ACTAS DEL VIII CONGRESO
DR. ANTONIO A. R. MONTEIRO
(2005), Páginas 99–119

COMBINATORIAL HOMOTOPY THEORY

ELIAS GABRIEL MINIAN

ABSTRACT. This article intends to summarize the main results of homotopy theory in algebraic or combinatorial settings. It includes new results in this area and investigates some old results with a new insight.

1. INTRODUCTION

In his beautiful paper *Higher Algebraic K-Theory I* ([Q]), Daniel Quillen defines the homotopy groups of a small category $\mathscr{C}$ as the homotopy groups of its classifying space $\mathscr{B}\mathscr{C}$. Recall that the classifying space of $\mathscr{C}$ is a CW-complex whose n-cells are in one to one correspondence with the n-tuples $(f_1,\dots,f_n)$ of composable maps in $\mathscr{C}$ such that none of them is an identity map. He shows that the fundamental group $\pi_1(C,x)$ can be defined algebraically without the use of topology and he remarks that "the existence of similar descriptions of the higher homotopy groups seems to be unlikely, because so far nobody has produced an algebraic definition of the homotopy groups of a simplicial complex".

In [M0] I introduced a homotopy model structure applicable in combinatorial settings, such as simplicial complexes, small categories, directed graphs, global actions and finite topological spaces. This homotopy theory is based on a family of natural cylinders and generalizes Baues' homotopy theory for I-categories [B].

There are applications of this theory in various directions. In K-theory, via small categories and global actions; in topology, by computing homotopy groups of CW-complexes and in category theory. A homotopy model structure on a combinatorial setting allows one to do homotopy theory in that particular setting and provides all the constructions and tools available for topological spaces.

In [T1] Thomason proved that the category $\mathscr{C}at$ of small categories admits a structure of closed model category in the sense of Quillen. This structure is lifted from the one defined on simplicial sets. In [M1] I proved that $\mathscr{C}at$ admits also this homotopy model structure based on a family of natural cylinders. This homotopy theory for $\mathscr{C}at$ differs from the one induced by the closed model structure given by Thomason. If we denote by $q:\mathscr{C}at\to\mathscr{H}o(\mathscr{C}at)$ the localization of $\mathscr{C}at$ with respect to the class of strong homotopy equivalences and by $\gamma:\mathscr{C}at\to\mathscr{H}o_T(\mathscr{C}at)$ the localization of $\mathscr{C}at$ with respect to the class of weak equivalences in the sense of Thomason (i.e. $\mathscr{H}o_T(\mathscr{C}at)$ is the homotopy category of $\mathscr{C}at$ as a closed model category), there exists a unique functor $F:\mathscr{H}o(\mathscr{C}at)\to\mathscr{H}o_T(\mathscr{C}at)$ such that $Fq=\gamma$ and this functor is not an equivalence.

2000 *Mathematics Subject Classification.* 55U10, 55P40, 55P48, 18G30.
Key words and phrases. Small categories, simplicial objects, global actions, homotopy theory.

Our approach to an axiomatic model structure has the advantage that all constructions, whether main ones or subsidiary ones, are done inside the category. This means that a solution of an algebraic problem can be followed step by step (algebraically) in the category.

This article summarizes the main results and constructions of this theory, which was developed in previous papers [M0, M1, M2, M3, M4]. We give some new examples and prove some known results with a new insight.

2. CYLINDERS AND SUBDIVISIONS

The classical homotopy theory for topological spaces is based on the existence of a natural cylinder. Given a topological space X, the cylinder of X is the topological space $IX = X \times I$, where I denotes the unit interval $[0,1] \subset \mathbb{R}$. The notion of homotopy between continuous maps $f, g : X \to Y$ is defined using the cylinder of X. It is a continuous map

$$H : IX \to Y$$

such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.

In general a natural cylinder (I, p, i_0, i_1) on a category $\mathscr{C}$, as defined by Baues in [B], is an endofunctor

$$I : \mathscr{C} \to \mathscr{C}$$

together with natural transformations

$$i_0, i_1 : id \to I \qquad p : I \to id$$

where id is the identity functor, such that $pi_0 = pi_1 = 1$ (the identity natural transformation).

The presence of a natural cylinder in the category $\mathscr{T}op$ of topological spaces, together with a suitable notion of *cofibration* allows us to develop the homotopy theory (cf. [B, KP]).

To develop homotopy theory in algebraic or combinatorial settings, such as categories, simplicial complexes, global actions, directed graphs and finite topological spaces, we require a family of natural cylinders instead of just one. In fact, in these settings, for any natural number n there exists a finite model I_n of the unit interval and all these finite models are needed to develop the theory.

In order to exemplify the use of the natural cylinders in these settings, we begin by recalling the classical notions of homotopy in $\mathscr{C}at$, the category of small categories.

There are three different notions of homotopy for categories. The notion of strong homotopy is the symmetric transitive closure of the relation given by: $f \sim g$ if there is a natural transformation between them. The notion of weak homotopy (studied in [Q] and [S]) is related to the classifying space functor

$$\mathscr{B} : \mathscr{C}at \to \mathscr{T}op.$$

Two functors f and g are weak homotopic if $\mathscr{B}f$ and $\mathscr{B}g$ are homotopic continuous maps. In [H1, H2], an intermediate notion of homotopy is introduced by using path categories.

In this paper we work with the notion of strong homotopy [M1]. We will introduce the family of natural cylinders in this category and reformulate this notion of homotopy via these cylinders. When we refer to homotopies of functors and homotopy equivalences of categories, we will mean strong homotopies and strong homotopy equivalences.

2.1. Definition. Let $f,g:\mathscr{C}\to\mathscr{D}$ be two functors. We say that f is *homotopic* to g if there is a finite sequence of functors $f=f_0,f_1,\dots,f_n=g$ such that for each $i=0,\dots,n-1$ there exists a natural transformation between f_i and f_{i+1}. We denote $f\simeq g$.

Note that this is an equivalence relation which is preserved by composition, i.e. if $f\simeq g:\mathscr{C}\to\mathscr{D}$ and $h\simeq i:\mathscr{D}\to\mathscr{E}$ then $hf\simeq ig:\mathscr{C}\to\mathscr{E}$.

2.2. Definition. A functor $f:\mathscr{C}\to\mathscr{D}$ is a *homotopy equivalence* if there exists $g:\mathscr{D}\to\mathscr{C}$ such that $fg\simeq 1_{\mathscr{D}}$ and $gf\simeq 1_{\mathscr{C}}$. A small category $\mathscr{C}$ is *contractible* if it is homotopy equivalent to the singleton category $*$.

2.3. Proposition. *If $f:\mathscr{C}\to\mathscr{D}$ has either a left or a right adjoint then it is a homotopy equivalence.*

Proof. If $g:\mathscr{D}\to\mathscr{C}$ is for example right adjoint of f, then there are natural transformations $fg\Rightarrow 1$ and $1\Rightarrow gf$ and therefore $fg\simeq 1$ and $gf\simeq 1$. ☐

2.4. Corollary. *If $\mathscr{C}$ has either an initial or a final object then it is contractible, since the functor from $\mathscr{C}$ to the category $*$ has an adjoint.*

We recall next from [M1], the definition of *interval categories* I_n $(n\in\mathbb{N})$. We show later that these categories constitute the family of natural cylinders in $\mathscr{C}at$ which is used to define the structure of a Λ-cofibration category [M0].

2.5. Definition. Given $n\in\mathbb{N}$, let I_n be the following category. The objects of I_n are the integers $0,1,\dots,n$ and the morphisms, other than the identities, are defined as follows. If r and s are two distinct objects in I_n there is exactly one morphism from r to s if r is even and $s=r-1$ or $s=r+1$ and no morphisms otherwise. The sketch of I_n is as follows (case n odd).

$$I_n:\qquad 0\longrightarrow 1\longleftarrow 2\longrightarrow 3\longleftarrow \cdots\cdots\longrightarrow n$$

By using the interval categories I_n one can reformulate 2.1 as follows.

2.6. Definition. Two functors $f,g:\mathscr{C}\to\mathscr{D}$ are homotopic if there is an $n\in\mathbb{N}$ and a functor $H:\mathscr{C}\times I_n\to\mathscr{D}$ such that $H(a,0)=f(a)$ and $H(a,n)=g(a)$ for all $a\in\mathscr{C}$. The functor H is called a *homotopy* from f to g and we denote $H:f\simeq g$.

2.7. Definition. Let $n,m\in\mathbb{N}$ with $m\geq n$. A functor $t:I_m\to I_n$ such that $t(0)=0$ and $t(m)=n$ will be called a *subdivision functor*.

2.8. *Remark.* Let $H:f\simeq g$ with $H:\mathscr{C}\times I_n\to\mathscr{D}$. If $m\geq n$ there exists at least one subdivision functor $t:I_m\to I_n$. Thus there is a homotopy $H':\mathscr{C}\times I_m\to\mathscr{D}$ from f to g taking $H'=H(1\times t)$.

In [H1, H2] a weaker notion of homotopy is presented by using textitpath categories. Hoff defines the category **N** whose objects are the nonnegative integers and the morphisms between two objects r and s are defined as in 2.5. A functor $f:\mathbf{N}\to\mathscr{C}$ is finite if there exists $m\in\mathbf{N}$ such that $f(n)=f(m)$ $\forall n\geq m$. The path category of $\mathscr{C}$, denoted by $\mathscr{C}h(\mathscr{C})$ consists of all finite functors from **N** to $\mathscr{C}$. Two functors $f,g:\mathscr{C}\to\mathscr{D}$ are homotopic in the sense of Hoff if there exists a functor $H : \mathscr{C} \to \mathscr{C}h(\mathscr{D})$ such that $\alpha H = f$ and $\omega H = g$ where $\alpha : \mathscr{C}h(\mathscr{D}) \to \mathscr{D}$ is the functor which assigns to each path its initial value and $\omega : \mathscr{C}h(\mathscr{D}) \to \mathscr{D}$ is the functor which assigns to each path its final value (see [H1, H2] for more details). We denote this equivalence relation by $f \simeq_H g$.

2.9. *Remark.* Let $f, g : \mathscr{C} \to \mathscr{D}$. If $f \simeq g$ then $f \simeq_H g$. For a map $H : \mathscr{C} \times I_n \to \mathscr{D}$ induces a map $H : \mathscr{C} \to \mathscr{D}^{I_n}$ where $\mathscr{D}^{I_n}$ can be seen as a subcategory of $\mathscr{C}h(\mathscr{D})$ whose set of objects consists of all finite functors $T : \mathbf{N} \to \mathscr{D}$ such that $T(m) = T(n)\ \forall m \geq n$.

It is easy to see that both notions of homotopy coincide if one considers categories with finite sets of objects. In general the notion of homotopy given in this paper is strictly stronger as it is shown in the following example.

2.10. Example. The category $\mathbf{N}$ is contractible in the sense of Hoff but it is not contractible in the sense of this paper. The functor $H : \mathbf{N} \to \mathscr{C}\mathbf{h}(\mathbf{N})$ defined as $H(n)(m) = m$ for $m \leq n$ and $H(n)(m) = n$ for $m \geq n$ induces a homotopy in the sense of Hoff between the identity of $\mathbf{N}$ and the constant map 0. Since there is no *finite* homotopy between the identity and a constant map, this category is not contractible in the sense of this paper.

There is a weaker notion of homotopy for functors defined using the *classifying space* functor

$$\mathscr{B} : \mathscr{C}at \to \mathscr{T}op$$

introduced in [S]. Two functors $f, g : \mathscr{C} \to \mathscr{D}$ are weak homotopic if $\mathscr{B}f, \mathscr{B}g : \mathscr{B}\mathscr{C} \to \mathscr{B}\mathscr{D}$ are homotopic continuous maps. We denote this equivalence relation by $f \simeq_B g$.

2.11. *Remark.* If $f \simeq_H g$ then $f \simeq_B g$ (see [H1, H2]). Thus

$$f \simeq g \Rightarrow f \simeq_H g \Rightarrow f \simeq_B g.$$

One can prove directly that the homotopy in our sense implies the weakest notion of homotopy using that the classifying space of any interval category is the topological unit interval.

Quillen's famous Theorem A ([Q]) gives sufficient conditions for a functor $f : \mathscr{C} \to \mathscr{C}'$ to be a homotopy equivalence in the weakest sense (i.e. $\mathscr{B}f$ is a homotopy equivalence between the classifying spaces). The following example shows that the strong version of this Theorem is not true.

2.12. Example. Consider the category $\mathbf{N}$ as in 2.10 and let $\mathbf{M}$ be the following category. The objects of $\mathbf{M}$ are the same as in $\mathbf{N}$ and the maps of $\mathbf{M}$ are the maps of $\mathbf{N}$ together with the following maps. If $r = 4k+2$, with $k \geq 0$, there is a map from r to $r-2$ such that the composition of this map with the map from $r-2$ to $r-1$ is the map from r to $r-1$. We can sketch $\mathbf{M}$ as follows.

$$\begin{tikzcd} \mathbf{M}: 0 \arrow[r] & 1 & 2 \arrow[l] \arrow[ll, bend left] \arrow[r] & 3 & 4 \arrow[l] \arrow[r] & 5 & 6 \arrow[l] \arrow[ll, bend left] \arrow[r] & 7 & \cdots \arrow[l] \end{tikzcd}$$

We consider now the inclusion $i : \mathbf{N} \to \mathbf{M}$. For $n \in Obj\ \mathbf{M}$, the category $n \backslash i$ is the category $*$ if n is odd, the category $(I_2)^{op}$ if $n = 4k$, the category I_3 if $n = 2$ and the category $(I_4)^{op}$ if $n = 4k+2$. Thus the categories $n \backslash i$ are strong contractible for all n. But the functor i is not a (strong) homotopy equivalence since there is no $g : \mathbf{M} \to \mathbf{N}$ such that $gi \simeq 1_{\mathbf{N}}$.

Now we consider the case of simplicial complexes. We will define the models of the interval in this setting and show that the classical notion of *contiguity classes*, as defined for example in [Sp], coincides with our notion of homotopy.

2.13. Definition. Given $n \in \mathbb{N}$, let I_n be the following simplicial complex. The vertices of I_n are the integers $0, 1, \ldots, n$. The simplices of I_n are the subsets $\{j\}$ for $j = 0, \ldots, n$ and the subsets $\{j, j+1\}$ for $j = 0, \ldots, n-1$.

2.14. Definition. Two simplicial maps $f, g : K \to L$ are homotopic if there is an $n \in \mathbb{N}$ and a simplicial map $H : K \times I_n \to L$ such that $H(a, 0) = f(a)$ and $H(a, n) = g(a)$ for all vertices $a \in K$.

We recall the definition of *contiguity classes* given in [Sp] and show that this definition coincides with our definition of homotopy.

2.15. Definition. Two simplicial maps $f, g : K \to L$ are contiguous if $f(s) \cup g(s)$ is a simplex in L for any simplex $s \in K$. We say that $f, g : K \to L$ have the same *contiguity class* if there exists a finite sequence $f = f_0, f_1, \ldots, f_n = g$ such that f_i, f_{i+1} are contiguous for each i. We denote $f \simeq_c g$.

2.16. Proposition. *Let $f, g : K \to L$ be simplicial maps. Then $f \simeq_c g$ if and only if $f \simeq g$.*

Proof. Suppose first that $f \simeq_c g$. Then there exists a finite sequence $f = f_0, f_1, \ldots, f_n = g$, with f_i, f_{i+1} contiguous.

Consider the map $H : K \times I_n \to L$, given by

$$H(a, m) = f_m(a)$$

By the definition of the product of K and I_n and since f_i, f_{i+1} are contiguous for each i, then H is a simplicial homotopy between f and g.

Conversely, given a homotopy $G : K \times I_n \to L$, consider the maps $f_i : K \to L$ defined by

$$f_i(a) = G(a, i).$$

$\square$

The category $\mathscr{C}at$ of small categories, the category $\mathscr{S}C$ of simplicial complexes and many other *combinatorial* categories are examples of Λ-cofibration categories.

The homotopy theory for Λ-cofibration categories was introduced in [M0]. We recall now some basic facts on this theory.

For a comprehensive exposition of definitions, examples, results and applications of Λ-cofibration categories we refer the reader to [M0] .

Let Λ denote a set with a relation $\geq$ which is reflexive, transitive and directed, i.e. for any $\alpha, \beta \in \Lambda$, there exists $\gamma \geq \alpha, \beta$.

2.17. Definition. Let $\mathscr{C}$ be a category. A Λ-*cylinder on* $\mathscr{C}$ consists of a family

$$\{(I_\alpha, p_\alpha, i_0^\alpha, i_1^\alpha) \mid \alpha \in \Lambda\}$$

of natural cylinders on $\mathscr{C}$ (as defined above) and for each pair $\beta \geq \alpha$ in Λ, a non-empty set $(\beta \geq \alpha)$ of natural transformations from I_β to I_α such that the following conditions hold:

(a) $ti_\varepsilon^\beta = i_\varepsilon^\alpha$ and $p_\alpha t = p_\beta \quad \forall\, t \in (\beta \geq \alpha)$, $\varepsilon = 0, 1$.

(b) If $t \in (\beta \geq \alpha)$ and $s \in (\gamma \geq \beta)$, then $ts \in (\gamma \geq \alpha)$.

(c) Given $t_1 \in (\alpha_1 \geq \alpha)$ and $t_2 \in (\alpha_2 \geq \alpha)$, there exist $\beta \geq \alpha_1, \alpha_2$, $s_1 \in (\beta \geq \alpha_1)$, and $s_2 \in (\beta \geq \alpha_2)$ such that $t_1 s_1 = t_2 s_2$.

The category $\mathscr{C}at$ has a Λ-cylinder with $\Lambda = \mathbb{N}$, with the usual order. For any $\alpha \in \mathbb{N}$, the α-cylinder of a category $\mathscr{C}$ is the product with the interval category

$$I_\alpha \mathscr{C} = \mathscr{C} \times I_\alpha.$$

The natural transformations $t \in (\beta \geq \alpha)$ are induced by the subdivision functors defined in 2.7.

The category $\mathscr{S}C$ of simplicial complexes and many other combinatorial categories admit a natural $\mathbb{N}$-cylinder in a similar way.

One of the most important notions in a category with a Λ-cylinder is the concept of *subdivision*. If an object such as a cone, a suspension or a mapping cylinder is constructed using a particular cylinder, it will be necessary to relate it to corresponding objects constructed using other cylinders from the family $\{I_\alpha :\ \alpha \in \Lambda\}$. To this end in [M0] and [M1] we introduced the concept of *adding points* or *subdivision* available in any category with a Λ-cylinder. We illustrate this idea with the following example.

2.18. Example. Let $\{I_\alpha \mid \alpha \in \Lambda\}$ be a Λ-cylinder on a pointed category $\mathscr{C}$ with base point $*$. Suppose that for some object A of $\mathscr{C}$ and for every $\alpha \in \Lambda$ the pushouts

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i_0^\alpha"'] \arrow[dr, phantom, "Push"] & * \arrow[d] \\ I_\alpha A \arrow[r, "k_\alpha"] & C_\alpha A \end{tikzcd}$$

exist. For any fixed $\alpha \in \Lambda$, $C_\alpha A$ is called the α-*cone* of A.

If one takes $\beta \geq \alpha$ and a transformation $t \in (\beta \geq \alpha)$, one gets the pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i_0^\beta"'] \arrow[dr, phantom, "Push"] & * \arrow[d] \\ I_\beta A \arrow[r, "k_\beta"] & C_\beta A \end{tikzcd}$$

and a morphism of diagrams

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i_0^\beta"'] \arrow[ddr, "1"] & * \arrow[ddr, "1"] & \\
I_\beta A \arrow[ddr, "t_A"'] & & \\
& A \arrow[r] \arrow[d, "i_0^\alpha"] & * \\
& I_\alpha A &
\end{tikzcd}$$

which induces a morphism

$$T : C_\beta A \to C_\alpha A.$$

The cone $C_\beta A$ is called a subdivision of $C_\alpha A$ and the map T is called a transformation. Note also that we have commutative diagrams

$$\begin{tikzcd}
A \arrow[r, "i_0^\beta"] \arrow[d, "1"'] & I_\beta A \arrow[d, "t"] \\
A \arrow[r, "i_0^\alpha"] & I_\alpha A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "*"] \arrow[d, "1"'] & C_\beta A \arrow[d, "t"] \\
A \arrow[r, "*"] & C_\alpha A
\end{tikzcd}
\qquad
\begin{tikzcd}
I_\beta A \arrow[r, "k_\beta"] \arrow[d, "t"'] & C_\beta A \arrow[d, "t"] \\
I_\alpha A \arrow[r, "k_\alpha"] & C_\alpha A
\end{tikzcd}$$

The maps $i_0^\beta : A \to I_\beta A$, $* : A \to C_\beta A$ and $k_\beta : I_\beta A \to C_\beta A$ are subdivisions of the corresponding maps i_0^α, $*$ and k_α.

The notion of homotopy in a category with a Λ-cylinder is clear:

2.19. Definition. Let $\{I_\alpha \mid \alpha \in \Lambda\}$ be a Λ-cylinder on a category $\mathscr{C}$. Two morphisms $f, g : A \to B$ in $\mathscr{C}$ are homotopic if there exists $\alpha \in \Lambda$ and a morphism $H : I_\alpha A \to B$ such that $Hi_0^\alpha = f$ and $Hi_1^\alpha = g$.

Note that, by definition, for any $\beta \geq \alpha$ there exists at least one natural transformation $t \in (\beta \geq \alpha)$. Therefore, for any given homotopy $H : I_\alpha A \to B$ we can find another homotopy $H' = Ht : I_\beta A \to B$ from f to g.

Note also that this notion does not induce, in general, an equivalence relation. For that, we need more hypotheses on the Λ-cylinder and the notion of *cofibration*.

3. Cofibrations

Besides the concepts of natural cylinders and subdivision, the other main ingredient in the theory of Λ-cofibration categories is the notion of *cofibration*.

The classical notion of cofibration of topological spaces corresponds to maps with the homotopy extension property. In the combinatorial case, since we have a family of cylinders instead of just one, the right notion of cofibration uses the concept of subdivision, i.e. the replacement of a given homotopy $H : I_\alpha C \to D$ by a homotopy $H' : I_\beta C \to D$ defined in a bigger cylinder I_β with $\beta \geq \alpha$.

Let $\mathscr{C}$ be a category with a Λ-cylinder.

3.1. Definition. A map $i : A \to B$ has the *homotopy extension property* if given $\alpha \in \Lambda$ and a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i_\varepsilon^\alpha"] \arrow[d, "i"'] & I_\alpha A \arrow[d, "H"] \\ B \arrow[r, "f"] & X \end{tikzcd}$$

there exist $\beta \geq \alpha$ and $t \in (\beta \geq \alpha)$, such that the commutative diagram

$$\begin{tikzcd} A \arrow[r, "i_\varepsilon^\beta"] \arrow[d, "i"'] & I_\beta A \arrow[d, "Ht"] \\ B \arrow[r, "f"] & X \end{tikzcd}$$

satisfies the following extension property: there exists a map $G : I_\beta B \to X$, with $Gi_\varepsilon^\beta = f$ and $GI_\beta(i) = Ht$.

We shall frequently set $H' = Ht$.

In [M0, M1] we defined the notion of *special arrow* in any category with a Λ-cylinder. A special arrow $f : A \to B$ is a map induced by a morphism of diagrams in $\mathscr{C}$, as in example 2.18, and these special arrows can be naturally subdivided, as it is shown also in that example. A subdivision of a map $f : A \to B$ is denoted $f' : A' \to B'$.

Another simple example of special arrow is the following. Let $A+A$ denote the coproduct of two copies of A and consider the map $f = (i_0^\alpha, i_1^\alpha) : A+A \to I_\alpha A$ which *geometrically* corresponds to the inclusion of each copy of A in the top and bottom of the cylinder.

This map is an example of a special arrow and for any $\beta \geq \alpha$, the map $f' = (i_0^\beta, i_1^\beta) : A+A \to I_\beta A$ is a subdivision of f.

3.2. Definition. We say that a map $i : A \to B$ has the *weak homotopy extension property* if after a suitable subdivision $i' : A' \to B'$, it satisfies the homotopy extension property.

In a Λ-cofibration category, cofibrations must satisfy the weak homotopy extension property. In fact, in most important examples (topological spaces, categories, simplicial complexes, global actions, etc), cofibrations are precisely the maps with this property.

In a similar way one can define the *homotopy lifting property*. This notion is not necessary for the development of the homotopy theory in Λ-categories. However, it is a very useful complement to the theory (cf. [M4]).

3.3. Definition. A map $p : E \to B$ has the *homotopy lifting property* if given $\alpha \in \Lambda$ and a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_\varepsilon^\alpha"'] & E \arrow[d, "p"] \\ I_\alpha A \arrow[r, "H"] & B \end{tikzcd}$$

there exist $\beta \geq \alpha$ and $t \in (\beta \geq \alpha)$, such that the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_\varepsilon^\beta"'] & E \arrow[d, "p"] \\ I_\beta A \arrow[r, "Ht"] & B \end{tikzcd}$$

satisfies the following: there exists a map $G : I_\beta A \to E$, with $Gi_\varepsilon^\beta = f$ and $pG = Ht$.

3.4. *Remark.* In $\mathscr{C}at$, functors with the weak homotopy lifting property are called *fibrations*. They are the analogues of Hurewicz fibrations in $\mathscr{T}op$. We showed in [M1] that any functor f admits a factorization of the form

$$f = ri$$

with i a cofibration and r a homotopy equivalence. In a similar way, it can be proved that f has a factorization $f = pr$ with p a fibration and r a homotopy equivalence. To prove this, one proceeds as in the case of the cofibration, but replaces cylinder categories by path categories.

We are now in condition to define the notion of Λ-cofibration category.

Fix Λ a directed set with relation $\geq$.

3.5. Definition. A Λ-*cofibration category* is a category $\mathscr{C}$ with structure

$$(\mathscr{C}, cof, \emptyset, (I_\alpha)_{\alpha\in\Lambda}, (\beta \geq \alpha))$$

where $\emptyset$ is an initial object in $\mathscr{C}$ and cof is a class of maps in $\mathscr{C}$ called *cofibrations*, such that the following axioms $(\Lambda1), \ldots, (\Lambda6)$ hold.

$(\Lambda1)$ $\{I_\alpha \mid \alpha \in \Lambda\}$ is a Λ-cylinder on $\mathscr{C}$ with natural transformations $(\beta \geq \alpha)$ for each pair $\beta \geq \alpha \in \Lambda$.

$(\Lambda2)$ If $i : A \to B$ is a cofibration and $f : A \to X$ is any map, the pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] \arrow[dr, phantom, "Push"] & X \arrow[d, "j"] \\ B \arrow[r] & B \underset{A}{\cup} X \end{tikzcd}$$

exists and j is also a cofibration. Moreover, for all $\alpha \in \Lambda$, the functor I_α carries pushouts into pushouts and $I_\alpha \emptyset = \emptyset$.

$(\Lambda3)$ Isomorphisms are cofibrations. For all objects X, the map $\emptyset \to X$ is a cofibration.

$(\Lambda4)$ Cofibrations satisfy the weak homotopy extension property (3.2). We call a cofibration *strong* if it satisfies the homotopy extension property (3.1). The composition of strong cofibrations is a strong cofibration. In general, compositions of cofibrations are cofibrations.

$(\Lambda 5)$ If $i : A \to B$ is a cofibration, then for each α, the map j_α defined below is also a cofibration.

$$\begin{tikzcd}
A+A \arrow[r, "i+i"] \arrow[d, "(i_0^\alpha, i_1^\alpha)"'] \arrow[dr, phantom, "Push"] & B+B \arrow[d] \arrow[ddr, bend left, "(i_0^\alpha, i_1^\alpha)"] & \\
I_\alpha A \arrow[r] \arrow[drr, bend right, "I_\alpha(i)"'] & B \underset{A}{\cup} I_\alpha A \underset{A}{\cup} B \arrow[dr, dotted, "\exists j_\alpha"] & \\
 & & I_\alpha B
\end{tikzcd} .$$

$(\Lambda 6)$ For all $\alpha, \beta \in \Lambda$, there exists a natural transformation

$$T_{\alpha\beta} : I_\alpha I_\beta \to I_\beta I_\alpha$$

such that, for all X:

$$T_{\alpha\beta} i_\varepsilon^\alpha (I_\beta X) = I_\beta (i_\varepsilon^\alpha (X)) : I_\beta X \to I_\beta I_\alpha X$$

and

$$T_{\alpha\beta} I_\alpha (i_\varepsilon^\beta (X)) = i_\varepsilon^\beta (I_\alpha X) : I_\alpha X \to I_\beta I_\alpha X.$$

Note that the category $\mathscr{T}op$ is a Λ-cofibration category with $\Lambda = *$ (only one cylinder), that is, $\mathscr{T}op$ is an I-category in the sense of Baues [B].

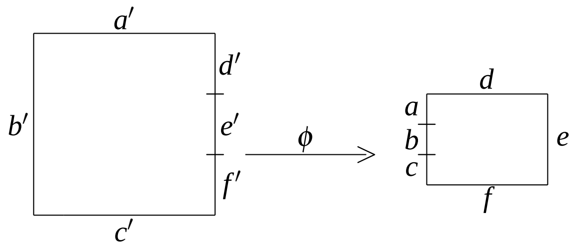
As we pointed out above, the categories $\mathscr{C}at$ and $\mathscr{S}C$ admit a Λ-cylinder with $\Lambda = \mathbb{N}$. These are examples of Λ-cofibration categories. In these cases, the natural cylinders are induced by taking products with the models of the interval, cofibrations are the maps with the weak homotopy extension property and the natural transformations $T : I_n I_m \to I_m I_n$ of $(\Lambda 6)$ are induced by the interchange maps

$$T : A \times I_n \times I_m \to A \times I_m \times I_n$$

defined by $T(a, r, s) = (a, s, r)$. With these definitions, axioms $(\Lambda 1)$–$(\Lambda 4)$ and $(\Lambda 6)$ are easily verified. As we showed in [M1], the proof of $(\Lambda 5)$ can be deduced from the following lemma (3.6).

Given $m, n \in \mathbb{N}$, the product category $I_n \times I_m$ will be sketched as a rectangle, where for example the upper side corresponds to the objects $(0,0), \dots, (0,m) \in I_n \times I_m$, the left side to the objects $(0,0), \dots, (n,0)$, etc. A functor $F : I_{n'} \times I_{m'} \to I_n \times I_m$ will be sketched as an $((n'+1) \times (m'+1))$-matrix (a_{ij}), where $a_{ij} = F((i,j))$. There will be no confusion about the values of F on maps since between two objects in $I_n \times I_m$ there exists at most one map.

3.6. Lemma. *Given $n, m \in \mathbb{N}$, there are m', n', with $n' \geq n$ and $m' \geq m$, and a functor $\phi : I_{n'} \times I_{m'} \to I_n \times I_m$ with the following sketches on the boundaries*



where $a' \to a$, $b' \to b$, etc are subdivision functors. Moreover, there exist $n'', m'' \geq n', m'$ and a functor $\phi' : I_{n''} \times I_{m''} \to I_{n'} \times I_{m'}$ with the opposite sketches on the boundaries, such that the composition $\phi\phi' : I_{n''} \times I_{m''} \to I_n \times I_m$ has the form $\phi\phi' = t_1 \times t_2$, where t_1 and t_2 are subdivision functors $t_1 : I_{n''} \to I_n$ and $t_2 : I_{m''} \to I_m$.

Proof. Let $n, m \in \mathbb{N}$. Suppose n and m are even (the other cases are similar). Take $n' = 2m + n$ and $m' = m$ and define $\phi : I_{n'} \times I_{m'} \to I_n \times I_m$ as follows.

$$\phi = \begin{array}{ccccc}
(0,0) & \dots & \dots & \dots & (0,0) \\
(0,0) & (0,1) & \dots & \dots & (0,1) \\
(0,0) & (0,1) & (0,2) & \dots & (0,2) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(0,0) & (0,1) & (0,2) & \dots & (0,m) \\
(1,0) & (1,1) & (1,2) & \dots & (1,m) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(n,0) & \dots & \dots & \dots & (n,m) \\
(n,0) & (n,1) & \dots & (n,m-1) & (n,m-1) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(n,0) & (n,0) & \dots & \dots & (n,0)
\end{array}$$

Now take $n'' = 4m + n$ and $m'' = m$ and define $\phi' : I_{n''} \times I_{m''} \to I_{n'} \times I_{m'}$ as follows.

$$\phi' = \begin{array}{ccccc}
(0,m) & (1,m) & \dots & \dots & (m,m) \\
(0,m-1) & (1,m-1) & \dots & (m-1,m-1) & (m,m) \\
(0,m-2) & (1,m-2) & \dots & (m-1,m-1) & (m,m) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(0,1) & (1,1) & \dots & (m-1,m-1) & (m,m) \\
(1,0) & (1,1) & (2,2) & \dots & (m,m) \\
(2,0) & (2,1) & (2,2) & \dots & (m,m) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(m,0) & (m,1) & \dots & \dots & (m,m) \\
(m+1,0) & (m+1,1) & \dots & \dots & (m+1,m) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(m+n,0) & (m+n,1) & \dots & \dots & (m+n,m) \\
(m+n+1,0) & (m+n+1,1) & \dots & \dots & (m+n,m) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(2m+n,0) & (2m+n-1,1) & \dots & \dots & (m+n,m) \\
(2m+n,1) & (2m+n-1,1) & \dots & \dots & (m+n,m) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(2m+n,m) & (2m+n-1,m) & \dots & \dots & (m+n,m)
\end{array}$$

It is easy to check that the composition $\phi\phi'$ is indeed $\phi\phi' = t_1 \times t_2$ for some subdivision functors t_1 and t_2. $\square$

4. HOMOTOPY THEORY FOR Λ-COFIBRATION CATEGORIES

Let $\mathscr{C} = (\mathscr{C}, cof, \emptyset, (I_\alpha)_{\alpha\in\Lambda}, (\beta \geq \alpha))$ be a Λ-cofibration category. By definition, one can easily verify the following properties.

(a) $\emptyset \to A$ is a cofibration for any object A.
(b) The natural "inclusions" $(i_0^\alpha, i_1^\alpha) : A+A \to I_\alpha A$ are cofibrations.
(c) Strong cofibrations are stable under pushouts.
(d) If $i : A \to B$ is a cofibration then $I_\alpha(i) : I_\alpha A \to I_\alpha B$ is a cofibration for each α.

In the previous section we defined the notion of homotopy for any category with a Λ-cylinder. For Λ-cofibration categories, homotopy is an equivalence relation (this is not necessarily true for general categories with Λ-cylinders).

4.1. Proposition. *In a Λ-cofibration category, homotopy of maps defines an equivalence relation.*

Proof. Take α big enough such that $H : I_\alpha A \to B$ and $G : I_\alpha A \to B$ are homotopies $H : f \simeq g$ and $G : f \simeq h$. Consider the commutative diagram

$$\begin{array}{ccc} A+A & \xrightarrow{i_0} & I_\alpha(A+A) = I_\alpha A + I_\alpha A\ . \\ \downarrow{\scriptstyle (i_0,i_1)} & & \downarrow{\scriptstyle (H,G)} \\ I_\alpha A & \xrightarrow{fp} & B \end{array}$$

Since (i_0, i_1) is a cofibration, there exists $\beta \geq \alpha$ and a homotopy extension $E : I_\beta I_\beta A \to B$. The map $Ei_1 : I_\beta A \to B$ is a homotopy from g to h.

This proves that the relation is transitive and symmetric (take $h = f$). □

In a pointed Λ-cofibration category (i.e. a Λ-cofibration category where the initial object is also a terminal object, which is denoted by $*$) the groups $\pi_n^A(U)$ for $n \geq 1$ are defined using the α-*suspensions* ($\alpha \in \Lambda$).

To exemplify this construction, let $\mathscr{C}at_*$ be the category of pairs $(\mathscr{C}, x)$ with $\mathscr{C} \in \mathscr{C}at$ and $x \in Obj\,\mathscr{C}$. This category is a pointed Λ-cofibration category and for any $m \in \mathbb{N}$, the m-suspension of an object $(\mathscr{C}, x)$ is the pushout

$$\begin{array}{ccc} (\mathscr{C},x)+(\mathscr{C},x) & \longrightarrow & * \\ {\scriptstyle (i_0^\alpha,i_1^\alpha)}\downarrow & \mathit{Push} & \downarrow \\ I'_m(\mathscr{C},x) & \longrightarrow & \Sigma_m\mathscr{C} \end{array}$$

where I'_m is the reduced cylinder

$$I'_m = (\mathscr{C}, x) = \mathscr{C} \times I_m / \{x\} \times I_m.$$

The groups $\pi_n^{(\mathscr{C},x)}(\mathscr{D}, y)$ (or simply $\pi_n^{\mathscr{C}}(\mathscr{D})$) are defined by

$$\pi_n^{\mathscr{C}}(\mathscr{D}) = \operatorname*{colim}_m [\Sigma_m^n \mathscr{C}, \mathscr{D}].$$

Note that this colimit is constructed using all (reduced) cylinders and all transformations from the structure of Λ-cylinder in $\mathscr{C}at$. In [M0] we proved that these groups are Abelian for $n \geq 2$.

Denote by S^0 the category with two objects and with no maps other than the identities. Let $\pi_n(\mathscr{D}) = \pi_n^{S^0}(\mathscr{D})$.

Since S^0 is finite, one can check that $\pi_n(\mathscr{D}) = \pi_n(\mathscr{B}\mathscr{D})$. This important result gives a method to compute higher homotopy groups of some topological spaces using combinatorial tools.

In [E] M. Evrard defines the groups $\pi_n(\mathscr{D})$ of a pointed small category $\mathscr{D}$ introducing a loop functor $\Omega : \mathscr{C}at_* \to \mathscr{C}at_*$ which is very similar to the loop functor defined by Hoff. Evrard constructs that functor using interval categories and he reproves Quillen's theorems A and B. The homotopy groups $\pi_n(\mathscr{D})$ defined above are isomorphic to the ones defined by Evrard and by Hoff.

We have similar definitions and results in the category $\mathscr{S}C$ of simplicial complexes. For instance:

4.2. Proposition. *The homotopy groups $\pi_n(K)$ of a simplicial complex K coincide with the homotopy groups $\pi_n(|K|)$ of its geometric realization.*

4.3. Example. Consider the border of the two simplex $\dot{s}$ with vertices $\{a,b,c\}$. Using the above result we give an elementary proof that $\pi_1(S^1) = \pi_1(\dot{s}) = \mathbb{Z}$. Let a be the base point of $\dot{s}$. The elements of $\pi_1(\dot{s})$ are, by definition, homotopy classes of finite sequences $[a_1,\dots,a_n]$ for $n \in \mathbb{N}$ with $a_i \in \{a,b,c\}$ and $a_1 = a_n = a$.

Since the transformations 'repeat points', it is easy to see that

$$[a_1 \dots a_i a_{i+1} \dots a_n] = [a_1 \dots a_i a_{i+2} \dots a_n] \qquad \text{if } a_i = a_{i+1}$$

and by the simplicial structure of $\dot{s}$, we have also:

$$[a_1 \dots a_{i-1} a_i a_{i+1} \dots a_n] = [a_1 \dots a_{i-1} a_{i+1} \dots a_n] \qquad \text{if } a_{i-1} = a_{i+1}.$$

Thus the elements of $\pi_1(\dot{s})$ are of the following three types: $[a]$, $[abcabc\dots a]$ or $[acbacb\dots a]$.

Note also that

$$[abca] + [acba] = [abcacba] = [a].$$

Therefore $\pi_1(\dot{s}) = \mathbb{Z}$ generated by $[abca]$.

In Λ-cofibration categories one can obtain the best known results of classical homotopy theory. For example, if $i : A \to B$ is a strong cofibration (i.e. a cofibration which satisfies the homotopy extension property) then for any object C there is a long exact sequence of the cofibre

$$\dots \to \pi^A_{n+1}(C) \to \pi^{B/A}_n(C) \to \pi^B_n(C) \to \pi^A_n(C) \to \dots$$

Recall that the cofibre of the map $i : A \to B$ is the pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & * \arrow[d] \\ B \arrow[r] & B/A \end{tikzcd}$$

We finish this section of the paper with a brief discussion about homotopy equivalences. In a Λ-cofibration category, there are different notions of homotopy equivalences.

A map $f : A \to B$ is a *homotopy equivalence in the strong sense* if there exists a homotopy inverse $g : B \to A$ such that $fg \simeq 1_B$ and $gf \simeq 1_A$.

A map $f : A \to B$ is a *homotopy equivalence in the weak sense* if there exists a subdivision $f' : A' \to B'$, transformation maps $T_A : A' \to A$ and $T_B : B' \to B$ (see 2.18) and a homotopy inverse $g : B \to A$ such that $gT_B f' \simeq T_A$ and $fT_A g' \simeq T_B$.

Note that the tranformation maps $T_A : A' \to A$ are not necessarily homotopy equivalences (even in the weak sense) since one cannot find in general homotopy inverses $A \to A'$, as we can see in the following example.

4.4. Example. Consider the following category T_3

$$\begin{tikzcd} * + * \arrow[r] \arrow[d, "(i_0,i_1)"'] \arrow[dr, phantom, "Push"] & * \arrow[d] \\ I_3 \arrow[r] & T_3 \end{tikzcd}$$

This category can be sketched as follows:

$$\begin{tikzcd} 0 \arrow[r, "\alpha"] & 1 & 2 \arrow[l, "\beta"'] \arrow[ll, bend right, "\gamma"'] \end{tikzcd}$$

with $\alpha\gamma \neq \beta$. Replacing I_3 by I_4 we construct in the same way T_4 whose sketch is as follows:

$$\begin{tikzcd} 0 \arrow[r, "a"] \arrow[rrr, bend right, "d"'] & 1 & 2 \arrow[l, "b"'] \arrow[r, "c"] & 3 \end{tikzcd}$$

The subdivision functor $t : I_4 \to I_3$ defined by $t(r) = r$ for $r = 0,1,2,3$ and $t(4) = 3$ induces a transformation map $T : T_4 \to T_3$ which is not a strong homotopy equivalence, since there is no $F : T_3 \to T_4$ with $TF \simeq 1$.

5. Complexes and Freudenthal Theorem

In this section we work specifically in the category $\mathscr{C}at_*$ of pointed categories. Note, however, that many constructions and results that we obtain here can be obtained in other settings like simplicial complexes.

Let $\mathrm{Mor}_*(A,B)$ denote the set of pointed functors from A to B. This is a pointed set; its base point is the functor $\tilde{b}_0 : A \to B$ which takes all maps of A to the identity of b_0.

Let $\mathscr{C}at_*(A,B)$ be the following pointed category. The set of objects is the pointed set $\mathrm{Mor}_*(A,B)$ and the morphisms are the natural transformations $\gamma : f \to g$ between such functors such that $\gamma(a_0) = 1_{b_0} : f(a_0) = b_0 \to g(a_0) = b_0$.

Given two pointed categories C and D, we denote by $C \vee D$ the following subcategory of $C \times D$. The objects of $C \vee D$ are all the objects of the form (c,d_0) and (c_0,d). The set of maps of $C \vee D$ is generated by the maps of the form $(f,1) : (c,d_0) \to (c',d_0)$ and $(1,g) : (c_0,d) \to (c_0,d')$ with $f : c \to c'$ in C and $g : d \to d'$ in D. We denote then by $C \wedge D$ the following pushout in $\mathscr{C}at$

$$\begin{tikzcd} C \vee D \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "Push"] & * \arrow[d] \\ C \times D \arrow[r] & C \wedge D \end{tikzcd}$$

The set of objects of this category can be described as the set of classes $[(c,d)]$ with $(c,d) \in C \times D$ such that $[(c,d)] = [(c',d')]$ if $c = c'$ and $d = d'$ or $(c,d),(c',d') \in C \vee D$. We consider $C \wedge D$ as a pointed category with base point $[(c_0,d_0)]$.

Using these definitions it is easy to prove the pointed exponential law for categories.

5.1. Proposition. *Given pointed categories A, B and C, there is an isomorphism of pointed sets*

$$\mathrm{Mor}_*(A \wedge B, C) \simeq \mathrm{Mor}_*(A, \mathscr{C}at_*(B,C))$$

which is natural in A, B and C.

Given $\alpha \in \mathbb{N}$, we denote $S^1_\alpha = \Sigma_\alpha S^0$ and define the α*-loop functor*

$$\Omega_\alpha = \mathscr{C}at_*(S^1_\alpha, -) : \mathscr{C}at_* \to \mathscr{C}at_*.$$

It is clear that Ω_α is right adjoint for Σ_α for any $\alpha \in \mathbb{N}$ in $\mathscr{H}o(\mathscr{C}at_*) = \mathscr{C}at_*/\simeq$, the strong homotopy category of $\mathscr{C}at_*$. Explicitly,

$$[\Sigma_\alpha A, B] \simeq [A, \Omega_\alpha B].$$

Given a pointed category B and $\alpha \in \mathbb{N}$, we can describe the α-loop category $\Omega_\alpha B$ as follows (case α even). The objects of $\Omega_\alpha B$ are the sequences of the form

$$b_0 \xrightarrow{f_0} b_1 \xleftarrow{f_1} b_2 \xrightarrow{f_2} \cdots\cdots \longrightarrow b_{\alpha-1} \xleftarrow{f_{\alpha-1}} b_0 \ .$$

The maps of $\Omega_\alpha B$ are commutative diagrams

$$\begin{tikzcd}
b_0 \arrow[r, "f_0"] \arrow[d, equal] & b_1 \arrow[d, "\gamma_1"] & b_2 \arrow[l, "f_1"'] \arrow[r, "f_2"] \arrow[d, "\gamma_2"] & \cdots\cdots \arrow[r] & b_{\alpha-1} \arrow[d, "\gamma_{\alpha-1}"] & b_0 \arrow[l, "f_{\alpha-1}"'] \arrow[d, equal] \\
b_0 \arrow[r, "f'_0"] & b'_1 & b'_2 \arrow[l, "f'_1"'] \arrow[r, "f'_2"] & \cdots\cdots \arrow[r] & b'_{\alpha-1} & b_0 \arrow[l, "f'_{\alpha-1}"']
\end{tikzcd}$$

5.2. *Remark.* For $\beta \geq \alpha$, any subdivision functor $t \in (\beta \geq \alpha)$ induces a map $t^* : \Omega_\alpha B \to \Omega_\beta B$. If we consider all $\alpha \in \mathbb{N}$ and all the subdivision functors, we can interpret the homotopy groups in terms of the α-loop functors

$$\pi^A_n(B) = \operatorname*{colim}_\alpha [A, \Omega^n_\alpha B].$$

5.3. Definition. Define a *big loop functor*

$$\Omega = \operatorname*{colim}_\alpha \Omega_\alpha : \mathscr{C}at_* \to \mathscr{C}at_*.$$

The maps $\Omega_\alpha B \to \Omega B$ induce a map

$$\varepsilon : \pi^A_n(B) \to \pi^A_{n-1}(\Omega B).$$

Note that this map is not in general an isomorphism.

A category A is called *finite* if its set of morphisms is finite. If A is a finite category then the natural map $\nu : \operatorname*{colim}_\alpha [A, \Omega_\alpha B] \to [A, \Omega B]$ is a bijection.

The following result is an immediate consequence of this remark.

5.4. Proposition. *If A is finite, the map $\varepsilon : \pi_n^A(B) \to \pi_{n-1}^A(\Omega B)$ is an isomorphism. In particular for $A = S^0$ we obtain the isomorphisms*

$$\pi_n(B) = \pi_{n-1}(\Omega B) = \pi_0(\Omega^n B)$$

where $\Omega^n B = \Omega(\Omega^{n-1}B) = \operatorname*{colim}_{\alpha} \Omega^n_\alpha B$.

Note that our big loop category ΩB is similar to the loop category introduced by Hoff [H2] which we denote by $\Omega_H B$. In fact there is a quotient map $\Omega_H B \to \Omega B$ which induces an isomorphism

$$\pi_0(\Omega_H B) \xrightarrow{\simeq} \pi_0(\Omega B)\ .$$

Given a topological space X, the adjunction map $X \to \Omega\Sigma X$ induces a map $\pi_r(X) \to \pi_{r+1}(\Sigma X)$. The classical Freudenthal suspension theorem says that this map is an isomorphism for $r \leq 2n$ and an epimorphism for $r = 2n+1$ if X is an n-connected CW-complex.

If A is a pointed small category and $r \in \mathbb{N}$, for every $\alpha \in \mathbb{N}$ the adjunction map $A \to \Omega_\alpha \Sigma_\alpha A$ induces a map

$$\Sigma_\alpha : \pi_r(A) \to \pi_r(\Omega_\alpha \Sigma_\alpha A) \to \operatorname*{colim}_{\beta} \pi_r(\Omega_\beta \Sigma_\alpha A) = \pi_{r+1}(\Sigma_\alpha A).$$

We know that $\pi_r(A) = \pi_r(\mathscr{B}(A))$ and in particular if A is an n-connected category then $\mathscr{B}(A)$ is an n-connected CW-complex. We will prove that under a certain cofibration condition on A, the spaces $\mathscr{B}(\Sigma_\alpha A)$ and $\Sigma\mathscr{B}(A)$ are homotopy equivalent and therefore in that case we obtain the analogue of the Freudenthal theorem for $\Sigma_\alpha : \pi_r(A) \to \pi_{r+1}(\Sigma_\alpha A)$. This cofibration condition is satisfied by any complex-category. Complex-categories are analogous to CW-complexes and were introduced in [M2].

If A and C are pointed categories it is not in general true that $\mathscr{B}(A \wedge C)$ and $\mathscr{B}(A) \wedge B(C)$ are homotopy equivalent, even if A or C are finite. This is because the natural map from the pushout of the nerve to the nerve of the pushout is not in general a weak equivalence of simplicial sets.

Under a certain cofibration condition on A and C the map $\Gamma : N(A) \wedge N(C) \to N(A \wedge C)$, which is defined via the identification $N(A \times C) = N(A) \times N(C)$, is a weak equivalence of simplicial sets and therefore in this case we obtain a homotopy equivalence $B(A) \wedge B(C) \to \mathscr{B}(A \wedge C)$.

5.5. *Remark.* Given a functor $F : K \to \mathscr{C}at$, recall the Grothendieck construction on F, $K \int F$ (cf. [T0]). A pushout

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ {\scriptstyle i}\downarrow & & \downarrow \\ X_0 & \longrightarrow & X_0 \cup_{X_1} X_2 \end{array}$$

in $\mathscr{C}at$ can be seen as the colimit of a functor $F : I_2^{op} \to \mathscr{C}at$, where I_2^{op} is the oposite category of I_2. If $i : X_1 \to X_0$ is a cofibration, then $I_2^{op} \int F$ and $X_0 \cup_{X_1} X_2$ are weak equivalent. This result is analogous to the well-known result about homotopy pushouts and pushouts of simplicial sets. In our case, the Grothendieck construction $K \int F$ plays the role of the homotopy colimit (see [T0]). To prove this, note that the cofibration i satisfies the extension property with respect to maps $X_1 \times I_2^{op} \to Y$ up to subdivision.

5.6. Proposition. *Given a pushout*

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ {\scriptstyle i}\downarrow & & \downarrow \\ X_0 & \longrightarrow & X_0 \cup_{X_1} X_2 \end{array} ,$$

if i is a cofibration then the natural map

$$NX_0 \cup_{NX_1} NX_2 \to N(X_0 \cup_{X_1} X_2)$$

is a weak equivalence of simplicial sets.

Proof. By [T0] and since Ni is a cofibration of simplicial sets one has a weak equivalence $NX_0 \cup_{NX_1} NX_2 \to N(I_2^{op} \int F)$, and by the last remark the map $N(I_2^{op} \int F) \to N(X_0 \cup_{X_1} X_2)$ is also a weak equivalence. □

5.7. Corollary. *If the inclusion $A \vee C \to A \times C$ is a cofibration and either A or C is a finite category, then $\mathscr{B}(A \wedge C) \simeq \mathscr{B}A \wedge \mathscr{B}C$.*

5.8. Definition. We say that a category A satisfies the S^1_α-cofibration condition if the inclusion $A \vee S^1_\alpha \to A \times S^1_\alpha$ is a cofibration. In this case, by the last corollary, $\mathscr{B}(\Sigma_\alpha A) \simeq \Sigma\mathscr{B}(A)$.

The most important example of categories which satisfy the S^1_α-cofibration condition are the complex-categories. We recall the definition of complexes in $\mathscr{C}at$ given in [M2].

Let S^0 be the discrete category with two points. For $n \in \mathbb{N}$, we define the *n-dimensional α-sphere* as the pointed category $S^n_\alpha = \Sigma^n_\alpha S^0$.

Given a pointed category A and $\alpha \in \mathbb{N}$, the *reduced α-cone* $C_\alpha A$ of A is the pushout

$$\begin{array}{ccc} A+A & \xrightarrow{(1,*)} & A \\ {\scriptstyle (i_0,i_1)}\downarrow & \textit{Push} & \downarrow \\ I_\alpha A & \longrightarrow & C_\alpha A \end{array}$$

Here $I_\alpha A$ denotes the reduced α-cylinder of A.

5.9. *Remark.* Note that the natural inclusion $j_\beta : S^n_\alpha \to C_\beta S^n_\alpha$ is a cofibration.

5.10. Definition. A category $\tilde{X}$ is obtained from X by attaching an n-cell of length (α,β) (or simply an $n-(\alpha,\beta)$-cell) if there exists a pushout

$$\begin{array}{ccc} S^{n-1}_\alpha & \xrightarrow{f} & X \\ {\scriptstyle j_\beta}\downarrow & \textit{Push} & \downarrow \\ C_\beta S^{n-1}_\alpha & \longrightarrow & \tilde{X} \end{array}$$

for some *attaching map* $f : S^{n-1}_\alpha \to X$.

Attaching a 0-cell will mean to add a disjoint point.

5.11. Definition. A pair of categories (X,A) is a relative complex category if there exists a sequence $A = X^{-1} \to X^0 \to X^1 \to \dots$ with $X = \operatorname{colim} X^n$ and X^n is obtained from X^{n-1} by attaching $n-(\alpha,\beta)$-cells.

For $A = *$ we denote the pair $(X,*)$ simply as X and call it a complex category.

We say that $dim(X,A) = n$ if $X = X^n$ and $X \neq X^{n-1}$. A relative complex (X,A) is finite if X is obtained by attaching a finite number of cells.

5.12. Example. S^n_α is a finite complex with the base point as 0-cell and S^n_α as n-cell.

The following result follows from the basic properties of cofibrations.

5.13. Proposition. *Let (X,A) be a relative complex. Then the inclusion $A \to X$ is a cofibration.*

As in the case of CW-complexes, we have proved in [M2] the following result. If $X = \operatorname{colim}(* \to X^0 \to X^1 \to \dots)$ is a complex, then for any n, the pair (X,X^n) is n-connected.

5.14. *Remark.* If X and Y are complexes, the category $X \times Y$ is also a complex and the inclusion $X \vee Y \to X \times Y$ is a cofibration. In particular if X is a complex, then it satisfies the S^1_α-cofibration condition.

5.15. Theorem. *Given a category A which satisfies the S^1_α-cofibration condition, there is a commutative diagram of groups*

$$\begin{tikzcd} \pi_r(A) \arrow[rr, "\Sigma_\alpha"] \arrow[d, "\simeq"] & & \pi_{r+1}(\Sigma_\alpha A) \arrow[d, "\simeq"] \\ \pi_r(\mathscr{B}(A)) \arrow[dr, "\Sigma"] & & \pi_{r+1}(\mathscr{B}(\Sigma_\alpha A)) \\ & \pi_{r+1}(\Sigma\mathscr{B}(A)) \arrow[ur, "\simeq"] & \end{tikzcd}.$$

Proof. Follows from 5.7 and 5.14. □

5.16. Corollary. (Freudenthal Theorem for $\mathscr{C}at$). If A is an n-connected complex in $\mathscr{C}at$, the maps $\Sigma_\alpha : \pi_r(A) \to \pi_{r+1}(\Sigma_\alpha A)$ are isomorphisms for all $r \leq 2n$ and epimorphisms for $r = 2n+1$.

Proof. Since A is an n-connected complex in $\mathscr{C}at$ then $\mathscr{B}(A)$ is an n-connected CW-complex. By the classical Freudenthal theorem for CW-complexes, the map

$$\Sigma : \pi_r(\mathscr{B}(A)) \to \pi_{r+1}(\Sigma\mathscr{B}(A))$$

is an isomorphism for $r \leq 2n$ and an epimorphism for $r = 2n+1$. Now the result follows from 5.15. □

The maps $\Sigma_\alpha : \pi_r(A) \to \pi_{r+1}(\Sigma_\alpha A)$ can be used to define the stable homotopy groups as follows.

5.17. Definition. Given a category A and $r \in \mathbb{N}$, we define the stable homotopy groups

$$\pi^{st}_r(A) = \operatorname*{colim}_n \pi_{r+n}(\Sigma^n_\alpha A) \qquad (\text{for any } \alpha).$$

5.18. *Remark.* Note that in general the stable homotopy groups do not coincide with the stable homotopy groups of the classifying spaces. But if A is a complex, then $\pi_r^{st}(\mathscr{B}(A)) \simeq \pi_r^{st}(A) \simeq \pi_{r+m}(\Sigma_\alpha^m A)$ for m big enough.

It is well known that CW-complexes are paracompact spaces. We have a similar result for complex categories. In order to understand the meaning of paracompact in this setting, we need to find the analog of an open covering for categories. To this end, we introduced in [M4] the concept of *family of strong generators.*

5.19. Definition. Let A and B be categories and let $\{A_i\}_{i\in J}$ be a family of categories together with monomorphisms $\phi_i : A_i \to A$ for every $i \in J$. A family of functors $\{f_i : A_i \to B\}$ is called *compatible* with the family $\{\phi_i\}$ if given any category H and any pair of functors $g_i : H \to A_i$ and $g_j : H \to A_j$ such that $i, j \in J$ and $\phi_i g_i = \phi_j g_j$, it follows that $f_i g_i = f_j g_j$.

5.20. Definition. Let A be a category. A family of categories $\{A_i\}_{i\in J}$ *strongly generates* A if there exists a family of monomorphisms $\phi_i : A_i \to A$ such that for any category B and any family $f_i : A_i \to B$ of compatible functors, there exists a unique $f : A \to B$ such that $f\phi_i = f_i$. The family $\{\phi_i : A_i \to A\}$ is called a *family of strong generators* of A.

5.21. Example. Let A be the category with three objects $1, 2, 3$ and with one morphism $1 \xrightarrow{a} 2$ and two morphisms $2 \xrightarrow{b} 3$ and $2 \xrightarrow{c} 3$ such that $ba \neq ca$. The inclusions of the subcategories $A_1 : 1 \xrightarrow{a} 2 \xrightarrow{b} 3$ and $A_2 : 2 \xrightarrow{c} 3$ strongly generate A. If B is the quotient of A by the relation $ba = ca$, then A_1 and A_2 do not strongly generate B.

5.22. *Remark.* Let $\{\phi_i : A_i \to A\}_{i\in J}$ be a family of monomorphisms. For any pair $i, j \in J$ we denote by $A_i \cap A_j$ the pullback

$$\begin{tikzcd} A_i \cap A_j \arrow[r] \arrow[d] \arrow[dr, phantom, "Pull"] & A_i \arrow[d] \\ A_j \arrow[r] & A \end{tikzcd} .$$

The family $\{\phi_i : A_i \to A\}_{i\in J}$ strongly generates A if A is the colimit of the diagram of categories induced by the $A_i's$ and all inclusions $A_i \cap A_j \to A_i$.

5.23. Definition. A category A is *contractibly generated* if there exists a family of contractible categories $\{A_i\}$ which strongly generates A.

5.24. Example. It is clear that a contractible category is contractibly generated (by itself). Note also that sphere-categories are contractibly generated but not contractible.

Now we can translate the concepts of compact and paracompact space into our context. First, we need the notion of refinement of a family of strong generators.

5.25. Definition. Let A be a category and let $\{\phi_i : A_i \to A\}_{i\in J}$ be a family of strong generators of A. A *refinement* of $\{\phi_i : A_i \to A\}$ is a family $\{B_k\}_{k\in K}$ together with inclusions $\psi_k : B_k \to A_{i_k}$ for some $i_k \in J$ $(\forall k \in K)$ such that $\{\phi_{i_k}\psi_k : B_k \to A\}$ strongly generates A. The refinement is called *finite* if K is finite and it is called *star-finite* if and only if for every $k_0 \in K$ the intersection of B_{k_0} with B_k (i.e. the pullback of the inclusions $B_{k_0} \to A$ and $B_k \to A$) is not empty for only a finite number of $k \in K$.

5.26. Definition. A category A is called *compact* if every family of strong generators of A admits a finite refinement. A is called *paracompact* if every family of strong generators admits a star-finite refinement.

5.27. *Remark.* It is clear that finite categories are compact. In particular the sphere categories S_α^{n-1} and the cones of the spheres $C_\beta S_\alpha^{n-1}$ are compact; in the same way, any finite complex category is compact. Note also that compact categories are paracompact.

5.28. Lemma. *Consider the pushout*

$$\begin{tikzcd} S_\alpha^{n-1} \arrow[r, "f"] \arrow[d, "j_\beta"'] \arrow[dr, phantom, "Push"] & X \arrow[d] \\ C_\beta S_\alpha^{n-1} \arrow[r] & \tilde{X} \end{tikzcd}$$

in $\mathscr{C}at$. *If* X *is contractibly generated, then* $\tilde{X}$ *is contractibly generated.*

Proof. Let $(C_\beta \overset{\circ}{S}{}_\alpha^{n-1})$ denote the full subcategory of $C_\beta S_\alpha^{n-1}$ on the set of objects of $C_\beta S_\alpha^{n-1} - S_\alpha^{n-1}$. Note that $(C_\beta \overset{\circ}{S}{}_\alpha^{n-1})$ is contractible.

If $\{A_i\}_{i\in J}$ is a contractible family strongly generating X, then $\{A_i\}_{i\in J} \cup (C_\beta \overset{\circ}{S}{}_\alpha^{n-1})$ strongly generates $\tilde{X}$. □

5.29. Theorem. *If* X *is a complex category, then it is contractibly generated.*

Proof. Since $X = \mathrm{colim}(* \to X^0 \to X^1 \to \ldots)$, it suffices to prove the theorem for X^m. Now this follows by induction and the previous lemma. □

5.30. Theorem. *Any complex category is paracompact.*

Proof. It is clear that a complex category can be strongly generated by a family of finite subcomplexes, each of which intersects only a finite number of members of the family.

Since every finite complex is compact, it follows that any complex category is paracompact (compare with [FP, Appendix A.2]). □

References

[B] H.-J.Baues. *Algebraic Homotopy.* Cambridge studies in advanced mathematics **15** (1989).

[DKP] T. tom Dieck, K. H. Kamps, D. Puppe. *Homotopietheorie*. Lecture Notes in Math. **157** (1970).

[E] M. Evrard. *Fibrations de petites catégories.* Bull. Soc. math. France **103** (1975), 241–265.

[FP] R. Fritsch, R.A. Piccinini. *Cellular structures in topology*. Cambridge studies in advanced mathematics **19** (1990).

[H1] G. Hoff. *Catégories fibrées et homotopie.* C.R. Acad. Sc. Paris **278** (1974).

[H2] G. Hoff. *Introduction à l'homotopie dans Cat.* Esquisses Mathématiques **23** (1975).

[KP] K.H. Kamps, T. Porter. *Abstract homotopy and simple homotopy theory.* World Scientific (1997).

[L] M.J. Lee. *Homotopy for functors.* Proc. Amer. Math. Soc. **36** (1972), 571–577.

[M0] E.G. Minian. *Generalized cofibration categories and global actions. K*-theory **20** (2000), 37–95.

[M1] E.G. Minian. *Cat as a Lambda-cofibration category*. J. Pure Appl. Algebra **167** (2002), 301–314.

[M2] E.G. Minian. *Complexes in Cat*. Topology and its Applications **119** (2002), 41–51.

[M3] E.G. Minian. *Loop and suspension functors for small categories and stable homotopy groups*. Applied Categorical Structures **11** (2003), 207–218.

[M4] E.G. Minian. *Numerably contractible categories*. K-theory (to appear).

[Q] D.G. Quillen. *Higher algebraic K-theory I.* Lecture notes in Math. **341** (1973), 85–147.

[S] G. Segal. *Classifying spaces and spectral sequences*. Publ. Math. Inst. des Hautes Etudes Scient. (Paris) **34** (1968), 105–112.

[Sp] E. Spanier. *Algebraic Topology.* McGraw Hill (1966).

[T0] R.W. Thomason. *Homotopy colimits in the category of small categories.* Mathematical Proceedings of the Cambridge Philosophical Society **85** (1979), 91–109.

[T1] R.W. Thomason. *Cat as a closed model category*. Cahiers Topologie Géom. Différentielle **XXI-3** (1980), 305–324.

DEPARTAMENTO DE MATEMÁTICA, FCEYN, UNIVERSIDAD DE BUENOS AIRES, BUENOS AIRES, ARGENTINA

E-mail: gminian@dm.uba.ar