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TOTAL CENTRAL CURVATURE OF CURVES IN THE 3-DIMENSIONAL LORENTZIAN SPACE

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ABSTRACT. Total central curvature of closed curves in Euclidean spaces has been studied by Thomas F. Banchoff, see Duke Math. Journal, 1969. In some papers, it has been related to Riemannian spaces, but this curvature has not been treated on spaces with indefinite metrics. In this work we generalize, by means of integral formulas, the total central curvature from Euclidean spaces with dimension 2 and 3 to Lorentzian spaces with dimension 2 and 3, respectively.

1. INTRODUCTION

In some papers, the total central curvature is related to Riemannian spaces; but this curvature has not been treated on spaces with indefinites metrics.

In [1], Thomas F. Banchoff stated that the total central curvature of a closed curve in 3-dimensional Euclidean space refers to the measure of curvedness of a space curve contained in a bounded ball. He obtained this curvature by averaging the total absolute curvature of the image curves under central projection from all points on the sphere, and he showed that the total central curvature agrees with the classical total absolute curvature of the original space.

In this work, we generalize, by means of integral formulas, the total central curvature from Euclidean spaces with dimension 2 and 3 to Lorentzian spaces with dimension 2 and 3, respectively.

In 2-dimensional Euclidean space, the total central curvature of a closed curve f with respect to the unit circle S^1, $tcc_2\left(f;S^1\right)$, is given by

$$tcc_2\left(f;S^1\right) = \frac{1}{2\pi}\int_{\xi\in S^1}\Upsilon_\xi\left(f\right)ds_{S^1}$$

where $\Upsilon_\xi\left(f\right)$ denotes the number of local support lines to f passing through the point ξ.

One of the problems that arise when we want to generalize this curvature to 2-dimensional Lorentzian space is the fact that S_1^1 has not finite length. For that reason we define the central curvature of a closed curve with respect to a connected arc $C_j \subset \left(S_1^1\right)_+$ with finite length, $cc_2\left(f,C_j\right)$, by

$$cc_2\left(f,C_j\right) = \frac{1}{length\left(C_j\right)}\int_{\xi\in C_j}\Upsilon_\xi\left(f\right)ds_{C_j},$$

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and the total central curvature, $tcc_2(f)$, is defined by

$$tcc_2(f) = \lim_{C_j \to \left(S_1^1\right)_+} cc_2(f, C_j),$$

if this limit exists; we say that $tcc_2(f) = \infty$ if this limit does not exist.

We will show that if f is a $\mathscr{C}^2$ simple closed curve, then $tcc_2(f)$ depends on the number of its local support lightlike lines.

In 3-dimensional Lorentzian space, we consider the central projection map π_p in the definition of central curvature of a closed curve g with respect to a connected region $S_j \subset S_1^2$ with finite area, $cc_3(g, S_j)$, as follows

$$cc_3(g, S_j) = \frac{1}{area(S_j)} \int_{p \in S_j} cc_2\left(\pi_p \circ i_3 \circ g, S_j \cap L_p^2\right) ds_{S_j},$$

where L_p^2 denotes a plane parallel to $T_p\left(S_1^2\right)$ which contains the origin of coordinates, and $i_3 : \left\{x \in L^3 : \langle x, x\rangle_L > 1\right\} \to L^3 - T_p\left(S_1^2\right)$ is the inclusion map.

The total central curvature, $tcc_3(g)$, is defined by

$$tcc_3(g) = \lim_{S_j \to S_1^2} cc_3(g, S_j),$$

if this limit exists; we say that $tcc_3(g) = \infty$ if this limit does not exist.

In section 2, Preliminaries, we will recall the basic notions in Lorentzian geometry.

2. Preliminaries

In the n-dimensional vector space $\mathbb{R}^n$, $n \in \{2; 3\}$, we denote the Euclidean inner product and the Euclidean norm with $\langle , \rangle$ and $\|.\|$, respectively.

In what follows, $n \in \{2; 3\}$.

Let x and y be two vectors in the n-dimensional vector space $\mathbb{R}^n$. As it is well known ([2, 5]), the *Lorentzian inner product* of x and y is defined by

$$\langle x, y\rangle_L = -x_1 y_1 + \sum_{i=2}^{n} x_i y_i.$$

Thus the square ds^2 of an element of arc-length is given by

$$ds^2 = -dx_1^2 + \sum_{i=2}^{n} dx_i^2.$$

The space $\mathbb{R}^n$ equipped with this metric is called a *n-dimensional Lorentzian space*, or Lorentz n-space. We write L^n or $\mathbb{R}_1^n$, [5], instead of $(\mathbb{R}^n, ds)$.

We say that a vector x in L^n is *timelike* if $\langle x, x\rangle_L < 0$, *spacelike* if $\langle x, x\rangle_L > 0$, and *null* if $\langle x, x\rangle_L = 0$. The null vectors are also said to be *lightlike*.

We say that x is *orthogonal* to y if $\langle x, y\rangle_L = 0$, $x \neq y \neq 0$.

Let x be a vector in L^n, then $\|x\|_L = \sqrt{|\langle x, x\rangle_L|}$ is called the *Lorentzian norm* of x. We say that x is a *unit vector* if $\|x\|_L = 1$, that is, if $\langle x, x\rangle_L = 1$ or $\langle x, x\rangle_L = -1$.

We shall give a surface M in L^3 by expressing its coordinates x_i as functions of two parameters in a certain interval. We consider the functions x_i to be real functions of real variables.

We say that M is a *non-lightlike surface* if at every $p \in M$ its tangent plane T_pM is equipped with a positive definite or Lorentzian metric, [2].

A parametrized curve is called timelike, spacelike or null curve if at every point, its tangent vector is timelike, spacelike or null, respectively.

We say that the curve $\mathscr{C}(O,r) = \left\{X : \left\langle \overrightarrow{OX}, \overrightarrow{OX} \right\rangle_L = r^2 \right\}$ is a *Lorentzian circle* with center O and radius r, where O is a point in the Lorentzian plane and r is a positive real number. We remark that $\mathscr{C}(O,r)$ has two branches and each of them is a timelike curve. Hence, the Lorentzian circle is a timelike curve.

In what follows, we will denote $\mathscr{C}((0,0),1)$ with S_1^1, and $\left(S_1^1\right)_+ = \left\{(x_1,x_2) \in S_1^1 : x_2 > 0\right\}$.

We now recall a well know definition, [1].

Definition 1. *Let $f : S^1 \to \mathbf{R}^2$ be a continuous map of the circle S^1 into the Euclidean plane. A* local support line *to f at x is a line containing x and bounding a closed half-plane which contains the image of a neighborhood of x in S^1.*

In the definition of local support line, we consider f as a continuous map of the circle S^1 into L^2 when we are working in the Lorentzian plane.

We denote the number of local support lines to f passing through the point $\xi \in L^2$ with $\Upsilon_\xi(f)$.

3. Total central curvature of plane curves

In [1], we find the definition of the curvature of a closed plane curve with respect to a circle in Euclidean plane.

Definition 2. *Let $f : S^1 \to \mathbf{R}^2$ be a continuous map of the circle S^1 into the Euclidean plane. The* curvature of f with respect to a circle *C is defined by*

$$tcc_2(f;C) = \frac{1}{length(C)} \int_{\xi \in C} \Upsilon_\xi(f)\, ds_C$$

where $\Upsilon_\xi(f)$ and ds_C denote the number of local support lines to f passing through the point ξ, and the element of arc of C so that $\int_{\xi \in C} ds_C = length(C)$, respectively.

That means that the curvature $tcc_2(f;C)$ is defined as the average value of $\Upsilon_\xi(f)$ for points $\xi \in C$.

Example 3. *We show three examples where $f =$ identity and C is a circle with centre $O = (0,0)$ and radius r.*

i) *If $0 < r < 1$, then $\Upsilon_\xi(f) \equiv 2$. Hence, $tcc_2(f;C) = 2$.*
ii) *If $r = 1$, then $\Upsilon_\xi(f) \equiv 1$. Hence, $tcc_2(f;C) = 1$.*
iii) *If $r > 1$, then $\Upsilon_\xi(f) \equiv 0$. Hence, $tcc_2(f;C) = 0$.*

If f is a convex closed curve and C is a circle with center O and radius r, then $tcc_2(f;C) \equiv 2$ if $\left\|\overrightarrow{OX}\right\| < r$, $\forall X \in f\left(S^1\right)$ (cf. [1]).

One of the problems that arise when we want to generalize Definition 2 from the Euclidean plane to the Lorentzian plane is the fact that the Lorenzian circle does not have finite length. For that reason, we will define first the curvature of a continous map f with respect to a connected arc $C_j \subset \left(S_1^1\right)_+$ which has finite length.

In what follows, we consider a continuous map $f : S^1 \to L^2$ of the circle S^1 into the Lorentzian plane such that $f\left(S^1\right) \cap \left(S_1^1\right)_+ = \emptyset$.

Definition 4. *Let $C_j \subset \left(S_1^1\right)_+$ be a connected arc such that $0 < length\left(C_j\right) < \infty$. The* central curvature *of f with respect to C_j is given by*

$$cc_2\left(f;C_j\right) = \frac{1}{length\left(C_j\right)} \int_{\xi \in C_j} \Upsilon_\xi\left(f\right)\, ds_{C_j},$$

where $\Upsilon_\xi\left(f\right)$ denotes the number of local support lines to f passing through the point ξ, and $\int_{\xi \in C_j} ds_{C_j} = length\left(C_j\right)$.

Remark 5. *Since f is a closed curve, $\Upsilon_\xi\left(f\right) \neq 0\ \forall \xi \in \left(S_1^1\right)_+$. Also, $\Upsilon_\xi\left(f\right) \neq \infty$ because $\xi \notin f\left(S^1\right)$.*

Now we define the total central curvature of f.

Definition 6. *Let $\left(C_j\right)_{j\geq 1}$ a sequence of connected arcs such that:*

i) $C_j \subset \left(S_1^1\right)_+$, $\forall j \geq 1$.
ii) $C_j \subset C_{j+1}$, $\forall j \geq 1$.
iii) $length\left(C_j\right) < \infty$, $\forall j \geq 1$, *and* $\lim\limits_{j\to\infty} \frac{length\left(C_{j+1}\right)}{length\left(C_j\right)} = 1$.
iv) $\lim\limits_{j\to\infty} C_j = \left(S_1^1\right)_+$.

The total central curvature *of f is defined by*

$$tcc_2\left(f\right) = \lim_{j\to\infty} cc_2\left(f;C_j\right) = \lim_{j\to\infty} \frac{1}{length\left(C_j\right)} \int_{\xi \in C_j} \Upsilon_\xi\left(f\right) ds_{C_j},$$

if this limit exists; we say that $tcc_2\left(f\right) = \infty$ if this limit does not exist.

The existence of a sequence $\left(C_j\right)_{j\geq 1}$ fulfilling the above mentioned conditions is shown by the following Theorem.

Theorem 7. *In L^2 there exists a sequence of connected arcs $\left(C_j\right)_{j\geq 1}$ such that:*

i) $C_j \subset \left(S_1^1\right)_+$, $\forall j \geq 1$.
ii) $C_j \subset C_{j+1}$, $\forall j \geq 1$.
iii) $length\left(C_j\right) < \infty$, $\forall j \geq 1$, *and* $\lim\limits_{j\to\infty} \frac{length\left(C_{j+1}\right)}{length\left(C_j\right)} = 1$.
iv) $\lim\limits_{j\to\infty} C_j = \left(S_1^1\right)_+$.

Proof. Let $\left\{a_1^j,\ b_1^j\right\}_{j\geq 1}$ be a sequence of pairs of real numbers such that:

a) $b_1^{j+1} < b_1^j < a_1^j < a_1^{j+1}$, $\forall j \geq 1$.
b) $\lim\limits_{j\to\infty} \frac{a_1^j}{a_1^{j+1}} = \lim\limits_{j\to\infty} \frac{b_1^j}{b_1^{j+1}} = \frac{1}{2}$.
c) $\lim\limits_{j\to\infty} a_1^j = +\infty$ and $\lim\limits_{j\to\infty} b_1^j = -\infty$.

We call $I_{AB} = \left\{I^+\left(B\right) \cap I^-\left(A\right) : A, B \in L^2\right\}$, where $I^+\left(B\right)$ and $I^-\left(A\right)$ are the chronological future of B and the chronological past of A, respectively; in particular, $I^+\left(B\right)$ and $I^-\left(A\right)$ are open sets in the Alexandrov topology of L^2, cf. [2] and [6].

Let $A_j = \left(a_1^j, \sqrt{1+\left(a_1^j\right)^2}\right)$ and $B_j = \left(b_1^j, \sqrt{1+\left(b_1^j\right)^2}\right)$. Hence $A_j, B_j \in \left(S_1^1\right)_+$ and $I_{A_jB_j} \subset I_{A_{j+1}B_{j+1}}, \forall j \geq 1$.

Hence, if we call $C_j = I_{A_jB_j} \cap \left(S_1^1\right)_+$ then $\left(C_j\right)_{j\geq 1}$ is a sequence of connected arcs with properties i)-iv). $\square$

Note that $C_j = \left\{\left(x_1^j, x_2^j\right) \in \left(S_1^1\right)_+ : b_1^j < x_1^j < a_1^j\right\}$.

Lemma 8. *Let $\left(C_j\right)_{j\geq 1}$ be a sequence of connected arcs as given in Theorem 7. If $\left(C_i\right)_{i\geq 1}$ is a subsequence of $\left(C_j\right)_{j\geq 1}$ with properties i)-iv), then*

$$\lim_{j\to\infty} cc_2\left(f;C_j\right) = \lim_{i\to\infty} cc_2\left(f;C_i\right).$$

Proof. Since the properties i)-iv) hold for $\left(C_j\right)_{j\geq 1}$and $\left(C_i\right)_{i\geq 1}$, and $\left(C_i\right)_{i\geq 1} \subset \left(C_j\right)_{j\geq 1}$, then:

$\forall i \geq 1\ \exists j_{1_i}, j_{2_i} \geq 1$ such that $C_{j_{1_i}} \subset C_i \subset C_{j_{2_i}}$ and $\forall j \geq 1\ \exists i_{1_j}, i_{2_j} \geq 1$ such that $C_{i_{1_j}} \subset C_j \subset C_{i_{2_j}}$.

Hence, $\lim\limits_{j\to\infty} cc_2\left(f;C_j\right) = \lim\limits_{i\to\infty} cc_2\left(f;C_i\right)$. $\square$

Theorem 9. *Let $\left(C_j\right)_{j\geq 1}$ and $\left(C_h'\right)_{h\geq 1}$ be two sequences of connected arcs as given in Theorem 7, then $\lim\limits_{j\to\infty} cc_2\left(f;C_j\right) = \lim\limits_{h\to\infty} cc_2\left(f;C_h'\right)$.*

Proof. By properties i)-iv), $\forall h \geq 1\ \exists j_{1_h}, j_{2_h} \geq 1$ such that $C_{j_{1_h}} \subset C_h' \subset C_{j_{2_h}}$ and $\left(C_{j_{1_h}}\right)_{h\geq 1}$, $\left(C_{j_{2_h}}\right)_{h\geq 1} \subset \left(C_j\right)_{j\geq 1}$ with properties i)-iv). Hence, we have that

$$\begin{aligned}
&\frac{1}{length\left(C_{j_{2_h}}\right)} \int_{\xi\in C_{j_{2_h}}} \Upsilon_\xi\left(f\right) ds_{C_{j_{2_h}}} < \\
&< \int_{\xi\in\left(C_{j_{2_h}}-C_h'\right)} \frac{\Upsilon_\xi(f)}{length\left(C_h'\right)} ds_{\left(C_{j_{2_h}}-C_h'\right)} + \int_{\xi\in C_h'} \frac{\Upsilon_\xi(f)}{length\left(C_h'\right)} ds_{C_h'} < \\
&< \int_{\xi\in\left(C_{j_{2_h}}-C_h'\right)} \frac{\Upsilon_\xi(f)}{length\left(C_{j_{1_h}}\right)} ds_{\left(C_{j_{2_h}}-C_h'\right)} + \int_{\xi\in C_h'} \frac{\Upsilon_\xi(f)}{length\left(C_h'\right)} ds_{C_h'} < \\
&< \int_{\xi\in\left(C_{j_{2_h}}-C_{j_{1_h}}\right)} \frac{\Upsilon_\xi(f)}{length\left(C_{j_{1_h}}\right)} ds_{\left(C_{j_{2_h}}-C_{j_{1_h}}\right)} + \int_{\xi\in C_{j_{1_h}}} \frac{\Upsilon_\xi(f)}{length\left(C_{j_{1_h}}\right)} ds_{C_{j_{1_h}}}.
\end{aligned}$$

By Lemma 8, we obtain

$$\begin{aligned}
&\lim_{j\to\infty} cc_2\left(f;C_j\right) \leq \\
&\leq \lim_{h\to\infty} \frac{1}{length\left(C_{j_{1_h}}\right)} \int_{\xi\in\left(C_{j_{2_h}}-C_h'\right)} \Upsilon_\xi\left(f\right) ds_{\left(C_{j_{2_h}}-C_h'\right)} + \lim_{h\to\infty} cc_2\left(f;C_h'\right) \leq \\
&\leq \lim_{h\to\infty} \frac{1}{length\left(C_{j_{1_h}}\right)} \int_{\xi\in\left(C_{j_{2_h}}-C_{j_{1_h}}\right)} \Upsilon_\xi\left(f\right) ds_{\left(C_{j_{2_h}}-C_{j_{1_h}}\right)} + \lim_{j\to\infty} cc_2\left(f;C_j\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\lim_{h\to\infty} \int_{\xi\in\left(C_{j_{2_h}}-C_{j_{1_h}}\right)} \Upsilon_\xi\left(f\right) ds_{\left(C_{j_{2_h}}-C_{j_{1_h}}\right)} = \text{constant} \Rightarrow \\
&\Rightarrow \lim_{h\to\infty} \frac{1}{length\left(C_{j_{1_h}}\right)} \int_{\xi\in\left(C_{j_{2_h}}-C_{j_{1_h}}\right)} \Upsilon_\xi\left(f\right) ds_{\left(C_{j_{2_h}}-C_{j_{1_h}}\right)} = 0.
\end{aligned}$$

Then,

$$\lim_{h\to\infty} cc_2\left(f;C_h^{'}\right) \leq \lim_{j\to\infty} cc_2\left(f;C_j\right). \tag{1}$$

Analogously, $\forall j \geq 1\ \exists h_{1_j}, h_{2_j} \geq 1$ such that $C'_{h_{1_j}} \subset C_j \subset C'_{h_{2_j}}$ and $\left(C'_{h_{1_j}}\right)_{j\geq 1}, \left(C'_{h_{2_j}}\right)_{j\geq 1} \subset \left(C_h^{'}\right)_{h\geq 1}$ with properties i)-iv). Hence,

$$\lim_{j\to\infty} cc_2\left(f;C_j\right) \leq \lim_{h\to\infty} cc_2\left(f;C_h^{'}\right). \tag{2}$$

By (1) and (2), we have that

$$\lim_{j\to\infty} cc_2\left(f,C_j\right) = \lim_{h\to\infty} cc_2\left(f,C'_h\right).$$

□

Theorem 10. *Let f be a $\mathscr{C}^2$ simple closed curve. According to Definition 6, we have that*

$$2 \leq tcc_2(f) \leq \frac{\eta}{2},$$

where η denotes the number of lightlike lines which are tangent to f and passing through some point of $\left(S_1^1\right)_+$.

The equality holds if f is a convex simple closed curve.

Proof. Let f be a $\mathscr{C}^2$ map. By [3] we know that $\eta \geq 4$ because f has at least four lightlike points.

By [6], we know that $\eta < \infty$ because f is a closed curve and it has a finite number of folds.

Denote $u : x_1 = x_2$ and $v : x_1 = -x_2$ which are two lightlike lines in L^2 passing through the point $(0,0)$.

Let $u_1,\ldots,u_h$ be lightlike lines such that u_j is local support line to f and u_j is parallel to u, and let $v_1,\ldots,v_t$ be lightlike lines such that v_j is local support line to f and v_j is parallel to v. Denote $U_j = v_j \cap u$ and $V_j = u_j \cap v$.

We denote the number of points U_j and the number of points V_j with η_u and η_v, respectively.

Let $U_i = \left(x_1^i, x_2^i\right)$ and $V_j = \left(y_1^j, y_2^j\right)$, and let $A = (a_1, a_2) \in \left(S_1^1\right)_+$ such that

$$a_1 = \max_{1\leq i\leq h,\ 1\leq j\leq t} \left\{\left|x_1^i\right|, \left|y_1^j\right|\right\}$$

and let $B = (-a_1, a_2)$. We denote the arc between A and B with AB.

Since the lightlike lines u and v are asymptotic lines of $\left(S_1^1\right)_+$, and u_j and v_j are parallel to u and v, respectively, then $\Upsilon_\xi(f) \leq h+t$, $\forall \xi \in \left(S_1^1\right)_+ - AB$.

Let $\left(C_j\right)_{j\geq 1}$ be a sequence of arcs as given in Definition 6. Without loss of generality, we consider C_j as simetric arcs A_jB_j; that means: $C_j = A_jO + OB_j$ and $length\left(C_j\right) = 2\, length\left(A_jO\right)$, where $O = (0,0)$.

There exists j_0 such that for every $j \geq j_0$, $AB \subset C_j$. Then

$\frac{1}{length\left(C_j\right)} \int_{\xi\in C_j} \Upsilon_\xi(f)\, ds_{C_j} =$

$= \frac{1}{length(C_j)} \left(\int_{\xi \in AB} \Upsilon_\xi (f)\, ds_{C_j} + \int_{\xi \in (C_j - AB)} \Upsilon_\xi (f)\, ds_{(C_j - AB)} \right).$

Since

$\frac{1}{length(C_j)} \int_{\xi \in (C_j - AB)} \Upsilon_\xi (f)\, ds_{(C_j - AB)} =$

$= \int_{\xi \in (A_j O - AO)} \frac{\Upsilon_\xi (f)}{2 length(A_j O)} ds_{(A_j O - AO)} + \int_{\xi \in (OB_j - OB)} \frac{\Upsilon_\xi (f)}{2 length(A_j O)} ds_{(OB_j - OB)},$

then

$\frac{1}{length(C_j)} \int_{\xi \in C_j} \Upsilon_\xi (f)\, ds_{C_j} =$

$= \frac{1}{length(C_j)} \int_{\xi \in AB} \Upsilon_\xi (f)\, ds_{C_j} + (\eta_u + \eta_v) \frac{length(A_j O - AO)}{2 length(A_j O)}.$

Hence,

$2 \leq tcc_2 (f) = \lim_{j \to \infty} \frac{1}{length(C_j)} \int_{\xi \in C_j} \Upsilon_\xi (f)\, ds_{C_j} \leq \frac{\eta_u + \eta_v}{2} = \frac{\eta}{2}.$

If f is convex then, by [3], $\eta = 4$. Hence, $tcc_2 (f) = 2$. □

Corollary 11. *Let f_1 and f_2 be two $\mathscr{C}^2$ simple closed curves such that $f_1 \left(S^1\right) \cap \left(S^1_1\right)_+ = f_2 \left(S^1\right) \cap \left(S^1_1\right)_+ = \emptyset$. If f_1 and f_2 differ from a translation, then $tcc_2 (f_1) = tcc_2 (f_2)$.*

Proof. The numbers η_u and η_v are invariants under translations. □

Note that, in this case, we cannot refer to the rotations because f_1 and f_2 are not two pure curves.

4. TOTAL CENTRAL CURVATURE OF CURVES IN THE LORENTZ 3-SPACE

In Thomas F. Banchoff's words, the total central curvature is related to the measure of curvedness of a space closed curve contained in the ball (bounded by an Euclidean sphere) obtained by averaging the total absolute curvatures of the image curves under central projection from all points on the sphere, [1] .

In 3-dimensional Lorentzian space, we find similar problems to the 2-dimensional case: S^2_1 is not a compact surface and its area is not finite either.

Analogous to section 3, we will define first the curvature of a continous map g with respect to a connected region $S_j \subset S^2_1$ which has finite area.

In what follows, we consider a one-to-one continuous map $g : S^1 \to L^3$ of the circle S^1 into the Lorentzian 3-space such that $\langle x, x \rangle_L > 1 \; \forall x \in g\left(S^1\right)$.

In [4], we studied some projection maps in Lorentzian 3-space.

Definition 12. *Let $T_p \left(S^2_1\right)$ be the tangent plane to S^2_1 at $p \in S^2_1$ and let L^2_p be the plane parallel to $T_p \left(S^2_1\right)$ through the center of S^2_1. The* central projection map $\pi_p : L^3 - T_p \left(S^2_1\right) \to L^2_p$ *is given by*

$$\pi_p (x) = \frac{1}{1 - \langle x, p \rangle_L} \left(x - \langle x, p \rangle_L \; p\right).$$

This projection map is a one-to-one and onto map (cf. [4] for more details). Note that π_p restricted to $S^2_1 - T_p \left(S^2_1\right)$ gives the stereographic projection.

Let us remark that L^2_p is congruent to the plane L^2.

Definition 13. *Let $S_j \subset S_1^2$ be a connected region such that $area(S_j) < \infty$. The* central curvature of g with respect to S_j *is defined by*

$$cc_3(g, S_j) = \frac{1}{area(S_j)} \int_{p \in S_j} cc_2\left(\pi_p \circ i_3 \circ g,\ \left(S_j \cap L_p^2\right)_+\right) ds_{S_j},$$

where $\int_{S_j} ds_{S_j} = area(S_j)$, $i_3 : \left\{x \in L^3 : \langle x, x \rangle_L > 1\right\} \to L^3 - T_p\left(S_1^2\right)$ is the inclusion map, and $\pi_p : L^3 - T_p\left(S_1^2\right) \to L_p^2$ is the central projection map.

In Theorem 19 we will show some properties of $\left(S_j \cap L_p^2\right)_+$.

Remark 14. *According to Definition 4, the curvature of g with respect to S_j is given by*

$$cc_3(g, S_j) = \int_{p \in S_j} \int_{\xi \in \left(S_j \cap L_p^2\right)_+} \frac{\Upsilon_\xi\left(\pi_p \circ i_3 \circ g\right)}{area(S_j)\ length\left(S_j \cap L_p^2\right)_+} ds_{\left(S_j \cap L_p^2\right)_+} ds_{S_j}.$$

We now define the total central curvature of g.

Definition 15. *Let $(S_j)_{j \geq 1}$ be a sequence of connected regions such that:*

i) $S_j \subset S_1^2$, $\forall j \geq 1$.
ii) $S_j \subset S_{j+1}$, $\forall j \geq 1$.
iii) $area(S_j) < \infty$, $\forall j \geq 1$, *and* $\lim\limits_{j \to \infty} \frac{area(S_{j+1})}{area(S_j)} = 1$.
iv) $\forall p \in S_1^2$, $\exists j_p \geq 1$ *such that* $\left(S_j \cap L_p^2\right)_+ \neq \emptyset$, $\forall j \geq j_p$.
v) $\lim\limits_{j \to \infty} S_j = S_1^2$.

The total central curvature *of g is defined by:*

$$tcc_3(g) = \lim_{j \to \infty} cc_3(g, S_j),$$

if this limit exists; we say that $tcc_3(g) = \infty$ if this limit does not exist.

The existence of a sequence $(S_j)_{j \geq 1}$ fulfilling the above mentioned conditions is shown by the following Theorem.

Theorem 16. *In L^3 there exists a sequence $(S_j)_{j \geq 1}$ of connected regions such that:*

i) $S_j \subset S_1^2$, $\forall j \geq 1$.
ii) $S_j \subset S_{j+1}$, $\forall j \geq 1$.
iii) $area(S_j) < \infty$, $\forall j \geq 1$, *and* $\lim\limits_{j \to \infty} \frac{area(S_{j+1})}{area(S_j)} = 1$.
iv) $\forall p \in S_1^2$, $\exists j_p \geq 1$ *such that* $\left(S_j \cap L_p^2\right)_+ \neq \emptyset$, $\forall j \geq j_p$.
v) $\lim\limits_{j \to \infty} S_j = S_1^2$.

Proof. There exists a sequence of pairs of suitable real numbers $\left\{a_1^j,\ b_1^j\right\}_{j \geq 1}$ such that:

a) $b_1^{j+1} < b_1^j < a_1^j < a_1^{j+1}$, $\forall j \geq 1$.
b) $\lim\limits_{j \to \infty} \frac{a_1^j}{a_1^{j+1}} = \lim\limits_{j \to \infty} \frac{b_1^j}{b_1^{j+1}} = \frac{1}{2}$.
c) $\lim\limits_{j \to \infty} a_1^j = +\infty$ and $\lim\limits_{j \to \infty} b_1^j = -\infty$.

Then $(S_j)_{j\geq 1}$ is a sequence of connected regions with properties i)-v), where $S_j = \left\{\left(x_1^j, x_2^j, x_3^j\right) \in S_1^2 : b_1^j < x_1^j < a_1^j\right\}$. □

Lemma 17. *Let $(S_j)_{j\geq 1}$ be a sequence of connected regions as given in Theorem 16. If $(S_i)_{i\geq 1}$ is a subsequence of $(S_j)_{j\geq 1}$ with properties i)-v), then*

$$\lim_{j\to\infty} cc_3(g;S_j) = \lim_{i\to\infty} cc_3(g;S_i).$$

Proof. This proof is analogous to the proof of Lemma 8. □

Theorem 18. *Let $(S_j)_{j\geq 1}$ and $\left(S_h^{'}\right)_{h\geq 1}$ be two sequences of connected regions as given in Theorem 16, then $\lim\limits_{j\to\infty} cc_3(g;S_j) = \lim\limits_{h\to\infty} cc_3\left(g;S_h^{'}\right)$.*

Proof. This proof is analogous to the proof of Theorem 9. □

We now study the arcs $\left(S_j \cap L_p^2\right)_+$.

Theorem 19. *Let $p \in S_1^2$ and let $(S_j)_{j\geq 1}$ be a sequence of connected regions as given in Theorem 16. If there exist $j_p \geq 1$ and $\left(\left(S_{h_p} \cap L_p^2\right)_+\right)_{h_p\geq j_p} \subset \left(\left(S_j \cap L_p^2\right)_+\right)_{j\geq j_p}$ such that $\lim\limits_{h_p\to\infty} S_{h_p} = S_1^2$ and $\lim\limits_{h_p\to\infty} \frac{length\left(S_{h_p+1}\cap L_p^2\right)_+}{length\left(S_{h_p}\cap L_p^2\right)_+} = 1$, then $\left(\left(S_{h_p} \cap L_p^2\right)_+\right)_{h_p\geq j_p}$ is a sequence of connected arcs of a branch $\left(S_1^2 \cap L_p^2\right)_+$ of the Lorentzian circle $\left(S_1^2 \cap L_p^2\right)$ in L_p^2 with properties i)-iv) of Theorem 7.*

Proof. Let p be a fixed point of S_1^2 and let $(S_j)_{j\geq 1}$ be a sequence of connected regions as given in Theorem 16. There exists $j_p \geq 1$ such that $p \in S_j$, $\forall j \geq j_p$, and there exists $\left(\left(S_{h_p} \cap L_p^2\right)_+\right)_{h_p\geq j_p} \subset \left(\left(S_j \cap L_p^2\right)_+\right)_{j\geq j_p}$ such that $\lim\limits_{h_p\to\infty} S_{h_p} = S_1^2$ and $\lim\limits_{h_p\to\infty} \frac{length\left(S_{h_p+1}\cap L_p^2\right)_+}{length\left(S_{h_p}\cap L_p^2\right)_+} = 1$.

Hence, $\forall h_p \geq j_p$, $\left(S_{h_p} \cap L_p^2\right)_+$ is a connected arc. Also:

i) Since $S_j \subset S_1^2$, then $\left(S_{h_p} \cap L_p^2\right)_+ \subset \left(S_1^2 \cap L_p^2\right)_+$, $\forall h_p \geq j_p$.

ii) Since $S_j \subset S_{j+1}$, then $\left(S_{h_p} \cap L_p^2\right)_+ \subset \left(S_{h_p+1} \cap L_p^2\right)_+$, $\forall h_p \geq j_p$.

iii) Since $area(S_j) < \infty$, then $length\left(S_j \cap L_p^2\right)_+ < \infty$, $\forall j \geq 1$. In particular,

$$length\left(S_{h_p} \cap L_p^2\right)_+ < \infty, \quad \forall h_p \geq j_p.$$

Let us recall that $\left(S_j \cap L_p^2\right)_+$ is a timelike curve in L_p^2 and

$$\left(S_j \cap L_p^2\right) = \left\{\left(x_1^j, x_2^j, x_3^j\right) \in S_1^2 \cap L_p^2 : b_1^j < x_1^j < a_1^j\right\}$$

if $S_j = \left\{\left(x_1^j, x_2^j, x_3^j\right) \in S_1^2 : b_1^j < x_1^j < a_1^j\right\}$.

iv) Since $\lim\limits_{h_p\to\infty} S_{h_p} = S_1^2$ and $\lim\limits_{S_{h_p}\to S_1^2} \left(S_{h_p} \cap L_p^2\right)_+ = \left(S_1^2 \cap L_p^2\right)_+$, then $\lim\limits_{h_p\to\infty} \left(S_{h_p} \cap L_p^2\right)_+ = \left(S_1^2 \cap L_p^2\right)_+$. □

We show the main theorem in L^3.

Theorem 20. *Let $(S_j)_{j\geq 1}$ be a sequence of connected regions as given in Theorem 16 such that $\forall p\in S_1^2$ there exist $j_p\geq 1$ and $\left(S_{h_p}\right)_{h_p\geq j_p}\subset (S_j)_{j\geq 1}$ as given in Theorem 19. Then we have that*

$$tcc_3(g)=\lim_{j\to\infty}\frac{1}{area(S_j)}\int_{p\in S_j}tcc_2(\pi_p\circ i_3\circ g)\,ds_{S_j}.$$

Proof. Let $\gamma_p=\pi_p\circ i_3\circ g$ and $tcc_3(g)=\lim\limits_{j\to\infty}cc_3(g,S_j)$.

According to definitions 6, 13 and 15 we have that:

$$\begin{aligned}
&\lim_{j\to\infty}\frac{1}{area(S_j)}\int_{p\in S_j}\left[tcc_2(\gamma_p)-cc_2\left(\gamma_p,\ \left(S_j\cap L_p^2\right)_+\right)\right]ds_{S_j}\\
&\quad=\lim_{j\to\infty}\frac{1}{area(S_j)}\int_{p\in S_j}\left[\lim_{t_p\to\infty}cc_2\left(\gamma_p,C_{t_p}\right)-cc_2\left(\gamma_p,\ \left(S_j\cap L_p^2\right)_+\right)\right]ds_{S_j}\\
&\quad=\lim_{j\to\infty}\frac{1}{area(S_j)}\int_{p\in S_j}\lim_{t_p\to\infty}\left[cc_2\left(\gamma_p,C_{t_p}\right)-cc_2\left(\gamma_p,\ \left(S_j\cap L_p^2\right)_+\right)\right]ds_{S_j}.
\end{aligned}$$

By Theorems 16 and 19, we may assume that $\left(C_{t_p}\right)_{t_p\geq j_p}=\left(\left(S_{h_p}\cap L_p^2\right)_+\right)_{h_p\geq j_p}$. Then,

$$\begin{aligned}
&\lim_{j\to\infty}\frac{1}{area(S_j)}\int_{p\in S_j}\left[tcc_2(\gamma_p)-cc_2\left(\gamma_p,\ \left(S_j\cap L_p^2\right)_+\right)\right]ds_{S_j}\\
&\quad=\lim_{j\to\infty}\frac{1}{area(S_j)}\int_{p\in S_j}\lim_{h_p\to\infty}\left[cc_2\left(\gamma_p,\ \left(S_{h_p}\cap L_p^2\right)_+\right)-cc_2\left(\gamma_p,\ \left(S_j\cap L_p^2\right)_+\right)\right]ds_{S_j}.
\end{aligned}$$

Hence, by Lemma 8,

$$\lim_{j\to\infty}\tfrac{1}{area(S_j)}\int_{p\in S_j}\left[tcc_2(\gamma_p)-cc_2\left(\gamma_p,\ \left(S_j\cap L_p^2\right)_+\right)\right]ds_{S_j}=0. \qquad \square$$

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