ldots x_{j_s}^{m'_{j_s}-\mu_{j_s}-\frac{1}{2}}\phi | \leq$$
$$\sup_{x\in C} | x^{m'-\mu-\frac{1}{2}}\phi(x) | \leq \sup_{x\in \mathbb{R}^n_+} | x^{m'-\mu-\frac{1}{2}}\phi(x) | \leq C_{m',0}.$$

Next, ϕ is rapid descent. The general case follows by induction on $| k |$ and the following equality, valid for $k\in\mathbb{N}^n$,

$$T^k\left\{x^{-\mu-\frac{1}{2}}\phi\right\} = x^{-\mu-\frac{1}{2}}\left\{\sum_{j=0}^{k} b_{k,j}\frac{D^j\phi}{x^{2k-j}}\right\}, \tag{9}$$

for some constants $b_{k,j}$.

Corollary 2.1.. *If $\phi\in\mathcal{H}_\mu$ and $\mu\geq-\frac{1}{2}$ then $\phi\in L^1(\mathbb{R}^n_+)$.*

Proof: If $\phi\in\mathcal{H}_\mu$ then $\gamma^\mu_{0,0}(\phi)<\infty$ and therefore $\phi(x)=x^{\mu+\frac{1}{2}}\psi(x)$ where ψ is a bounded function in $\mathbb{R}^n_+$. Next, ϕ is bounded in a neighborhood of 0 and by the lemma 2.1 is rapid descent in ∞, then $\phi\in L^1(\mathbb{R}^n_+)$.

Lemma 2.2.. *$\mathcal{H}_\mu$ is a Fréchet space.*

Proof: Let $\{\phi_\nu\}_{\nu\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{H}_\mu$, then if $m,k\in\mathbb{N}^n_0$ and $\varepsilon>0$, exists $N_{\varepsilon,m,k}\in\mathbb{N}$ such that if $\nu,\eta\geq N_{\varepsilon,m,k}$, we have

$$\gamma^\mu_{m,k}(\phi_\nu-\phi_\eta)<\varepsilon. \tag{10}$$

Consequently, from (10) we obtain, for $m=0$, that

$$\sup_{x\in\mathbb{R}^n_+} | T^k\{x^{-\mu-\frac{1}{2}}(\phi_\nu-\phi_\eta)\} | < \varepsilon \qquad \nu,\eta\geq N_{\varepsilon,0,k}\ . \tag{11}$$

Considering the case $k=0$ in (11), we obtain the uniform convergence of $\{\phi_\nu\}$ on compact subsets $K\subset\mathbb{R}^n_+$ whereas $x^{-\mu-\frac{1}{2}}$ is continuous on K for each $\mu\in\mathbb{R}$. By induction on $| k |$ and equality (9) we obtain the uniform convergence on compact subsets of $\mathbb{R}^n_+$ for $\{D^k\phi_\nu\}_{\nu\in\mathbb{N}}$. Therefore, there is a $\phi\in C^\infty(\mathbb{R}^n_+)$ such that $D^k\phi_\nu(x)\to D^k\phi(x)$ when $\nu\to\infty$ for each $k\in\mathbb{N}^n_0$ and $x\in\mathbb{R}^n_+$. Taking $\eta\to\infty$ in (10), we obtain

$$\gamma^\mu_{m,k}(\phi_\nu-\phi)\leq\varepsilon \qquad \forall\nu>N_{\varepsilon,m,k}\ . \tag{12}$$

Moreover, $\gamma^\mu_{m,k}(\phi_\nu)$ are uniformly bounded for $\nu\in\mathbb{N}$ because $\{\phi_\nu\}_{\nu\in\mathbb{N}}$ is a Cauchy sequence. Next, from inequality (12) and the following inequality

$$\gamma^\mu_{m,k}(\phi)\leq\gamma^\mu_{m,k}(\phi-\phi_\nu)+\gamma^\mu_{m,k}(\phi_\nu),$$

we obtain that $\phi \in \mathcal{H}_\mu$.

Observation 2.4.. *$\mathcal{H}_\mu$ with the collection of seminorms $\{\gamma^\mu_{m,k}\}_{m,k\in\mathbb{N}^n}$, is a testing-function space on $\mathbb{R}^n_+$,* ([1], §2.4).

$\mathcal{H}'_\mu$ denote the dual space of $\mathcal{H}_\mu$.

Observation 2.5.. *$\mathcal{H}'_\mu$ is also complete* ([1], §1.8).

Example 2.1.. *For $n = 1$ and $\mu \in \mathbb{R}$, the function $\varphi(x) = x^{\mu+\frac{1}{2}}e^{-x^2} \in \mathcal{H}_\mu$. For $n > 1$ the function $\gamma(x) = x^{\mu+\frac{1}{2}}e^{-|x|^2} = x_1^{\mu_1+\frac{1}{2}} \dots x_n^{\mu_n+\frac{1}{2}}e^{-(x_1^2+\dots+x_n^2)} \in \mathcal{H}_\mu$.*

The following properties are valid for $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$:

1. *Let $\{e_i\}_{i=1,\dots,n}$ be the canonical basis of $\mathbb{R}^n$, then for each even positive integer q, $\mathcal{H}_{\mu+qe_i} \subset \mathcal{H}_\mu$ for $i = 1,\dots,n$.*

   To see this, we first consider $q = 2$. Let be $\phi \in \mathcal{H}_{\mu+2e_i}$ with $\mu + 2e_i = (\mu_1,\dots,\mu_i+2,\dots,\mu_n)$ and $k \in \mathbb{N}^n$, $k = (k_1,\dots,k_n)$, from (3) we obtain

$$T^k\{x^{-\mu-\frac{1}{2}}\phi(x)\} = T^k\{x_i^2 x^{-(\mu+2e_i)-\frac{1}{2}}\phi(x)\} :=$$

$$= \sum_{j=0}^{k}\binom{k}{j}T^{k-j}\{x^{-(\mu+2e_i)-\frac{1}{2}}\phi(x)\}.T^j\{x_i^2\}. \tag{13}$$

   The terms $T^j\{x_i^2\}$ are zero if $j \neq 0$ and $j \neq e_i$. So, we obtain

$$T^k\{x^{-\mu-\frac{1}{2}}\phi(x)\} =$$

$$= \binom{k}{0}T^0\{x_i^2\}T^k\{x^{-(\mu+2e_i)-\frac{1}{2}}\phi(x)\} + \binom{k}{e_i}T^{e_i}\{x_i^2\}T^{k-e_i}\{x^{-(\mu+2e_i)-\frac{1}{2}}\phi(x)\} =$$

$$= x_i^2T^k\{x^{-(\mu+2e_i)-\frac{1}{2}}\phi(x)\} + 2k_iT^{k-e_i}\{x^{-(\mu+2e_i)-\frac{1}{2}}\phi(x)\}.$$

   Multiplying by x^m, $m \in \mathbb{N}^n$, the last formula, we have

$$\gamma^\mu_{m,k}(\phi) \leq \gamma^{\mu+2e_i}_{m+2e_i,k}(\phi) + 2k_i\,\gamma^{\mu+2e_i}_{m,k-e_i}(\phi).$$

   Wherefrom $\phi \in \mathcal{H}_\mu$ y $\mathcal{H}_{\mu+2e_i} \subset \mathcal{H}_\mu$. The general case follows by induction.

2. *The space $\mathcal{D}(\mathbb{R}^n_+)$, (the set of infinitely differentiable functions whose support is a compact set contained in $\mathbb{R}^n_+$), is a subspace of $\mathcal{H}_\mu$ for each $\mu \in \mathbb{R}^n$. Moreover, $\mathcal{D}(\mathbb{R}^n_+)$ is not dense in $\mathcal{H}_\mu$.*

   Since $x^mT^k\{x^{-\mu-\frac{1}{2}}\phi(x)\}$ has compact support for all k and m multiindices and $\phi \in \mathcal{D}(\mathbb{R}^n_+)$, it is clear that $\mathcal{D}(\mathbb{R}^n_+) \subset \mathcal{H}_\mu$. To prove the second statement, we consider the function $\gamma(x) = x^{\mu+\frac{1}{2}}e^{-|x|^2}$ (example 2.9) and the neighborhood of γ:

$$\mathcal{B}_\gamma = \left\{\phi \in \mathcal{H}_\mu : \gamma^\mu_{0,0}(\phi-\gamma) < \frac{1}{2}\right\}.$$

Next, if $\psi \in \mathcal{D}(I\!R^n_+)$, whose support is $K \subset I\!R^n_+$, then we have

$$\gamma^\mu_{0,0}(\psi - \gamma) = \sup_{x \in I\!R^n_+} \mid x^{-\mu-\frac{1}{2}}(\psi(x) - \gamma(x)) \mid \geq$$

$$\geq \sup_{x \in I\!R^n_+ - K} \mid x^{-\mu-\frac{1}{2}}(\psi(x) - \gamma(x)) \mid = \sup_{x \in I\!R^n_+ - K} \mid e^{-|x|^2} \mid = 1.$$

The conclusion is

$$\mathcal{B}_\gamma \bigcap \mathcal{D}(I\!R^n_+) = \emptyset.$$

3. *It is clear that the convergence in $\mathcal{D}(I\!R^n_+)$ implies the convergence in $\mathcal{H}_\mu$, wherefrom we deduce that the restriction of $f \in \mathcal{H}'_\mu$ to $\mathcal{D}(I\!R^n_+)$ is a member of $\mathcal{D}'(I\!R^n_+)$.*
4. *Since $\mathcal{D}(I\!R^n_+) \subset \mathcal{H}_\mu \subset \mathcal{E}(I\!R^n_+)$, where $\mathcal{E}(I\!R^n_+) = \{f : I\!R^n_+ \to \mathbb{C}, f \in C^\infty\}$, $\forall \mu \in I\!R^n$, we deduce the density of $\mathcal{H}_\mu$ in $\mathcal{E}(I\!R^n_+)$.*
5. *The topology of $\mathcal{H}_\mu$ generated by the collection of seminorms $\{\gamma^\mu_{m,k}\}_{m,k \in I\!N}$ is stronger than that induced on it by $\mathcal{E}(I\!R^n_+)$. By density of $\mathcal{H}_\mu$ in $\mathcal{E}(I\!R^n_+)$ we obtain that $\mathcal{E}'(I\!R^n_+)$ is a subspace of $\mathcal{H}'_\mu$ for all $\mu \in I\!R^n$.*

To see this, we consider the collection of seminorms in $\mathcal{E}(I\!R^n_+)$ given by $R = \{\chi_{K',k}\}_{K' \in C, k \in I\!N^n}$, where C denote the class of all the compact subsets of $I\!R^n_+$, so for $\psi \in \mathcal{E}(I\!R^n_+)$, we arrive at

$$\chi_{K',k}(\psi) = \sup_{x \in K'} \mid D^k \psi \mid .$$

Let S the collection of seminorms defined in $\mathcal{H}_\mu$ by (2). Let us show that the following property is valid

$$\forall \chi \in R,\ \exists \gamma_1, \ldots, \gamma_r \in S\ :\ \chi(\phi) \leq C(\gamma_1(\phi) + \cdots + \gamma_r(\phi)), \quad \forall \phi \in \mathcal{H}_\mu \tag{14}$$

Let K' be a compact set and $k = (0, \ldots, 0) = 0$. If $\phi \in \mathcal{H}_\mu$, then

$$\chi_{K',0}(\phi) = \sup_{x \in K'} \mid \phi(x) \mid \leq C \sup_{x \in I\!R^n_+} \mid x^{-\mu-\frac{1}{2}} \phi(x) \mid = C\gamma^\mu_{0,0}(\phi), \tag{15}$$

where $C = \sup_{x \in K'} \mid x^{\mu+\frac{1}{2}} \mid$. If $\mid k \mid = 1$, $k = e_i$ we obtain, by (9), that

$$T^{e_i}\{x^{-\mu-\frac{1}{2}}\phi\} = x^{-\mu-\frac{1}{2}}\left\{b_{e_i,0}\frac{\phi}{x_i^2} + b_{e_i,e_i}\frac{D^{e_i}\phi}{x_i}\right\}.$$

Hence

$$\sup_{x \in K'} \mid D^{e_i}\phi(x) \mid \leq$$

$$\leq M\left\{\frac{c_1}{b_{e_i,e_i}} \sup_{x \in K'} \mid T^{e_i}\{x^{-\mu-\frac{1}{2}}\phi(x)\} \mid + c_2 \frac{b_{e_i,0}}{b_{e_i,e_i}} \sup_{x \in K'} \mid \phi(x) \mid\right\},$$

where $M = \sup_{x\in K'} \mid x_i \mid$, $c_1 = \sup_{x\in K'} \mid x^{\mu+\frac{1}{2}} \mid$, $c_2 = \sup_{x\in K'} \mid \frac{1}{x_i^2} \mid$. Next, taking into account (16), we obtain

$$\chi_{K',e_i}(\phi) \leq M\left\{c'_1\gamma^{\mu}_{e_i,0}(\phi) + c'_2\chi_{K',0}(\phi)\right\} \leq C_1\left\{\gamma^{\mu}_{e_i,0}(\phi) + \gamma^{\mu}_{0,0}(\phi)\right\}.$$

The property (14) is obtained by induction on $\mid k \mid$ and the equatity (9). Finally, we obtain that the topology generated by S is stronger than that generated by R.

6. *Let $f : \mathbb{R}^n_+ \to \mathbb{C}$ be a locally integrable function on $\mathbb{R}^n_+$, such that f is of slow growth at infinite ($\exists r \in \mathbb{N}$ such that $\mid f(x) \mid= O(\mid x \mid^r)$ as $\mid x \mid\to \infty$) and $x^{\mu+\frac{1}{2}}f(x)$ is absolutely integrable on $Q = (0,1)^n \subset \mathbb{R}^n_+$ for $\mu \in \mathbb{R}^n$. Then, f define a regular generalized function in $\mathcal{H}'_\mu$ given by:*

$$(f,\phi) = \int_{\mathbb{R}^n_+} f(x)\phi(x)\,dx, \text{ para cada } \phi \in \mathcal{H}_\mu .$$

7. *$\mathcal{H}_\mu$ can be identified with a subspace of $\mathcal{H}'_\mu$ if $\mu \geq -\frac{1}{2}$.*

Given a function $f \in \mathcal{H}_\mu$, by property 6, Lemma 2.1 and Corollary 2.1, it can be considered as an element of $\mathcal{H}'_\mu$. Since the functions of $\mathcal{H}_\mu$ are continuous , if $f, g \in \mathcal{H}_\mu$ such that $f \neq g$, then there exists an open set of $\mathbb{R}^n_+$ where $f \neq g$ and then there is a function $\phi \in \mathcal{D}(\mathbb{R}^n_+)$ such that $(f,\phi) \neq (g,\phi)$.

3. The space $\mathcal{O}$

Let $\mathcal{O}$ be the space of functions $\theta \in C^\infty(\mathbb{R}^n_+)$ with the property that for every $k \in \mathbb{N}^n$ there exists $n_k \in \mathbb{Z}$ and $C \in \mathbb{R}_+$ such that

$$\mid (1+ \mid x \mid^2)^{n_k} T^k\theta \mid< C \quad \forall x \in \mathbb{R}^n_+.$$

Observation 3.1.. *The product of members of $\mathcal{O}$ is a member of $\mathcal{O}$.*

It follows from (3).

Lemma 3.1.. *The operator $\phi \mapsto \theta\phi$, where $\theta \in \mathcal{O}$, is a continuous operator of $\mathcal{H}_\mu$ into itself. Moreover, the adjoint operator $f \mapsto \theta f$ defined on $\mathcal{H}'_\mu$ by*

$$\langle\theta f,\phi\rangle = \langle f,\theta\phi\rangle \quad f \in \mathcal{H}'_\mu,\ \theta \in \mathcal{O},\ \phi \in \mathcal{H}_\mu ,$$

is a lineal and continuous operator of $\mathcal{H}'_\mu$ into itself.

Proof: Let $\theta \in \mathcal{O}$, $\phi \in \mathcal{H}_\mu$ and $m, k \in \mathbb{N}^n$. In view of (3), we obtain

$$\mid x^m T^k\{x^{-\mu-\frac{1}{2}}\theta\phi(x)\} \mid=\mid x^m \sum_{j=0}^{k}\binom{k}{j} T^j\theta.T^{k-j}\{x^{-\mu-\frac{1}{2}}\phi(x)\} \mid\leq$$

$$\leq \sum_{j=0}^{k}\binom{k}{j} C_j \mid (1+ \mid x \mid^2)^{-n_k} x^m T^{k-j}\{x^{-\mu-\frac{1}{2}}\phi(x)\} \mid, \tag{16}$$

where the constant C_j satisfies $|(1+|x|^2)^{n_j} T^j\theta| < C_j \quad \forall x \in I\!R^n_+$. Next, for every multiindex j there exist a finite set of multiindex Γ_j such that

$$|x^m T^k\{x^{-\mu-\frac{1}{2}}\theta\phi(x)\}| \leq \sum_{j=0}^{k} \binom{k}{j} C_j \left(\sum_{\alpha\in\Gamma_j} \gamma^{\mu}_{\alpha,k-j}(\phi) \right). \tag{17}$$

Therefore, we deduce that $\theta\phi \in \mathcal{H}_\mu$ and the continuity of $\phi \mapsto \theta\phi$.

Lemma 3.2.. *Let $P(x)$ and $Q(x)$ be polynomials of one variable such that $Q(x)$ has no zeros on $0 \leq x < \infty$. Then $\frac{P(|x|^2)}{Q(|x|^2)} \in \mathcal{O}$ for $x = (x_1, \ldots, x_n) \in I\!R^n$.*

Proof: If $k \in I\!N^n$, then

$$T^k\Big\{P(|x|^2)\Big\} = 2^{|k|}P^{|k|}(|x|^2),$$

where $P^{|k|}$ denotes the derivative of order $|k|$ of P. Therefore $P(|x|^2) \in \mathcal{O}$. On the other hand, we have

$$T^k\left\{\frac{1}{Q}(|x|^2)\right\} = 2^{|k|}\left(\frac{1}{Q}\right)^{|k|}(|x|^2).$$

The expression of $\frac{1}{Q}^{|k|}$ has the form $\frac{N}{Q^{2^{|k|}}}$ where $gr(N) < gr(Q^{2^{|k|}})$. Since Q has no zeros in $[0, \infty)$ then $\frac{1}{Q(|x|^2)} \in \mathcal{O}$, and by observation 3.1 we conclude that $\frac{P(|x|^2)}{Q(|x|^2)} \in \mathcal{O}$.

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