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SOME REMARKS ON OCKHAM CONGRUENCES

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ABSTRACT. In this work we shall describe the lattice of congruences of an Ockham algebra whose quotient algebras are in the Urquhart clases $\mathbf{P}_{m,n}$. This description is obtained using the Duality for Ockham algebras given by Urquhart (see [3]). This work is a natural generalization for some of the results obtained by Rodriguez and Silva in [5].

1. PRELIMINARIES

In [5] Rodriguez and Silva describe the lattice of congruences of an Ockham algebra whose quotient algebras are Boolean. Given an Ockham algebra, they characterize them in two different ways, one by means of pro-boolean ideals and the other using the set of fixed points of the dual space. Here we will give a generalization of this results describing the lattice of congruences whose quotient algebras belong to the subvarieties of Ockham algebras defined by Urquhart (see [3]). We will see that this congruences do not admit a description by means of ideals, but they can be described by means of some subsets of the dual space.

In this section we will recall the definitions, results and notations that will be needed in the rest of the paper.

In section 2 we will introduce the set $Con_{m,n}(\mathbf{O})$ for every Ockham algebra and develop the main results of this paper.

Given $\langle X, \leq \rangle$ a poset, we will say that a subset $Y \subseteq X$ is *increasing* if for every $y \in Y$ and for every $x \in X$ such that $y \leq x$, then $x \in Y$. A map

$$g : X \longrightarrow X$$

is an *order reversing* map if for every $x, y \in X$ such that $x \leq y$, $g(y) \leq g(x)$.

If X is a set and $Y \subseteq X$, when there is no risk of misunderstanding, we will note $Y^c = X \backslash Y$.

Given a lattice $\mathbf{L}$ we will note the set of atoms of $\mathbf{L}$ by $At(\mathbf{L})$, and with $CoAt(\mathbf{L})$ the set of co-atoms of $\mathbf{L}$.

Definition 1. *An algebra* $\mathbf{O} = \langle O, \wedge, \vee, f, 0, 1 \rangle$ *of type* $(2, 2, 1, 0, 0)$ *is an Ockham algebra if it verifies the following conditions:*

O1 $\langle O, \wedge, \vee, 0, 1 \rangle$ *is a bounded distributive lattice.*
O2 $f(0) = 1$, $f(1) = 0$.
O3 $f(a \wedge b) \approx f(a) \vee f(b)$.
O4 $f(a \vee b) \approx f(a) \wedge f(b)$.

For the rest of the paper $\mathbf{O} = \langle O, \wedge, \vee, f, 0, 1 \rangle$ will be an arbitrary Ockham algebra.

Let $\emptyset \neq F \subseteq O$. We will say that F is a filter (prime filter) of $\mathbf{O}$ if and only if F is a filter (prime filter) of the lattice reduct of $\mathbf{O}$. We will note by $X(\mathbf{O})$ the set of all prime filters of $\mathbf{O}$.

Let us denote $Con(\mathbf{O})$ the congruence lattice of $\mathbf{O}$. As usual, we denote by ω the least element of $Con(\mathbf{O})$, and by ι the greatest element of $Con(\mathbf{O})$.

In [3] Urquhart develops a Priestley style duality for the algebraic category of Ockham algebras. Here we will recall some of these results. For the proof of the results in this section see [3] or see [2, Chapter 4].

Let us recall that a totally order-disconnected topological space is a triple $\langle X, \leq, \tau \rangle$ such that $\langle X, \leq \rangle$ is a poset, $\langle X, \tau \rangle$ is a topological space and given $x, y \in X$ such that $x \nleq y$ there is a clopen increasing set U such that $x \in U$ and $y \notin U$. A *Priestley space* is a compact totally order-disconnected topological space.

Definition 2. *A structure $\mathbf{X} = \langle X, \leq, \tau, g \rangle$ is an Ockham space if the following conditions hold:*

1. *$\langle X, \leq, \tau \rangle$ is a Priestley space,*
2. *$g : X \to X$ is a continuous order-reversing map.*

We denote $\varphi(a) = \{P \in X(\mathbf{O}) : a \in P\}$. Then the structure $\langle X(\mathbf{O}), \subseteq, \tau_{\mathbf{O}}, g_{\mathbf{O}} \rangle$, where $\tau_{\mathbf{O}}$ is the topology generated by the base

$$B = \{\varphi(a), X(\mathbf{O}) \setminus \varphi(a) : a \in \mathbf{O}\}$$

and $g_{\mathbf{O}}$ is defined by:

$$g_{\mathbf{O}}(P) = \{a \in O : f(a) \notin P\},$$

for each $P \in X(\mathbf{O})$, is an Ockham space called the *dual space* of $\mathbf{O}$.

Conversely if $\mathbf{X}$ is an Ockham space, then $\langle O(\mathbf{X}), \cap, \cup, f, \emptyset, X \rangle$, where

$$O(\mathbf{X}) = \{U \subseteq X : U \text{ is a clopen increasing subset of } X\}$$

and

$$f(U) = X \setminus g^{-1}(U),$$

for each $U \in O(\mathbf{X})$, is an Ockham algebra. Moreover, these constructions give a dual equivalence. The arrow part of the duality will not be developed because it plays no relevance for the aim of this work.

A subset Y of an Ockham space $\mathbf{X}$ is called a *g-set* if for every $x \in Y$, $g(x) \in Y$. We will say that Y is a *g-closed* set when it is closed (in the topology) and a g-set. For every $x \in X$, we will note $g^{\omega}(x) = \{g^n(x) : n \in \mathbb{N}\}$. It is easy to see that $Y \subseteq X$ is a g-set if and only if for every $x \in Y$, $g^{\omega}(x) \subseteq Y$. We will note by $G(\mathbf{X})$ the set of all g-closed subsets of X.

Theorem 3. *Let consider the following map*

$$\Phi : G(X(\mathbf{O})) \longrightarrow Con(\mathbf{O})$$

defined by

$$\Phi(Y) = \{(a,b) \in O \times O : \varphi(a) \cap Y = \varphi(b) \cap Y\},$$

for each $Y \in G(X(\mathbf{O}))$. Then ϕ is a dual isomorphism from the lattice $G(X(\mathbf{O}))$ to the lattice $Con(\mathbf{O})$. The inverse map of Φ is the map

$$\mathscr{C} : Con(\mathbf{O}) \longrightarrow G(X(\mathbf{O}))$$

defined by

$$\mathscr{C}(\theta) = \left\{\rho_\theta^{-1}(Q) : Q \in X(\mathbf{O}/\theta)\right\},$$

where $\theta \in Con(\mathbf{O})$, and $\rho_\theta : \mathbf{O} \longrightarrow \mathbf{O}/\theta$ is the the canonical projection.

For $m,n \in \mathbb{N}$ and $m > n$, Urquhart introduces the class $\mathbf{P}_{m,n}$ of Ockham algebras formed by those algebras whose dual space satisfies $g^m = g^n$.

Remark 4. *It is easy to see that if $m,n,p,q \in \mathbb{N}$ such that $m > n$ and $p > q$, then $\mathbf{P}_{p,q} \subseteq \mathbf{P}_{m,n}$ if and only if $p-q \mid m-n$, $q \leq n$ and $p \leq q$.*

For a proof of the next results see [3].

Theorem 5. *For every $m,n \in \mathbb{N}$ and $m > n$, $\mathbf{P}_{m,n}$ has only finitely many subdirectly irreducible algebras, all of which are themselves finite.*

Theorem 6. *Let $m,n \in \mathbb{N}$ with $m > n$, and let $\mathbf{O} \in \mathbf{P}_{m,n}$. If $\mathbf{O}$ is simple, then $\mathbf{O} \in \mathbf{P}_{m-n,0}$.*

The following Theorem characterizes the classes $\mathbf{P}_{m,n}$.

Theorem 7. *Let $m,n \in \mathbb{N}$ such that $m > n$. Then $\mathbf{O} \in \mathbf{P}_{m,n}$ if and only if it satisfies the following properties:*

1. *If $m-n$ is even, for every $a \in O$*
   (a) $f^m(a) = f^n(a)$.
2. *If $m-n$ is odd, for every $a \in O$*
   (a) $f^m(a) \vee f^n(a) = 1$.
   (b) $f^m(a) \wedge f^n(a) = 0$.

The previous Theorem proves that the classes $\mathbf{P}_{m,n}$ are subvarieties of the variety of Ockham algebras. Clearly $\mathbf{P}_{1,0}$ is the variety of Boolean algebras.

For every $p,q \in \mathbb{N}$, Berman (see [1] or [2, Chapter 1]) introduces the Berman classes $\mathbf{K}_{p,q}$ of Ockham algebras. An Ockham algebra belongs to the Berman class $\mathbf{K}_{p,q}$ if and only if it satisfies the equation

$$f^{2.p+q}(a) = f^q(a).$$

It follows that $\mathbf{K}_{p,q} = \mathbf{P}_{2p+q,q}$. Moreover an Urquhart class $\mathbf{P}_{m,n}$ is a Berman class if and only if $m-n$ is even.

The following result generalizes the Corollary of Theorem 2.7 of [2, Chapter 2] and will be useful in the next section.

Theorem 8. *Let $m,n \in \mathbb{N}$ such that $m > n$. If $\mathbf{O} \in \mathbf{P}_{m,n}$ then the following propositions are equivalent:*

1. *f is injective.*
2. $\mathbf{O} \in \mathbf{P}_{m-n,0}$.

Proof. If $m-n$ is even, the result follows from the Corollary above mentioned, because in this case $\mathbf{O}$ belongs to the Berman class $\mathbf{K}_{(m-n)/2,n}$.

Consider that $m-n$ is odd.

If we suppose that f is injective, then by Theorem 7, we have that for every $a \in O$

$$f^m(a) \vee f^n(a) = 1,$$
$$f^m(a) \wedge f^n(a) = 0.$$

Suppose that n is even. Then

$$f^n\left(f^{m-n}(a)\vee a\right)=f^m(a)\vee f^n(a)=1.$$

Since f is injective, we have that

$$f^{m-n}(a)\vee a=1$$

for every $a\in O$. In the same way we obtain that $f^{m-n}(a)\wedge a=0$ for every $a\in O$. Then $\mathbf{O}\in\mathbf{P}_{m-n,0}$. If n is odd the proof is similar.

To prove the converse suppose that $\mathbf{O}\in\mathbf{P}_{m-n,0}$. By Theorem 7 we have that every $a\in O$ is a complemented element and its complement if $f^{m-n}(a)$. If $a,b\in O$ are such that $f(a)=f(b)$, then $f^{m-n}(a)=f^{m-n}(b)$. Since $\mathbf{O}$ is a distributive lattice and a and b have the same complement, $a=b$. Therefore f is injective. ■

2. THE LATTICE $Con_{m,n}(\mathbf{O})$

Given $m,n\in\mathbb{N}$ such that $m>n$ we will consider the following subset of $Con(\mathbf{O})$

$$Con_{m,n}(\mathbf{O})=\{\theta\in Con(\mathbf{O}):\mathbf{O}/\theta\in\mathbf{P}_{m,n}\}$$

In [5] Rodriguez and Silva study the lattice of congruences of an Ockham algebra whose quotient algebras are Boolean algebras, this is clearly the lattice $Con_{1,0}(\mathbf{O})$.

For completeness we will prove the following Lemma that generalizes item 2 of Theorem 2 of [5].

Lemma 9. *Let $\mathscr{V}$ be variety of algebras of type $\mathscr{F}$ and $\mathbf{A}$ an algebra of type $\mathscr{F}$. Consider*

$$Con_{\mathscr{V}}(\mathbf{A})=\{\theta\in Con(\mathbf{A}):\mathbf{A}/\theta\in\mathscr{V}\}.$$

Then $Con_{\mathscr{V}}(A)$ is a complete filter of $Con(\mathbf{A})$.

Proof. Since $\mathscr{V}$ is a non empty variety, every trivial algebra of type $\mathscr{F}$ belongs to $\mathscr{V}$. Then $\mathbf{A}/\iota\in\mathscr{V}$, i.e., $\iota\in Con_{\mathscr{V}}(\mathbf{A})$.

If $\theta\in Con_{\mathscr{V}}(\mathbf{A})$ and $\theta\subseteq\phi\in Con(\mathbf{A})$, then by the Correspondence Theorem (see [4] Theorem 6.20) there exists $\sigma\in Con(\mathbf{A}/\theta)$ such that $\mathbf{A}/\phi$ is isomorphic to $(\mathbf{A}/\theta)/\sigma$. Since $\mathscr{V}$ is a variety we conclude that $\mathbf{A}/\phi\in\mathscr{V}$, i.e., $\phi\in Con_{\mathscr{V}}(\mathbf{A})$.

Let $\{\theta_i\}_{i\in I}$ be an arbitrary set of elements of $Con_{\mathscr{V}}(\mathbf{A})$ and let Σ be a set of equations such that $\mathscr{V}$ is axiomatized by Σ. Consider $\theta=\bigcap_{i\in I}\theta_i$. If $p(x_1,\ldots,x_n)\approx q(x_1,\ldots,x_n)$ is an equation in Σ, then for every $i\in I$ and every $a_1,\ldots,a_n\in A$ we have that

$$(p(a_1,\ldots,a_n),q(a_1,\ldots,a_n))\in\theta_i.$$

Thus $(p(a_1,\ldots,a_n),q(a_1,\ldots,a_n))\in\theta$. It follows that $\mathbf{A}/\theta$ satisfies $p\approx q$ for every equation $p\approx q$ in Σ. Therefore $\mathbf{A}/\theta\in\mathscr{V}$, i.e., $\theta\in Con_{\mathscr{V}}(\mathbf{A})$. ■

Theorem 10. *The following propositions hold:*

1. *For every $m,n\in\mathbb{N}$ such that $m>n$, $Con_{m,n}(\mathbf{O})$ is a complete filter.*
2. *For every $m,n,p,q\in\mathbb{N}$ such that $m>n$ and $p>q$, $Con_{m,n}(\mathbf{O})\subseteq Con_{p,q}(\mathbf{O})$ if and only if $p-q\mid m-n$, $q\leq n$ and $p\leq q$.*
3. *If $\theta\in Con_{m,n}(\mathbf{O})$ is a co-atom of $Con_{m,n}(\mathbf{O})$, then $\mathbf{O}/\theta$ is finite and $\theta\in Con_{m-n,0}(\mathbf{O})$.*

Proof. 1. It follows directly from Theorem 7 and Lemma 9.

2. It follows directly from Remark 4.

3. If θ is a co-atom of $Con_{m,n}(\mathbf{O})$, then $\mathbf{O}/\theta$ is simple and belongs to $\mathbf{P}_{m,n}$. By Theorem 5 $\mathbf{O}/\theta$ is finite. By Corollary 6 we have that $\mathbf{O}/\theta \in \mathbf{P}_{m-n,0}$. ■

Remark 11. By item 1 of the previous Theorem we have that the lattice $Con_{m,n}(\mathbf{O})$ is a bounded lattice. We will call $\sigma_{m,n}$ its first element. It is an easy consequence of the Correspondence Theorem that the lattice $Con_{m,n}(\mathbf{O})$ is isomorphic to the lattice $Con(\mathbf{O}/\sigma_{m,n})$.

Consider the map

$$\mathscr{I} : Con(\mathbf{O}) \longrightarrow Id(\mathbf{O})$$

as defined by

$$\mathscr{I}(\theta) = [0]_\theta$$

for each $\theta \in Con(\mathbf{O})$, where $Id(\mathbf{O})$ is the set of lattice ideals of $\mathbf{O}$ and if $a \in O$, $[a]_\theta$ is the congruence class of a. In [5] Rodriguez and Silva prove that $\mathscr{I}$ restricted to the congruences whose quotient algebras are Boolean, is an order isomorphism and its image are the pro-boolean ideals of $\mathbf{O}$. In the next example we will see that there are Ockham algebras where this property does not hold if we restrict $\mathscr{I}$ to $Con_{m,n}(\mathbf{O})$ when $(m,n) \neq (1,0)$.

Example 12. Consider the following Ockham algebra $\mathbf{O}$,

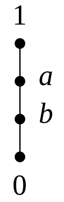

Fig. 1

where $f(a) = b$ and $f(b) = a$. By Theorem 7, $\mathbf{O} \in \mathbf{P}_{2,0}$. Since $\mathbf{P}_{2,0}$ is a variety, $Con_{2,0}(\mathbf{O}) = Con(\mathbf{O})$. Consider

$$\theta = \{(0,0),(a,a),(b,b),(1,1),(a,b),(b,a)\}$$

and let ω be the least element of $Con(\mathbf{O})$. Clearly $[0]_\theta = \{0\} = [0]_\omega$. Thus $\mathscr{I}$ is not injective.

Moreover, note that $[1]_\theta = \{1\} = [1]_\omega$. Thus a congruence in $Con_{2,0}(\mathbf{O})$ is not determined neither by $[0]_\theta$ nor $[1]_\theta$.

Given an Ockham space $\mathbf{X} = \langle X, \leq, \tau, g \rangle$ and $m,n \in \mathbb{N}$ with $m > n$, we will consider the following set

$$Fix_{m,n}(\mathbf{X}) = \{x \in X : g^m(x) = g^n(x)\}.$$

It is easy to see that for every $m,n \in \mathbb{N}$, $Fix_{m,n}(\mathbf{X})$ is a closed g-subset of $\mathbf{X}$. Consider $\Omega_{m,n}(\mathbf{X})$ the set of closed g-subsets of $Fix_{m,n}(\mathbf{X})$. Clearly

$$\Omega_{m,n}(\mathbf{X}) = (Fix_{m,n}(\mathbf{X})] \cap G(\mathbf{X}).$$

Theorem 13. *Let $m,n \in \mathbb{N}$ such that $m > n$. Then the lattice $Con_{m,n}(\mathbf{O})$ is dually isomorphic to $\Omega_{m,n}(\mathbf{X}(\mathbf{O}))$.*

Proof. Let $\mathscr{C} : Con(\mathbf{O}) \longrightarrow Fix(\mathbf{X}(\mathbf{O}))$ defined in Theorem 3. We only have to prove that the image of $Con_{m,n}(\mathbf{O})$ is $\Omega_{m,n}(\mathbf{X}(\mathbf{O}))$.

Let $\theta \in Con_{m,n}(\mathbf{O})$. If $P \in \mathscr{C}(\theta)$, then there exists $Q \in X(\mathbf{O}/\theta)$ such that $\rho_\theta^{-1}(Q) = P$. Since $\mathbf{O}/\theta \in P_{m,n}$,

$$\begin{aligned}(g_{\mathbf{O}})^m(P) &= (g_{\mathbf{O}})^m\left(\rho_\theta^{-1}(Q)\right)\\ &= \rho_\theta^{-1}\left((g_{\mathbf{O}}/\theta)^m(Q)\right)\\ &= \rho_\theta^{-1}\left((g_{\mathbf{O}}/\theta)^n(Q)\right) = (g_{\mathbf{O}})^n(P).\end{aligned}$$

Thus $\mathscr{C}(\theta) \in \Omega_{m,n}(\mathbf{X}(\mathbf{O}))$.

If $Y \in \Omega_{m,n}(\mathbf{X}(\mathbf{O}))$, then $\mathbf{O}/(\theta(Y)) \cong O\langle Y, \subseteq, \tau_Y, g_y\rangle$ where τ_Y is the restriction of $\tau_{\mathbf{O}}$ to the subset Y, and $g_Y = g|_Y$. Since $Y \in \Omega_{m,n}(\mathbf{X}(\mathbf{O}))$, $O\langle Y, \subseteq, \tau_Y, g_y\rangle \in P_{m,n}$, and the result follows. ■

Since $Fix_{m,n}(\mathbf{X}(\mathbf{O}))$ is the last element of $\Omega_{m,n}(\mathbf{X}(\mathbf{O}))$, from the previous theorem we have that

$$\mathscr{C}(\sigma_{m,n}) = Fix_{m,n}(\mathbf{X}(\mathbf{O})).$$

Theorem 14. *Let $\mathbf{X}$ be an Ockham space and $m,n \in \mathbb{N}$ such that $m > n$. Then the set of atoms of $\Omega_{m,n}(\mathbf{X})$ is the set*

$$At(\Omega_{m,n}(\mathbf{X})) = \{g^\omega(x) : x \in Fix_{m-n,0}(\mathbf{X})\}.$$

Proof. It follows directly from Theorems 3, 10 and 13. ■

Note that if $x \in \Omega_{m,n}(\mathbf{X}) \setminus \Omega_{m-n,0}(\mathbf{X})$, then

$$g^\omega(x) = C \neq \bigcup_{\substack{F \in At(\Omega_{m,n}(\mathbf{X}))\\ F \subseteq C}} F.$$

We conclude the following Corollary.

Corollary 15. *Let $m,n \in \mathbb{N}$ such that $m > n$. Then the following propositions are equivalent:*

1. *For every $\theta \in Con_{m,n}(\mathbf{O})$,*

$$\theta = \bigcap\{\phi : \phi \in CoAt(Con_{m,n}(\mathbf{O})) \text{ and } \theta \subseteq \phi\}.$$

2. $Con_{m,n}(\mathbf{O}) = Con_{m-n,0}(\mathbf{O})$.
3. $\Omega_{m,n}(\mathbf{X}(\mathbf{O})) = \Omega_{m-n,0}(\mathbf{X}(\mathbf{O}))$.
4. $Fix_{m,n}(\mathbf{X}(\mathbf{O})) = Fix_{m-n,0}(\mathbf{X}(\mathbf{O}))$.

Proof. By the previous observation and Theorem 14, 1 and 2 are equivalent.

Clearly by Theorem 13 and the definitions of $Fix_{m,n}(\mathbf{X}(\mathbf{O}))$ and $\Omega_{m,n}(\mathbf{X}(\mathbf{O}))$, items 2, 3 and 4 are equivalent. ■

Theorem 16. *Let $m,n \in \mathbb{N}$ such that $m > n$. Then the following propositions are equivalent:*

1. *$Con_{m,n}(\mathbf{O})$ is a Boolean lattice.*
2. *$Fix_{m,n}(\mathbf{X}(\mathbf{O}))$ is finite and $Fix_{m,n}(\mathbf{X}(\mathbf{O})) = Fix_{m-n,0}(\mathbf{X}(\mathbf{O}))$.*

3. $\Omega_{m,n}(\mathbf{X}(\mathbf{O}))$ *is finite and* $\Omega_{m,n}(\mathbf{X}(\mathbf{O})) = \Omega_{m-n,0}(\mathbf{X}(\mathbf{O}))$.
4. $Con_{m,n}(\mathbf{O})$ *is finite and* $Con_{m,n}(\mathbf{O}) = Con_{m-n,0}(\mathbf{O})$.

Proof. In this proof we will omit the subscript $\mathbf{O}$, and note g instead of $g_{\mathbf{O}}$.

Clearly 2, 3 and 4 are equivalent.

Suppose that 1 holds. First we will prove that $Fix_{m,n}(\mathbf{X}(\mathbf{O})) = Fix_{m-n,0}(\mathbf{X}(\mathbf{O}))$.

If $n = 0$ the result is obvious. Suppose that $n \neq 0$ and that there exists $x \in Fix_{m,n}(\mathbf{X}(\mathbf{O})) \setminus Fix_{m-n,0}(\mathbf{X}(\mathbf{O}))$. Then $x \notin g^{\omega}(g^{n}(x))$. By Theorem 13, $\Omega_{m,n}(\mathbf{X}(\mathbf{O}))$ is a Boolean lattice. So

$$(g^{\omega}(g^{n}(x)))^{c} \in \Omega_{m,n}(\mathbf{X}(\mathbf{O})),$$

but $(g^{\omega}(g^{n}(x)))^{c}$ is not a g-set since $x \in (g^{\omega}(g^{n}(x)))^{c}$ and $g^{\omega}(x) \nsubseteq (g^{\omega}(g^{n}(x)))^{c}$, which is a contradiction. Thus $Fix_{m,n}(\mathbf{X}(\mathbf{O})) \subseteq Fix_{m-n,0}(\mathbf{X}(\mathbf{O}))$. By Theorems 10 and 13 we have that $Fix_{m-n,0}(\mathbf{X}(\mathbf{O})) \subseteq Fix_{m,n}(\mathbf{X}(\mathbf{O}))$. Thus $Fix_{m,n}(\mathbf{X}(\mathbf{O})) = F_{m-n,0}(\mathbf{X}(\mathbf{O}))$.

Now we will prove that $\Omega_{m,n}(\mathbf{X}(\mathbf{O}))$ is finite. We already proved that $Fix_{m,n}(\mathbf{X}(\mathbf{O})) = Fix_{m-n,0}(\mathbf{X}(\mathbf{O}))$. Since $\Omega_{m,n}(\mathbf{X}(\mathbf{O}))$ is a Boolean lattice, $Fix_{m,n}(\mathbf{X}(\mathbf{O})) \setminus (g^{\omega}(x))$ is a closed subset of $Fix_{m,n}(\mathbf{X}(\mathbf{O}))$, for every $x \in Fix_{m,n}(\mathbf{X}(\mathbf{O}))$. Then $g^{\omega}(x)$ is relatively open in $Fix_{m,n}(\mathbf{X}(\mathbf{O}))$. Clearly

$$Fix_{m,n}(\mathbf{X}(\mathbf{O})) = \bigcup_{x \in Fix_{m,n}(\mathbf{X}(\mathbf{O}))} g^{\omega}(x),$$

and since $Fix_{m,n}(\mathbf{X}(\mathbf{O}))$ is closed, it is compact and there exist $\{x_1, \ldots, x_n\} \subseteq Fix_{m,n}(\mathbf{X}(\mathbf{O}))$ such that

$$Fix_{m,n}(\mathbf{X}(\mathbf{O})) = \bigcup_{i=1}^{n} g^{\omega}(x_i).$$

Since $g^{\omega}(x_i)$ is finite for every x_i, $Fix_{m,n}(\mathbf{X}(\mathbf{O}))$ is finite.

For the converse, suppose that $\Omega_{m,n}(\mathbf{X}(\mathbf{O}))$ is finite and $\Omega_{m,n}(\mathbf{X}(\mathbf{O})) = \Omega_{m-n,0}(\mathbf{X}(\mathbf{O}))$. Clearly $Fix_{m-n,0}(\mathbf{X}(\mathbf{O}))$ is finite. We only have to prove that $\Omega_{m-n,0}(\mathbf{X}(\mathbf{O}))$ is a boolean lattice.

Let $C \in \Omega_{m-n,0}(\mathbf{X}(\mathbf{O}))$. If we suppose that there exists $x \in C^{c} \cap Fix_{m-n,0}(\mathbf{X}(\mathbf{O}))$ such that $g^{\omega}(x) \cap C \neq \emptyset$, then there exists $k \leq m-n$ such that $g^{k}(x) \in C$. Thus

$$x = g^{m-n}(x) = g^{(m-n)-k}\left(g^{k}(x)\right) \in C,$$

because C is a g-set, which is a contradiction. Then $C^{c} \cap Fix_{m-n,0}(\mathbf{X}(\mathbf{O}))$ is a g-set.

Since $C^{c} \cap Fix_{m-n,0}(\mathbf{X}(\mathbf{O}))$ is finite, it is closed. Thus

$$C^{c} \cap Fix_{m-n,0}(\mathbf{X}(\mathbf{O})) \in \Omega_{m-n,0}(\mathbf{X}(\mathbf{O})). \quad \blacksquare$$

The following Corollary gives a generalization of Theorem 4.2 of [2, Chapter 4].

Corollary 17. *Let* $m, n \in \mathbb{N}$ *such that* $m > n$. *Let* $\mathbf{O} \in P_{m,n}$. *Then the following propositions are equivalent:*

1. $Con(\mathbf{O})$ *is a Boolean lattice.*
2. $Con(\mathbf{O})$ *is finite and* $\mathbf{O} \in P_{m-n,0}$.
3. $Con(\mathbf{O})$ *is finite and* f *is injective.*

Proof. Since $\mathbf{O} \in P_{m,n}$, $Con(\mathbf{O}) = Con_{m.n}(\mathbf{O})$. Thus the result follows directly from the previous Theorem and Theorem 8. ∎

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