ACTAS DEL VIII CONGRESO
DR. ANTONIO A. R. MONTEIRO
**(2005)**, Páginas 33–37

# QUANTUM DYNAMICS ON THE $SU(2)$ GROUP

GUILLERMO CAPOBIANCO AND WALTER REARTES

ABSTRACT. In this paper we study the quantum evolution of a wave function defined on the $SU(2)$ group using path integrals. We use an intrinsic approach based on integration over tangent spaces of the group which represents the infinitesimal contribution to the integral. We obtain finally an expression for the Feynman propagator.

## 1. INTRODUCTION

In 1942 Feynman presented for the first time the method of path integrals ([5], [6], [7]). This method has been a generator of new ideas in physics and in mathematics, see for example the books by Schulman [19] and Kleinert [13] for a full treatment of the subject. Nevertheless the advance on this topic has been slow, mainly because of the great difficulties in the calculation process.

The method of temporary slices, founded on the results of Lie-Trotter ([10], [18]), is the method par excellence from a practical point of view in non-relativistic quantum mechanics. These iterated integrals may be performed in phase space or in configuration space.

Until recently only quadratic Lagrangians could be solved exactly. An important advance took place when Duru and Kleinert [4] solved the hydrogen atom, extending the class of Lagrangians that can be calculated exactly.

Path integrals on Lie groups have been previously studied, especially on the compact groups $SO(3)$, $SU(2)$ and also non-compact groups such as $SU(1,1)$, for example in the papers by Junker and Böhm [1] and Duru [3].

Finding the correct path integral for a particle in a space with curvature is a non-trivial and ambiguous problem. Several authors have obtained corrections to the Schrödinger equation which differ by multiples of the scalar curvature, see for example [17], [16], [15], [14], [12], [11], [9], [2].

In this paper we develop the path integral on the $SU(2)$ group using hyperspherical coordinates coming from the 3-sphere. In section 2 we give the general settings for any compact Lie group and in section 3 the case of $SU(2)$ is solved.

## 2. THE GENERAL SETTING

Let us consider a compact Lie group $G$ of dimension $d$, equipped with its natural biinvariant Riemannian metric. To begin with, we define the one-step propagator corresponding to a small time slice $\varepsilon$. Instead of integrating over the group itself we perform the

2000 *Mathematics Subject Classification.* Primary: 81S40, Secondary: 22E05.
*Key words and phrases.* Path integrals, Lie groups.

integration over the tangent space as follows

$$S_\varepsilon \psi(g) = \left(\frac{1}{2\pi\varepsilon i}\right)^{d/2} \int_{T_gG} \exp\left(\frac{i\|\eta\|^2}{2\varepsilon}\right) \psi(\exp_g(\eta)) \exp_g^* d\mu(\eta). \tag{1}$$

In this scheme the wave function is lifted by the exponential map to the tangent plane at the point $g$ and then integrated to calculate its contribution to the value at time $\varepsilon$. We note that using the invariance of the metric the last expression can be evaluated as an integral over the Lie algebra of the group, that is

$$S_\varepsilon \psi(g) = \left(\frac{1}{2\pi\varepsilon i}\right)^{d/2} \int_{\mathfrak{g}} \exp\left(\frac{i\|\eta\|^2}{2\varepsilon}\right) \psi(g\exp(\eta)) \exp^* d\mu(\eta). \tag{2}$$

By successive application of the one-step propagator we obtain what we call the $N$-step propagator, which is given by

$$S_\varepsilon^N \psi(g) = \left(\frac{1}{2\pi\varepsilon i}\right)^{\frac{Nd}{2}} \int_{T_gG} \cdots \int_{T_{g_{N-1}}G} \exp\left(\frac{i}{2\varepsilon}\sum_{j=1}^{N}\|\eta_j\|^2\right) \psi(\exp_{g_N}(\eta_N)) \prod_{j=1}^{N-1} \exp_{g_{j-1}}^* d\mu(\eta_j). \tag{3}$$

With these definitions the evolution operator can be represented by a limit of discrete operators as follows

$$U_t = \lim_{N\to\infty} S_{t/N}^N. \tag{4}$$

Some words could be added about the one-step propagator. First we note that $S_\varepsilon$ is continuous to the right at $\varepsilon = 0$ by defining $S_0\psi(g) = \psi(g)$. Furthermore, it can be proved that the expansion of $S_\varepsilon\psi(g)$ to first order in $\varepsilon$ is given by

$$S_\varepsilon \psi(g) = \psi(g) + i\frac{\varepsilon}{2}\left(\Delta\psi(g) - \frac{1}{3}R(g)\psi(g)\right) + o(\varepsilon). \tag{5}$$

In other words the Schrödinger equation obtained from this propagator contains a curvature-dependent potential added to the Laplace-Beltrami operator.

## 3. THE PROPAGATOR FOR $SU(2)$

Our goal is to calculate the propagator (1) or (2) in the group $SU(2)$ for a sufficiently large class of functions. First we observe that $SU(2)$ has a natural Riemannian structure with constant curvature $R = 6$. Furthermore we use the fact that the group manifold $SU(2)$ is diffeomorphic to the three sphere $S^3$. Spherical coordinates in $\mathbb{R}^4$ are used. They are given by

$$\begin{aligned} x^1 &= \sin\rho\sin\theta_1\cos\theta_2 \\ x^2 &= \sin\rho\sin\theta_1\sin\theta_2 \\ x^3 &= \sin\rho\cos\theta_1 \\ x^4 &= \cos\rho \end{aligned} \tag{6}$$

With these coordinates the tangent space to $S^3$ over the point with coordinates $e = (0,0,0,1)$ can be parametrized as follows

$$\begin{aligned} \eta^1 &= \rho \sin\theta_1 \cos\theta_2 \\ \eta^2 &= \rho \sin\theta_1 \sin\theta_2 \\ \eta^3 &= \rho \cos\theta_1 \end{aligned} \tag{7}$$

The Haar measure, written in these coordinates is

$$d\mu = \sin^2\rho \, \sin\theta_1 \, d\rho \, d\theta_1 \, d\theta_2. \tag{8}$$

Finally, the one-step propagator for an arbitrary function $\psi$ is given by

$$S_\varepsilon \psi(e) = \left(\frac{1}{2\pi\varepsilon i}\right)^{3/2} \int_0^\infty \int_0^\pi \int_0^{2\pi} \exp\left(\frac{i\rho^2}{2\varepsilon}\right) \psi(\rho,\theta_1,\theta_2)\, d\mu, \tag{9}$$

where we have slightly simplified the notation.

Before going to more general functions we consider the case of constant $\psi$, for example $\psi = 1$. In this case the integral in (9) gives

$$S_\varepsilon 1 = \frac{\sin\varepsilon}{\varepsilon} \exp(-i\varepsilon). \tag{10}$$

Of course it does not depend on the point were it is calculated.

It is easy to construct the propagator for finite time $t$, it is given by

$$U_t 1 = \lim_{N\to\infty} S_{t/N}^N 1 = e^{-it} = e^{-i\frac{R}{6}t}. \tag{11}$$

Now we turn our attention to the eigenfunctions of the Laplace operator in the group. This operator is given by

$$\Delta = \frac{\partial^2}{\partial\rho^2} + 2\cot\rho \frac{\partial}{\partial\rho} + \csc^2\rho \left(\frac{\partial^2}{\partial\theta_1} + \cot\theta_1 \frac{\partial}{\partial\theta_1} + \csc^2\theta_1 \frac{\partial^2}{\partial\theta_2^2}\right). \tag{12}$$

The Laplace operator has eingenfunctions $u_{nlm}$ with eigenvalues

$$-(n^2-1), \quad n = 1,2,\ldots. \tag{13}$$

That is

$$\Delta u_{nlm} = -(n^2-1) u_{nlm} \tag{14}$$

These functions, conveniently normalized, are given by

$$u_{nlm}(\rho,\theta_1,\theta_2) = \\ i^{n-1-l} 2^{l+1} l! \left(\frac{n(n-l-1)!}{2\pi(n+l)!}\right)^{1/2} C_{n-l-1}^{l+1}(\cos(\rho)) \sin^l(\rho)\, Y_{lm}(\theta_1,\theta_2). \tag{15}$$

where $C_{n-l-1}^{l+1}$ are Gegenbauer polynomials and $Y_{lm}$ are the usual three-dimensional spherical harmonics. For a thorough treatment of harmonic analysis on groups see the books by Helgason [8] or Vilenkin and Klimyk [20].

The value of the one-step propagator (9) at the point $e$ can be evaluated giving the neat expression

$$S_\varepsilon u_{nlm}(e) = \exp\left(\frac{-i(n^2+1)\varepsilon}{2}\right) \frac{\sin(n\varepsilon)}{n\varepsilon} u_{nlm}(e). \tag{16}$$

We must stress that both members of this equation vanish simultaneously unless $l = 0$ and $m = 0$.

After some calculations it is possible to prove that $u_{nlm}$ is an eigenfunction of $S_\varepsilon$ with eigenvalue $\lambda_n(\varepsilon)$, that is

$$S_\varepsilon u_{nlm} = \lambda_n(\varepsilon) u_{nlm} \tag{17}$$

is valid for any $u_{nlm}$.

For these functions it is easy to evaluate the limit in (4), it is given by

$$U_t u_{nlm} = \lim_{N\to\infty} S^N_{t/N} u_{nlm} = \exp\left(-i\frac{(n^2-1)t}{2}\right)\exp\left(-i\frac{Rt}{6}\right) u_{nlm}. \tag{18}$$

Finally, using the closure relation

$$\delta(\cos\rho - \cos\rho')\delta(\theta-\theta')\delta(\phi-\phi') = \sum_{n=1}^{\infty}\sum_{l=0}^{n-1}\sum_{m=-l}^{l} \overline{u}_{nlm}(\rho',\theta',\phi')u_{nlm}(\rho,\theta,\phi), \tag{19}$$

we can write down an expression for the Feynman propagator, which is given by

$$K(\rho,\theta,\phi,\rho',\theta',\phi';t) = \sum_{n=1}^{\infty}\sum_{l=0}^{n-1}\sum_{m=-l}^{l} \exp\left(-i\frac{(n^2-1)t}{2}\right)\exp\left(-i\frac{Rt}{6}\right)\overline{u}_{nlm}(\rho',\theta',\phi')u_{nlm}(\rho,\theta,\phi). \tag{20}$$

## REFERENCES


[1] M. Böhm and G. Junker. Path integration over compact and noncompact rotation groups. *J. Math. Phys.*, 28(9):1978–1994, 1987.

[2] K. S. Cheng. Quantization of a general dynamical system by Feynman's path integration formulation. *J. Math. Phys.*, 13(11):1723–1726, 1972.

[3] I. H. Duru. Path integral over the $SU(2)$ manifold and related potentials. *Phys. Rev. D*, 30:2121–2125, 1984.

[4] I. H. Duru and H. Kleinert. Solution of the path integral for the h-atom. *Physics Letters*, 84B:185–188, 1979.

[5] R. P. Feynman. *The Principle of Least Action in Quantum Mechanics*. PhD thesis, Princeton University, 1942.

[6] R. P. Feynman. Space-time approach to non-relativistic quantum mechanics. *Reviews of Modern Physics*, 20(2):367–387, 1948.

[7] R. P. Feynman and A. R. Hibbs. *Quantum Mechanics and Path Integrals*. McGraw-Hill, 1965.

[8] S. Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces*. Academic Press, 1978.

[9] L. Kaplan, N. T. Maitra, and E. J. Heller. Quantizing constrained systems. *Phys. Rev. A*, 56(4):2592–2599, 1997.

[10] T. Kato. Fundamental properties of Hamiltonian operators of Schrödinger type. *Trans. Am. Math. Soc.*, 70:195, 1951.

[11] H. Kleinert. Quantum equivalence principle for path integrals in spaces with curvature and torsion. *Mod. Phys. Lett. A*, 4:2329, 1989.

[12] H. Kleinert. Path integral on spherical surfaces in $D$ dimensions and on group spaces with charged and Dirac monopoles. *Phys. Lett. B*, 236:315, 1990.

[13] H. Kleinert. *Path Integrals in Quantum Mechanics Statistics and Polymer Physics*. World Scientific, 2nd edition, 1995.

[14] H. Kleinert and S. V. Shabanov. Proper Dirac quantization of free particle on D-dimensional sphere. *Phys. Lett. A*, 232(3):482, 1996.

[15] Ali Mostafazadeh. Supersymmetry and the Atiyah-Singer index theorem : The scalar curvature factor in the Schrödinger equation: I. *J. Math. Phys.*, 35:1025–1138, 1994.
[16] Ali Mostafazadeh. Supersymmetry and the Atiyah-Singer index theorem : The scalar curvature factor in the Schrödinger equation: II. *J. Math. Phys.*, 35:1095–1124, 1994.
[17] Ali Mostafazadeh. Scalar curvature factor in the Schrödinger equation and scattering on a curved surface. *Phys.Rev. A*, 54:1165–1170, 1996.
[18] E. Nelson. Feynman integrals and the Schödinger equation. *J. Math. Phys.*, 5:332–343, 1964.
[19] L. S. Schulman. *Techniques and Applications of Path Integration*. Wiley, 1981.
[20] N. Ja. Vilenkin and A. U. Klimyk. *Representation of Lie Groups and Special Functions*, volume 1. Kluwer Academic Publishers, 1991.

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDAD NACIONAL DEL SUR AND CONICET, AV. ALEM 1253, 8000 BAHÍA BLANCA, ARGENTINA
*E-mail*: `capobian@criba.edu.ar`

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDAD NACIONAL DEL SUR, AV. ALEM 1253, 8000 BAHÍA BLANCA, ARGENTINA
*E-mail*: `reartes@uns.edu.ar`