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THE MATHEMATICS OF ANTÓNIO ANICETO MONTEIRO

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1. The beginning

Monteiro graduated in Mathematics at the University of Lisbon in 1930, when he was 23 years old.

He got a fellowship from the *"Instituto para a Alta Cultura"* of the Portuguese Ministry of Education that allowed him to study at the *Institut Henri Poincaré* in Paris, from November 1931 to July 1936.

In 1936 obtained the degree "Docteur d'État" from the Sorbonne, after completing his thesis *"Sur l'additivité des noyaux de Fredholm"*. His adviser was *Maurice Fréchet*.

2. The years in Paris

During his years in Paris, Monteiro was in touch with some of the leaders of the classical French School of Analysis, like *E. Borel*, *H. Lebesgue*, *J. Hadamard*, and, simultaneously, was a witness to the modern trends in the study of algebraic and topological structures.

In particular, his adviser Maurice Fréchet was one of the main contributors to the theory of abstract spaces (in a long paper published in 1906 he introduced and developed the theory of metric spaces, and he was one of the first to consider measures in abstract spaces).

Integral equations, subject matter of Monteiro doctoral thesis, were the principal motivation for the introduction of linear compact operators on Banach spaces.

Together with a group of his colleagues in Paris he studied the then recently published book by *B.L. van der Waerden* on modern algebra.

3. Return to Lisbon

The first papers Monteiro wrote after his return to Portugal in 1936 were on the foundations of general topology, a subject that was being developed at that time.

One advantage of this subject was that it offered a great number of not too difficult problems, which Monteiro used to develop enthusiasm in young people for mathematical research, as *Hugo Ribeiro*, a witness of Monteiro's activities on that period, told in an article published in Portugaliae Mathematica (vol 39 (1980), pp. V–VII) .

Let me remark that Monteiro's passion for transmitting his own enthusiasm for mathematical research remained unchanged through his entire life.

Between 1940 and 1945 Monteiro published a series of papers, some of them in collaboration with Hugo Ribeiro, characterizing topological spaces and continuous functions from different primitive notions, such as neighborhoods, closures and derived sets.

This paper reproduces the slides prepared for the inaugural talk delivered on the IX Congreso Dr. António Monteiro, Universidad Nacional del Sur, Bahía Blanca, May 30 2007.

4. Research in Portugal

Most of these papers on general topology appeared in *Portugaliae Mathematica*, a journal that Monteiro had founded in 1937. They are more algebraic than analytic, and several results were presented in terms of partially ordered sets and monotonic functions.

During that time Monteiro became familiar with the works of *Garret Birkhoff* on lattice theory and universal algebra, of *Marshal Stone* on the topological representation of Boolean algebras and distributive lattices, of *Henry Wallman* on the compactification of topological spaces, of *Alfred Tarski* on Boolean algebras and the relations between deductive systems and closure operators. All of them had a decisive influence in his own researches.

In Monteiro's own words, *he devoted his work to the study of topological spaces, lattices, and the relations among them.*

5. Activities in Portugal

Besides his work on his own research, Monteiro was very active in promoting the development of mathematics in Portugal.

In her talk in this meeting, Elza Amaral will describe those activities. Let me just mention that with a small group of colleagues, among them *Ruy Luis Gomes*, who was also professor at this University between 1958 and 1961, gave courses on different topics of advanced mathematics, founded the *"Gazeta da Matemática"*, devoted to the dissemination of mathematics movement and the renewal of methods and themes of study, organized mathematical clubs and radio talks to explain, at a popular level, the importance of mathematics for the scientific development of the country.

Together with *José da Silva Paulo*, wrote *"Aritmética Racional"*, a very interesting book for secondary school published in 1945, long before the boom of "modern mathematics" in the nineteen sixties.

But it is important to point out that Monteiro's intense activities had no official support. Since he refused to sign a document in support of the (fascist) principles of the *"Estado Novo"*, he was unable to get a position at portuguse universities or even secondary schools. He had to earn his living by giving private lessons, and working in a center for cataloging the scientific publications existing in Portugal. He took this last job very seriously, and he learned a lot about the organization of libraries.

6. Brazil

Feeling that his situation in Portugal was unsustainable, Monteiro accepted a position as Professor of General Analysis at the *"Facultade Nacional de Filosofia"* in Rio de Janeiro.

Monteiro and his coworkers' research in Brazil was mainly devoted to the study of the relationship between a lattice and the lattice of its filters.

The integers $\mathbf{Z}$ form a lattice when ordered by divisibility. The meet of two numbers is their greatest common divisor, and the join is their smallest common multiple.

The filters of $\mathbf{Z}$ as a lattice are precisely the ideals of $\mathbf{Z}$ as a ring. The maximal filters are the sets of multiples of prime numbers and the prime filters are the sets of multiples of prime powers.

The basic arithmetic properties of $\mathbf{Z}$ can be expressed in terms of filters. For instance, *the decomposition of an integer into prime factors is equivalent to the fact that each filter in the lattice $\mathbf{Z}$ is a finite intersection of prime filters.*

Thus lattices can be considered as generalization of the integers, and the study of the properties of the filters of a lattice can be considered as an *"arithmetic"* for this lattice. This was Monteiro's point of view.

For instance Monteiro proved that *a lattice is distributive if and only if every proper filter is an intersection of prime filters*. Hence distributive lattices are those in which the analogue of the factorization of an integer holds.

Leopoldo Nachbin, then working with Monteiro, characterized Boolean algebras as distributive lattices such that all prime filters are maximal.

The results of Monteiro and his Brazilian coworkers on lattice theory are exposed in the two small volumes *"Filtros e Ideais". I and II* of the series *"Notas de Matemática"*, that was founded by Monteiro and was latter continued under the direction of Nachbin and published by North-Holland.

One problem on the arithmetic of filters and ideals that remained open in these notes were solved many years latter, independently and by different techniques, by *Raymond Balbes*(Algebra Universalis, vol. 2 (1972), 389–392) and *M. E. Adams* (Colloquium Mathematicum, vol.30 (1974), pp. 61–67).

Besides his work on lattice theory, Monteiro taught courses on Functional Analysis and continued his researches on general topology, publishing papers with *Mauricio Peixoto* on uniform continuity.

Monteiro also had an important activity in developing mathematics in Brazil.

His original four year contract was not renewed, probably due to the influence of the Portuguese Embassy in Brazil, and Monteiro accepted a position as a professor of Mathematical Analysis in the *"Facultad de Ingeniería"* of the *Universidad Nacional de Cuyo*, in San Juan, Argentina.

On December 5, 1949, Monteiro and his family arrived in Buenos Aires.

7. Arithmetic of topological spaces

The work by Monteiro and his Portuguese and Brazilian disciples on general topology and lattice theory culminated in his paper *"Arithmétique des espaces topologiques"*, that was submitted in 1950 to the French Mathematical Society for a contest in honor of Maurice Fréchet. It was chosen among the four best paper presented.

According to Monteiro's explanation, the paper was not published at that time because since it was rather long, he was asked to separate it in two parts. And Monteiro never found the time to carry out such separation.

Finally the paper was published twenty years later, as a volume of the *"Notas de Lógica Matemática"* of this University.

In any case, Monteiro prepared a detailed summary of the paper, with a few new results added, that was published in the Proceedings of a Symposium organized by UNESCO in Villavicencio, Mendoza, Argentina, in 1954.

I will try to sketch the main ideas of the paper.

Topological spaces generalize the properties of the real numbers concerning convergence and continuity. But real numbers are in turn a generalization of the integers.

It is well known that the closed subsets of a topological space X, ordered by inclusion, form a complete distributive lattice, that I shall denote by $L(X)$.

Inspired by the interpretation of the arithmetic properties of the integers in terms of filters, Monteiro's idea was that the filters of $L(X)$ should be considered as the generalized integers in the space X.

He showed how some of the separation axioms currently considered in topology can be interpreted in terms of arithmetic properties of $L(X)$.

For instance, *X is a normal space if and only if each prime filter of $L(X)$ is contained in a unique maximal filter.*

In the lattice $\mathbf{Z}$ this property is obvious because prime filters correspond to prime powers, and maximal filters to prime numbers.

X is a completely normal space if and only if given a prime filter P of $L(X)$, the set of filters F of $L(X)$ such that $P \subseteq F$ is totally ordered by inclusion.

This property is stronger than the previous one, and it is also a property of the lattice $\mathbf{Z}$.

These two examples illustrate Monteiro's idea: *the spaces X such that $L(X)$ has arithmetic properties closer to those of the lattice $\mathbf{Z}$, should be considered better generalizations of the integers.*

8. Heyting algebras

Although it was written in San Juan, the paper on arithmetic of topological spaces reflects Monteiro's previous work. But the summary written in 1954 contains some results on Heyting algebras obtained in San Juan.

The lattices of closed sets are more than distributive lattices: they are *Brouwerian algebras*. Their duals, *Heyting algebras*, are the algebraic counterparts of the intuitionistic propositional calculus, and they play for intuitionistic logic the same role as that of Boolean algebras for classical logic.

In the 1954 summary, Monteiro considered Brouwerian and Heyting algebras in some detail. For instance, he gave the interesting algebraic result that the class of Boolean algebras coincides with the class of semi-simple Heyting algebras, and in a paper published in *"Revista de la Unión Matemática Argentina"* in 1955, he gave a set of independent axioms for the variety of Heyting algebras.

9. Monadic Heyting algebras

At the beginning of the nineteen fifties, *Paul R. Halmos* introduced monadic Boolean algebras as a tool for the algebraic analysis of quantifiers.

Immediately Monteiro, in collaboration with *Oscar Varsavsky* showed that it was possible to give a non-trivial generalization of the theory of monadic Boolean algebras to monadic Heyting algebras.

Recently the theory of monadic Heyting algebras was considered in detail in a series of papers by *Guram Bezhanishvili*. In particular, in a joint paper with *John Harding* published in Algebra Universalis (vol. 48 (2002), pp. 1–10) they proved, by using rather sophisticated topological tools, *that every monadic Heyting algebra is isomorphic to a functional monadic Heyting algebra*, solving a problem posed by Monteiro and Varsavsky in 1957.

10. Departamento de Investigaciones Científicas

Monteiro also modernized the teaching of the calculus for engineers, and, more important, he promoted the creation in Mendoza, a city near (relative to Argentinean distances)

San Juan, an Institute, depending of the National University of Cuyo, entirely dedicated to mathematical research and the preparation of Ph. D. students.

This Institute was a novelty in the Argentinean Universities of that time, and many important Argentinean mathematicians were members or students of it. For instance, *Mischa Cotlar* was its director.

Unfortunately, at the end of 1955 the institute was deactivated by the new authorities of the university.

In 1956 Monteiro moved to Bahía Blanca, as a Professor of the recently created *Universidad Nacional del Sur*.

11. Algebra of Logic

Throughout the study of Boolean and Heyting algebras, Monteiro became interested in the algebraic aspects of logic, this interest having been aroused by direct contact with *Roman Sikorski* and *Helena Rasiowa* when they visited Bahía Blanca in 1958.

During this visit, Rasiowa gave a series of talks about the algebraic systems corresponding to Nelson's *constructive logic with strong negation*. These algebras were called *N-lattices* or *quasi pseudo Boolean algebras* by Rasiowa, and *Nelson algebras* by Monteiro.

Nelson algebras are distributive lattices equipped with an involution of period 2 and a binary operation that interprets implication.

Distributive lattices equipped with an involution of period 2 were called *De Morgan algebras* by Monteiro, who gave a representation of finite De Morgan algebras by partially ordered sets with an involution.

Monteiro used this representation to give a characterization of finite Nelson algebras as De Morgan algebras such that their prime lattice filters satisfy certain simple condition, and also to provide examples of non-regular characteristic matrices for classical propositional calculus, solving a problem posed by the logician *Alonzo Church*.

Monteiro also initiated the study of the algebraic counterpart of the implicative fragment of intuitionistic logic, that he called *Hilbert Algebras*. This study culminated with the Doctoral Thesis of *Antonio Diego*.

Since Hilbert algebras do not have a lattice structure, new techniques had to be developed, that were models for the investigation of other structures.

Monteiro discovered that the semisimple Nelson algebras coincide with the *three-valued Łukasiewicz algebras* that had been introduced by *G. Moisil* as the algebraic counterpart of *Łukasiewicz three-valued logic*.

Using the theory of Nelson algebras, Monteiro showed that *it is possible to define from each monadic Boolean algebra A a three-valued Łukasiewicz algebra $L(A)$, and that each three-valued Łukasiewicz algebra is isomorphic to $L(A)$ for a suitable monadic Boolean algebra A*.

Since it was shown by Halmos that monadic Boolean algebras are the algebraic counterpart of classical first order monadic calculus, Monteiro considered that the representation of three-valued Łukasiewicz algebras into monadic Boolean algebras gives a proof of the consistency of Łukasiewicz three-valued logic relative to classical logic.

It is fair to say that Monteiro's results on three-valued Łukasiewicz algebras inspired most of the research done in the theory of n-valued Łukasiewicz algebras.

Since the end of the nineteen sixties, Monteiro and some of his disciples studied the *symmetric Heyting algebras*, that correspond to Moisil's *symmetric modal propositional calculus*. In particular, *symmetric Boolean algebras* were considered.

The long paper *"Sur les algèbres de Heyting symétriques"* contains a systematic study of these algebras, and also a presentation of many of the results obtained by Monteiro and his disciples in Bahía Blanca. This paper was prepared while Monteiro was visiting Lisbon, and obtained the *Prize Gulbenkian on Science and Technology* corresponding to the year 1978. This paper, edited by Manuel Abad and Luiz Monteiro, was published in an special issue of Portugaliae Mathematica corresponding to the year 1980.

12. Final remarks

I tried to summarize Monteiro's main lines of research, exemplifying them with a few results that I consider specially representative.

In particular, I tried to show how Monteiro's interest in algebra of logic developed almost continuously from his early works on the foundations of general topology.

A constant that one can perceive through the whole mathematical work done by Monteiro is his preference for the finitistic methods, that allow concrete constructions and algorithms. This tendency is already noticeable in his doctoral thesis, a long part of which is dedicated to finite matrices, as used to approximate the kernels of integral equations

When considering a new class of algebras, it was a basic question for him to decide if the finitely generated free algebras were finite, and if so, to find explicitly the number of their elements as a function of the number of generators. In general, to achieve this goal it is necessary to have a deep understanding of the structure of the algebras in the given class.

Leopoldo Nachbin, in a paper on the influence of Monteiro in the development of mathematics in Brazil, published in *Portugalia Mathematica* (vol. 39, 1980, pp. XV–XVII) expressed the following opinion:

> Although I now look at Monteiro as having been fundamentally a mathematical logician, by heart and blood, he still was an analyst when he arrived to Brazil, at least as a result of his doctoral training in Paris, maybe by heart and blood, too.

I do not agree with regarding Monteiro as a logician. I consider that he was mainly an algebraist, and that his interest for logic was as a source of algebraic problems, principally those capable of computable results.

He was convinced that these methods were the most appropriate to follow the advances in informatics, which he believed would have strong influence in the development of mathematics.

In any case, it is clear that Monteiro was a very competent mathematician, who also did statistical work for a Brazilian airline, and in San Juan, in cooperation with geologists.

His passion for mathematics had no limits. He belived in the possibility of a more just and egalitarian world, and he though that the scientific and technological development was indispensable to achieve this goal. And he dedicated his life to the prosecution of this ideal.

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