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# A DUALITY FOR MONADIC $(n+1)$-VALUED $MV$-ALGEBRAS

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ABSTRACT. Categorical equivalences between the varieties of monadic $(n+1)$-valued $MV$-algebras and the classes of monadic Boolean algebras endowed with certain family of their filters are given. Using these equivalences, it is proved that every monadic $(n+1)$-valued $MV$-algebra can be represented by a rich algebra.

## 1. INTRODUCTION AND PRELIMINARIES

Wajsberg algebras (see [7, 11, 23]) are an equivalent reformulation of Chang $MV$-algebras based on implication instead of disjunction. $MV$-algebras were introduced by Chang [4, 5] to prove the completeness of the infinite valued Łukasiewicz propositional calculus. The classes of $(n+1)$-valued $MV$-algebras were introduced by R. Grigolia in [13], who also gave their equational characterization. For each $n > 0$, this variety is generated by the chain of length $n+1$ and the algebras belonging to this variety are the algebraic models of the $(n+1)$-valued Łukasiewicz propositional calculus. Lukasiewicz 3-valued and 4-valued algebras coincide with 3-valued and 4-valued $MV$-algebras, respectively.

Y. Komori [16] introduced the $CN$-algebras as algebraic models of Łukasiewicz infinite-valued propositional calculus formulated in terms of the operations implication and negation. A. J. Rodriguez [23] called Wajsberg algebras what was previously known as $CN$-algebras (see also [11]). $(n+1)$-valued Wajsberg algebras are equivalent to $(n+1)$-valued $MV$-algebras. The variety of $(n+1)$-bounded $W$-algebras is generated by chains of length less or equal than $n+1$. In this paper Wajsberg algebras will be used instead of MV-algebras.

For each integer $n > 0$, it is shown in [19] that there exists a categorical equivalence between the variety of $(n+1)$-valued MV-algebras and the class of Boolean algebras endowed with a certain family of filters. Another similar categorical equivalence is given by A. Di Nola and A. Lettieri in [9]. In this paper, the mentioned equivalence is extended to the variety of monadic $(n+1)$-valued MV-algebras. Using this equivalence, it is proved that every monadic $(n+1)$-valued MV-algebra can be represented by a rich algebra. When $n = 2$, the results given by Luiz Monteiro in [21] about the representation of monadic 3-valued Lukasiewicz algebras by rich algebras are obtained.

The basic results about $MV$-algebras can be found, for instance, in [7]. For a reformulation in the context of Wajsberg algebras (or $CN$-algebras) see [23, 11, 16].

A Wajsberg algebra (or $W$-algebra, for short) is an algebra $A = \langle A, \rightarrow, \neg, 1\rangle$ of type $(2,1,0)$ satisfying the following identities: $1 \rightarrow x = x$, $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$, $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$ and $(\neg y \rightarrow \neg x) \rightarrow (x \rightarrow y) = 1$. The reduct $(A, \vee, \wedge, \neg, 0, 1)$ is a Kleene algebra where $0 = \neg 1$, $x \vee y = (x \rightarrow y) \rightarrow y$, $x \wedge y = \neg(\neg x \vee \neg y)$ and $x \leq y$ if and only if $x \rightarrow y = 1$. If we set $x \oplus y = \neg y \rightarrow x$ and $x \odot y = \neg(x \rightarrow \neg y)$ then $\langle A, \oplus, \odot, 0\rangle$ is an $MV$-algebra. The set $B(A) = \{x \in A : x \odot x = x\}$ is a Boolean algebra. Indeed, $B(A)$ is the Boolean algebra of the complemented elements of the lattice reduct of $A$. The elements of

$B(A)$ are called the boolean elements of $A$. For all $x \in A$ and each non negative integer $m$ we set:

$0x = 0$ and $(m+1)x = (mx) \oplus x$;
$x^0 = 1$ and $x^{m+1} = (x^m) \odot x$.

For every $x \in A$ and all integer $m \geq 0$, the following properties hold:
(W1) $\neg(x^m) = m(\neg x)$,
(W2) $(p \to q)^m \leq mp \to mq$.

A subset $F \subseteq A$ is an *implicative filter* of $A$ if $1 \in F$ and for all $a, b \in A$, $a, a \to b \in F$ implies $b \in F$. Implicative filters are lattice filters which are closed by the operation $\odot$. The family of all implicative filters of $A$ is an algebraic lattice under set-inclusion, and it is isomorphic to the algebraic lattice of all congruence relations on $A$. For every implicative filter $F$ of $A$ and each $x \in A$ we represent with $[x]_F$ the set of all elements $y \in A$ such that $x$ and $y$ are $F-congruent$. An implicative filter of $A$ is *prime* if it is a lattice prime filter of $A$. We denote by $\chi(A)$ the set of all prime implicative filters of $A$. An implicative filter $P$ of $A$ is prime if and only if $A/P$ is a chain.

In what follows let $n \geq 1$ be an integer.

The unit interval $[0,1]$ endowed with the operations $x \to y := min\ \{1, 1-x+y\}$ and $\neg x := 1-x$ is a Wajsberg algebra. We denote by $L_{n+1}$ the subalgebra of $[0,1]$ whose universe is $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$. It is verified that $L_{t+1}$ is a subalgebra of $L_{n+1}$ if and only if $t$ divides $n$.

An $(n+1)$-bounded Wajsberg algebra $A$ is a Wajsberg algebra which verifies $x^n = x^{n+1}$, for every $x \in A$.

An $(n+1)$-valued Wajsberg algebra $A$ is an $(n+1)$-bounded Wajsberg algebra which verifies $n(x^j \oplus (\neg x \odot \neg x^{j-1})) = 1$, for every $x \in A$ and $1 < j < n$ does not divide $n$.

If $\langle A, \to, \neg, 1\rangle$ is an $(n+1)$-valued Wajsberg algebra then $\langle A, \vee, \wedge, \neg, \sigma_1, \sigma_2, \ldots, \sigma_n, 0, 1\rangle$ is an $(n+1)$-valued Łukasiewicz algebra, where the operators $\sigma_i$, for $1 \leq i \leq n$, are defined in terms of the Wajsberg operations (see [15]).

The following results are developed in [19] and establish the equivalences mentioned above.

Let $B$ be a Boolean algebra. We denote by $B^{[n]}$ the set of all increasing monotone functions from $\{1, 2, \ldots, n\}$ into $B$. $B^{[n]}$ with the operations of the lattice defined pointwise, the chain of constants $0 = c_0 < c_1 < \ldots < c_{n-1} < c_n = 1$ where, for each $0 \leq k \leq n$, $c_k(i)$ is equal to 1 if $i \geq n+1-k$ and equal to 0 otherwise, the negation defined by $(\neg f)(i) = \neg f(n+1-i)$ for each $1 \leq i \leq n$ and the modal operators $\sigma_i(f)(j) = f(i)$ for all $1 \leq i \leq n$ and $1 \leq j \leq n$, is a Post algebra of order $n+1$ [2]; therefore it is an $(n+1)$-valued Wajsberg algebra [24]. In Theorem 1.1 a direct proof of this results is given, showing explicitly the form of operations. In every $(n+1)$-valued Wajsberg algebra, the prime filters occur in finite and disjoint chains, then by the Martínez's Unicity Theorem [20] the implication is determined by the order.

**Theorem 1.1.** [19] *Let $B$ be a Boolean algebra and $n \geq 1$ be an integer. Then $\langle B^{[n]}, \mapsto, \neg, \mathbb{I}\rangle$ is an $(n+1)$-valued Wajsberg algebra where $B^{[n]} =$*

$\{f:\{1,2,\ldots,n\}\longrightarrow B : f(i)\leq f(j)$ *for all* $i,j$ *such that* $i\leq j\}$, $\mathbb{I}$ *is the constant function equal to* 1 *and, for* $f,g\in B^{[n]}$ *and* $1\leq k\leq n$, $(\neg f)(k)=\neg f(n+1-k)$ *and* $(f\mapsto g)(k)=\bigwedge_{i=1}^{n-k+1}(f(i)\rightarrow g(i+k-1))$.

**Remark 1.1.** We denote by $Div(n)$ the set of all positive divisors of $n$. Let $d\in Div(n)$. For each integer $j$, $1\leq j\leq n$, there exists an only integer $q_{d,j}$, $1\leq q_{d,j}\leq d$, such that $(q_{d,j}-1)\frac{n}{d}<j\leq q_{d,j}\frac{n}{d}$. Indeed, $q_{d,j}$ is the first element of the set $X=\{q\in\mathbb{N}:1\leq q\leq d,\ j\leq q\frac{n}{d}\}$. That is to say that the only block corresponding to the divisor $d$ of $n$ that contains $j$ is that determined by $q_{d,j}$. Thus, for any $d\in Div(n)$, we can *think* an $n$-tuple to be composed by $d$ blocks, each one of them with $\frac{n}{d}$ elements.

In what follows, for each $f\in B^{[n]}$, $d\in Div(n)$ and any integer $1\leq q\leq d$, we shall write $\xi_{d,q}(f)$ instead of $f(q\frac{n}{d})\rightarrow f((q-1)\frac{n}{d}+1)$.

**Corollary 1.1.** [19] *Let B be a Boolean algebra, let* $n\geq 1$ *be an integer and let h be a function from the lattice of divisors of n into the lattice of filters of B. The set* $\{f\in B^{[n]}:\xi_{d,q}(f)\in h(d)$, *for each* $d\in Div(n)$ *and all* $1\leq q\leq d\}$ *is denoted by* $M(B,h)$. *Then* $\langle M(B,h),\mapsto,\neg,\mathbb{I}\rangle$ *is an* $(n+1)$*-valued Wajsberg subalgebra of* $B^{[n]}$. *Also, if* $h(d)=B$ *for each* $d\in D=Div(n)-\{n\}$ *then* $M(B,h)$ *is a Post algebra of order* $n+1$.

**Theorem 1.2.** [19] *Let* $\langle A,\rightarrow,\neg,1\rangle$ *be an* $(n+1)$*-valued Wajsberg algebra. For each* $d\in Div(n)$ *let* $h_A(d)=P_d\cap B(A)$, *where* $P_d=\bigcap\{P\in\chi(A):A/P\subseteq L_{d+1}\}$. *Then* $\varphi:A\longrightarrow M(B(A),h_A)$ *is a W-isomorphism, being* $\varphi(x)(i)=\sigma_i(x)$ *for all* $x\in A$ *and every integer* $1\leq i\leq n$.

**Definition 1.1.** (a) *A pair* $\langle B,h\rangle\in B^{n+1}$ *if B is a Boolean algebra and h is a function from the lattice of divisors of n into the lattice of filters of B such that* $h(n)=\{1\}$ *and* $h(gcd\{d,r\})=h(d)\vee h(r)$, *for every* $d,r\in Div(n)$ ($gcd\{d,r\}$ *is the greatest common divisor of the set* $\{d,r\}$).

(b) *Objects* $\langle B_1,h_1\rangle$ *and* $\langle B_2,h_2\rangle$ *in* $B^{n+1}$ *are isomorphic if there exists a boolean isomorphism* $\varphi:B_1\longrightarrow B_2$ *which verifies* $\varphi^{-1}(h_2(d))=h_1(d)$ *for all* $d\in Div(n)$.

**Remark 1.2.** Let $\langle A,\rightarrow,\neg,1\rangle$ be an $(n+1)$-valued Wajsberg algebra. Then $\langle B(A),h_A\rangle\in B^{n+1}$, where $h_A(d)=P_d\cap B(A)$ being $P_d=\bigcap\{P\in\chi(A):A/P\subseteq L_{d+1}\}$, for each $d\in Div(n)$.

**Theorem 1.3.** [19] *Let* $\langle B,h\rangle\in B^{n+1}$ *and let* $A=M(B,h)$. *Then* $\langle B,h\rangle$ *and* $\langle B(A),h_A\rangle$ *are isomorphic objects in* $B^{n+1}$.

Let $\mathscr{W}^{n+1}$ be the category of $(n+1)$-valued $W$-algebras and $W$-homomorphisms. Let $\mathscr{B}^{n+1}$ be the category whose objects are pairs in $B^{n+1}$ and whose morphisms are defined in the following way: if $O_1=\langle B_1,h_1\rangle$ and $O_2=\langle B_2,h_2\rangle$ are objects in this category, $\theta$ is a morphism from $O_1$ into $O_2$ if it is a boolean homomorphism from $B_1$ into $B_2$ which verifies $h_1(d)\subseteq\theta^{-1}(h_2(d))$ for any $d\in Div(n)$.

It is easy to see that $\theta$ is an isomorphism from $O_1$ onto $O_2$ if it is a boolean isomorphism from $B_1$ onto $B_2$ which verifies $h_1(d)=\theta^{-1}(h_2(d))$ for each $d\in Div(n)$.

Let $B$ be the functor from $\mathscr{W}^{n+1}$ to $\mathscr{B}^{n+1}$ defined in the following way:

(i) For each object $\mathscr{A}=\langle A,\rightarrow,\neg,1\rangle$ in $\mathscr{W}^{n+1}$, $B(\mathscr{A})=\langle B(A),h_A\rangle$, where $B(A)$ is the set of boolean elements of $A$ and for all $d$ divisor of $n$, $h_A(d)=P_d\cap B(A)$, being $P_d=\bigcap\{P\in\chi(A):A/P\subseteq L_{d+1}\}$.

(ii) If $\mathscr{A}_1$ and $\mathscr{A}_2$ are objects in the category $\mathscr{W}^{n+1}$ and $g : \mathscr{A}_1 \longrightarrow \mathscr{A}_2$ is a $\mathscr{W}^{n+1}$-morphism, $B(g) : \langle B(A_1), h_{A_1}\rangle \longrightarrow \langle B(A_2), h_{A_2}\rangle$ is defined by $B(g) = g/_{B(A_1)}$.

Let $M$ be the functor from $\mathscr{B}^{n+1}$ to $\mathscr{W}^{n+1}$ defined in the following way:

(i) For each object $\langle B,h\rangle$ in $\mathscr{B}^{n+1}$, let $M(\langle B,h\rangle) = \langle M(B,h), \mapsto, \neg, \mathbb{I}\rangle$.

(ii) If $\langle B_1,h_1\rangle$ and $\langle B_2,h_2\rangle$ are objects in the category $\mathscr{B}^{n+1}$ and $g$ is a $\mathscr{B}^{n+1}$-morphism from $\langle B_1,h_1\rangle$ into $\langle B_2,h_2\rangle$ let $M(g) : M(B_1,h_1) \longrightarrow M(B_2,h_2)$ where $M(g)(f) = g \circ f$, for any $f \in M(B_1,h_1)$.

From Theorems 1.2 and 1.3 the functors $B$ and $M$ define a natural equivalence between the categories $\mathscr{W}^{n+1}$ and $\mathscr{B}^{n+1}$.

Monadic $(n+1)$-valued $W$-algebras [25, 26, 12, 10, 1] are defined as follows.

**Definition 1.2.** *An algebra $\langle A, \rightarrow, \neg, \forall, 1\rangle$ is a monadic Wajsberg algebra if $\langle A, \rightarrow, \neg, 1\rangle$ is a Wajsberg algebra and $\forall : A \longrightarrow A$ is a function which verifies the following identities:*

(U1) $\forall x \rightarrow x = 1$,

(U2) $\forall(\forall x \rightarrow y) = \forall x \rightarrow \forall y$,

(U3)$\forall(\neg x \rightarrow x) = \neg\forall x \rightarrow \forall x$.

Observe that identity U3 can be write $\forall(2x) = 2\forall x$.

Let $\langle A, \rightarrow, \neg, \forall, 1\rangle$ be a monadic Wajsberg algebra. Often we will write $A$ or $\langle A, \forall\rangle$ instead of $\langle A, \rightarrow, \neg, \forall, 1\rangle$. If $X \subseteq A$, $\forall(X) = \{\forall x : x \in X\}$. Algebras $\forall(A)$ and $B(A)$ are monadic Wajsberg subalgebras of $A$. In particular $\langle B(A), \forall\rangle$ is a monadic Boolean algebra. For all $x, y \in A$ and all integer $m \geq 0$, the following properties hold:

(U4) $\forall\forall x = \forall x$,

(U5) $x \leq y$ implies $\forall x \leq \forall y$,

(U6) $\forall(x \wedge y) = \forall x \wedge \forall y$,

(U7) $\forall(x \rightarrow y) \leq \forall x \rightarrow \forall y$,

(U8) $\forall\neg\forall x = \neg\forall x$,

(U9) $\forall(x \odot \forall y) = \forall x \odot \forall y$,

(U10) $(\forall x)^m \leq \forall(x^m)$.

**Definition 1.3.** *A monadic Wajsberg algebra $\langle A, \rightarrow, \neg, \forall, 1\rangle$ is a monadic $(n+1)$-valued Wajsberg algebra ($MW^{n+1}$-algebra, for short) if $\langle A, \rightarrow, \neg, 1\rangle$ is an $(n+1)$-valued Wajsberg algebra.*

The varieties of monadic $(n+1)$-valued Wajsberg algebras will be denoted by $\mathbf{MW^{n+1}}$.

In [18] the classes of $(n+1)$-bounded Wajsberg algebras with a $U$-operator (or $UW_{n+1}$-algebras) are defined as $(n+1)$-bounded Wajsberg algebras with an operator which verifies the properties (U1) and (U2). With $\mathbf{UW_{n+1}}$ we denote the varieties of $(n+1)$-bounded Wajsberg algebras with a $U$-operator.

**Lemma 1.1.** $\mathbf{MW^{n+1}} \subseteq \mathbf{UW_{n+1}}$, *for all* $n \geq 1$.

**Remark 1.3.** (i) If $\langle A, \rightarrow, \neg, \forall, 1\rangle$ is a monadic Wajsberg algebra then $\langle A, \oplus, \odot, \neg, \exists, 0, 1\rangle$ is a monadic $MV$-algebra (see [10, 1, 25, 12]) where for each $x \in A$, $\exists x = \neg\forall\neg x$.

(ii) If $\langle A, \oplus, \odot, \neg, \exists, 0, 1\rangle$ is a monadic $MV$-algebra then $\langle A, \rightarrow, \neg, \forall, 1\rangle$ is a monadic Wajsberg algebra where for each $x \in A$, $\forall x = \neg\exists\neg x$.

**Theorem 1.4.** [10, Corollary 14] *If $\langle A,\forall\rangle$ is a totally ordered monadic Wajsberg algebra, then $\forall$ is the identity.*

The following result is consequence of [18, Theorem 2.2] and Lemma 1.1.

**Lemma 1.2.** *The variety* $\mathbf{MW^{n+1}}$ *is semisimple.*

Theorem 2.3 in [18] for $UW_{n+1}$-algebras yields the following result in the class of monadic $(n+1)$-valued Wajsberg algebras.

**Theorem 1.5.** *Let $A$ be a non trivial $MW^{n+1}$-algebra. Then $A$ is a simple $MW^{n+1}$-algebra if, and only if, $\forall(A)$ is a simple $(n+1)$-valued Wajsberg algebra if, and only if, $\forall(A)\cap B(A)$ is simple Boolean algebra.*

The following properties hold for every non trivial Wajsberg algebra $A$.

(P1) $A$ is a simple $(n+1)$-valued Wajsberg algebra if and only if $A$ is isomorphic to $L_{r+1}$ for some integer $r\geq 1$, $r$ divisor of $n$.

(P2) $A$ is an $(n+1)$-valued Wajsberg algebra if and only if $A$ can be represented (as subdirect product) in $\prod\limits_{i/n} L_{i+1}^{\chi_{i+1}}$, where $\chi_{i+1}=\{D\in\chi(A): A/D\simeq L_{i+1}\}$.

**Corollary 1.2.** *$\langle L_{n+1}^{I},\forall\rangle$ is a simple $MW^{n+1}$-algebra, where $I$ is a nonempty set and for each $f: I\longrightarrow L_{n+1}$, $\forall f$ is the constant function defined by $(\forall f)(x)=inf\{f(x): x\in I\}$.*

**Theorem 1.6.** *If $A$ is a simple $MW^{n+1}$-algebra, then it is isomorphic to a subalgebra of $\langle L_{n+1}^{I},\forall\rangle$, for some nonempty set $I$.*

*Proof.* The proof is a special case of Theorem 2.4 in [18] using Theorem 1.5, properties (P1) and (P2), Corollary 1.2 and Theorem 1.4. □

**Corollary 1.3.** *Let $\langle A,\forall\rangle$ be an $MW^{n+1}$-algebra. Then $\forall(kx)=k\forall x$ for every $x\in A$ and all integer $1\leq k\leq n$.*

*Proof.* It is easy to prove that the identities are valid in a simple $MW^{n+1}$-algebra; so they are valid in all $MW^{n+1}$-algebra, follows from Lemma 1.2. □

**Lemma 1.3.** *Let $\langle A,\forall\rangle$ be an $MW^{n+1}$-algebra. Then for every $x\in A$ the following properties hold:*

(U11) $\forall(x^k)=(\forall x)^k$, *for each integer* $1\leq k\leq n$,

(U12) $\forall(\sigma_i(x))=\sigma_i(\forall x))$, *for every* $i\in\{1,2,\ldots,n\}$.

*Proof.* (U11) follows from properties W1, W2, U5, U8, U9 and U10. (U12) follows from Corollary 1.3, U11 and [15, Theorem 5.23]. □

It is proved in [12] that monadic $(n+1)$-valued $MV$-algebras are polynomially equivalent to monadic $(n+1)$-valued Łukasiewicz algebras for $n=2$ and $n=3$, respectively.

## 2. The duality for monadic $(n+1)$-valued Wajsberg algebras

**Theorem 2.1.** *Let $\langle B,\forall\rangle$ be a monadic Boolean algebra and $n\geq 1$ be an integer. Then $\langle B^{[n]},\mapsto,\neg,\mathbb{V},\mathbb{I}\rangle$ is a monadic $(n+1)$-valued Wajsberg algebra where $B^{[n]}=\{f:\{1,2,\ldots,n\}\longrightarrow B: f(i)\leq f(j)$ for all $i,j$ such that $i\leq j\}$, $\mathbb{I}$ is the constant function equal to $1$ and, for $f,g\in B^{[n]}$ and $1\leq k\leq n$, $(\neg f)(k)=\neg f(n+1-k)$, $(f\mapsto g)(k)=\bigwedge\limits_{i=1}^{n-k+1}(f(i)\rightarrow g(i+k-1))$ and $(\mathbb{V}f)(i)=\forall(f(i))$.*

*Proof.* From Theorem 1.1 $\langle B^{[n]}, \mapsto, \neg, \mathbb{I}\rangle$ is an $(n+1)$-valued Wajsberg algebra. Moreover, for every $f,g \in B^{[n]}$ and integers $i,k$, $1 \leq i,k \leq n$, the following properties hold:

(1) $\mathbb{V}f \leq f$

$$(\mathbb{V}f)(i) = \forall f(i) \leq f(i)$$

(2) $\mathbb{V}(f \mapsto \mathbb{V}g) = \mathbb{V}f \mapsto \mathbb{V}g$

$$(\mathbb{V}(f\mapsto \mathbb{V}g))(k) = \forall((f\mapsto \mathbb{V}g)(k)) = \forall\left(\bigwedge_{i=1}^{n-k+1}(f(i)\rightarrow(\mathbb{V}g)(i+k-1))\right)$$
$$= \forall\left(\bigwedge_{i=1}^{n-k+1}(f(i)\rightarrow\forall(g(i+k-1))\right) = \bigwedge_{i=1}^{n-k+1}\forall(f(i)\rightarrow\forall(g(i+k-1)))$$
$$= \bigwedge_{i=1}^{n-k+1}(\forall(f(i))\rightarrow\forall(g(i+k-1))) = \bigwedge_{i=1}^{n-k+1}((\mathbb{V}f)(i)\rightarrow(\mathbb{V}g)(i+k-1)))$$
$$= (\mathbb{V}f\mapsto\mathbb{V}g)(k).$$

(3) $\mathbb{V}(\neg f \mapsto f) = \neg\mathbb{V}f \mapsto \mathbb{V}f.$

$$(\mathbb{V}(\neg f\mapsto f))(k) = \forall((\neg f\mapsto f)(k))$$
$$= \forall\left(\bigwedge_{i=1}^{n-k+1}(\neg f(n+1-i)\rightarrow f(i+k-1))\right)$$
$$= \bigwedge_{i=1}^{n-k+1}\forall(f(n+1-i)\vee f(i+k-1)). \quad (1)$$

On the other hand,

$$(\neg\mathbb{V}f \mapsto \mathbb{V}f)(k) = \bigwedge_{i=1}^{n-k+1}(\neg(\mathbb{V}f)(n+1-i)\rightarrow(\mathbb{V}f)(i+k-1)) =$$
$$\bigwedge_{i=1}^{n-k+1}(\forall f(n+1-i)\vee\forall f(i+k-1)). \quad (2)$$

If $i \leq \lfloor\frac{n-k}{2}\rfloor$ ($\lfloor x\rfloor$ denotes the largest integer less or equal to $x$, for a real number $x$) then $i+k-1 \leq n+1-i$ and the equality follows from (1), (2) and U5. Similarly if $i > \lfloor\frac{n-k}{2}\rfloor$ because $n+1-i \leq i+k-1$.

□

**Remark 2.1.** Let $\langle B,\forall\rangle$ be a monadic Boolean algebra. Algebras $(\forall(B))^{[n]}$ and $\mathbb{V}\left(B^{[n]}\right)$ are isomorphic algebras. Indeed, $(\forall(B))^{[n]} = \{f : \{1,2,\ldots,n\} \longrightarrow \forall(B) : f(i) \leq f(j)$ for all $i,j$ such that $i \leq j\}$ and $\mathbb{V}\left(B^{[n]}\right) = \{f \in B^{[n]} : \mathbb{V}f = f\} = \{f \in B^{[n]} : \forall(f(i)) = f(i),$ for all $i \in \{1,2,\ldots n\}\}$. It is clear that $f \in (\forall(B))^{[n]}$ if and only if $f$ is an increasing function from the set $\{1,2,\ldots,n\}$ into $B$ such that $f(i) \in \forall(B)$ for every $1 \leq i \leq n$; if and only if $f \in B^{[n]}$ and $\forall(f(i)) = f(i)$ for every $1 \leq i \leq n$; if and only if $f \in \mathbb{V}\left(B^{[n]}\right)$.

**Corollary 2.1.** *Let $\langle B,\forall\rangle$ be a monadic Boolean algebra, $n \geq 1$ be an integer and $h^M$ be a function from the lattice of divisors of $n$ into the lattice of monadic filters of $B$. Let $M(B,h^M)$ be the set $\{f \in B^{[n]} : f(q\frac{n}{d}) \rightarrow f((q-1)\frac{n}{d}+1) \in h^M(d)$, for each $d \in Div(n)$ and all $1 \leq q \leq d\}$. Then $\langle M(B,h^M), \mapsto, \neg, \mathbb{V}, \mathbb{I}\rangle$ is a monadic $(n+1)$-valued Wajsberg subalgebra of $B^{[n]}$.*

*Proof.* From Corollary 1.1 we only shall prove that $\mathbb{V}$ is closed into $M\left(B,h^M\right)$. Let $f \in M\left(B,h^M\right)$, then $f(q\frac{n}{d}) \rightarrow f((q-1)\frac{n}{d}+1) \in h^M(d)$, for every $d \in Div(n)$ and all integer $q$, $1 \leq q \leq d$. Since $h^M(d)$ is a monadic filter, using U7 we have $(\mathbb{V}f)(q\frac{n}{d}) \rightarrow (\mathbb{V}f)((q-1)\frac{n}{d}+1) = \forall(f(q\frac{n}{d})) \rightarrow \forall(f((q-1)\frac{n}{d}+1)) \geq \forall(f(q\frac{n}{d}) \rightarrow f((q-1)\frac{n}{d}+1))$; then $\mathbb{V}f \in M\left(B,h^M\right)$.
□

**Remark 2.2.** Let $\langle B,\forall\rangle$ be a monadic Boolean algebra, $n\geq 1$ be an integer and $h^M$ be a function from the lattice of divisors of $n$ into the lattice of monadic filters of $B$. Then, for each $f\in M\left(B,h^M\right)$, $\mathbb{V}f$ is the last element of the set $(f]\cap M\left(B,h^M\right)\cap(\forall(B))^{[n]}$.

**Corollary 2.2.** *Let $\langle B,\forall\rangle$ be a monadic Boolean algebra, $n\geq 1$ be an integer and $h$ be a function from the lattice of divisors of $n$ into the lattice of filters of $B$. Then $\langle M(B,h),\mapsto,\neg,\mathbb{V},\mathbb{I}\rangle$ is a monadic $(n+1)$-valued Wajsberg algebra where $(\mathbb{V}f)(i)=\forall(f(i))$, for each $f\in M(B,h)$ and $1\leq i\leq n$.*

*Proof.* Let $\langle B,\forall\rangle$ be a monadic Boolean algebra. If $F$ be a filter of $B$, then $\forall^{-1}F$ is a monadic filter of $B$ and $\forall^{-1}F\subseteq F$. Moreover, $\forall^{-1}F$ is maximal among all the monadic filters of $B$ included in $F$. Let $h^M$ be the function from the lattice of divisors of $n$ into the lattice of monadic filters of $B$ defined by $h^M(d)=\forall^{-1}h(d)$, for each $d\in Div(n)$. From Corollary 2.1 and Remark 2.2 we have that $\langle M(B,h^M),\mapsto,\neg,\mathbb{V},\mathbb{I}\rangle$ is a monadic $(n+1)$-valued Wajsberg algebra where, for each $f\in M\left(B,h^M\right)$, $\mathbb{V}f$ is the last element of the set $(f]\cap M\left(B,h^M\right)\cap(\forall(B))^{[n]}$. Moreover, $\langle M(B,h^M),\mapsto,\neg,\mathbb{I}\rangle$ is a $W$-subalgebra of $\langle M(B,h),\mapsto,\neg,\mathbb{I}\rangle$ because for every $f\in M(B,h^M)$ is $f(q\frac{n}{d})\to f((q-1)\frac{n}{d}+1)\in h^M(d)\subseteq h(d)$, for each $d\in Div(n)$ and all $1\leq q\leq d$. Let $f\in M(B,h)$; then $\mathbb{V}f$ is the last element of the set $(f]\cap M\left(B,h^M\right)\cap(\forall(B))^{[n]}$ because, if there exists $g\in M(B,h)$ such that $g\leq f$ and $g\in M\left(B,h^M\right)\cap(\forall(B))^{[n]}$, then $g=\mathbb{V}g\leq\mathbb{V}f$. Therefore $\mathbb{V}$ is the quantifier onto $M(B,h)$ determined by the subalgebra $M\left(B,h^M\right)\cap(\forall(B))^{[n]}$. □

**Theorem 2.2.** *Let $\langle A,\to,\neg,\forall,1\rangle$ be a monadic $(n+1)$-valued Wajsberg algebra. Let $h_A$ be the function from the lattice of divisors of $n$ into the lattice of filters of $B(A)$ where, for each $d\in Div(n)$, $h_A(d)=P_d\cap B(A)$, being $P_d=\bigcap\{P\in\chi(A):A/P\subseteq L_{d+1}\}$. Then $\langle M(B(A),h_A),\mapsto,\neg,\mathbb{V},\mathbb{I}\rangle$ and $\langle A,\to,\neg,\forall,1\rangle$ are isomorphic monadic $(n+1)$-valued Wajsberg algebras.*

*Proof.* From Theorem 1.2 the function $\varphi:A\longrightarrow M(B(A),h_A)$ is a $W$-isomorphism, being $\varphi(x)(i)=\sigma_i(x)$ for all $x\in A$ and every integer $1\leq i\leq n$; moreover, $\mathbb{V}\varphi(x)=\varphi(\forall x)$ because from U12 we have $(\mathbb{V}\varphi(x))(i)=\forall(\varphi(x)(i))=\forall(\sigma_i(x))=\sigma_i(\forall x)=(\varphi(\forall x))(i)$. □

**Definition 2.1.** (i) *A 3-tuple $\langle B,\forall,h\rangle\in MB^{n+1}$ if $\langle B,\forall\rangle$ is a monadic Boolean algebra and $h$ is a function from the lattice of divisors of $n$ into the lattice of filters of $B$ such that $h(n)=\{1\}$ and $h(gcd\{d,r\})=h(d)\vee h(r)$, for every $d,r\in Div(n)$ ($gcd\{d,r\}$ is the greatest common divisor of the set $\{d,r\}$).*

(ii) *3-tuples $\langle B_1,\forall_1,h_1\rangle$ and $\langle B_2,\forall_2,h_2\rangle$ in $MB^{n+1}$ are isomorphic if there exists a monadic boolean isomorphism $\varphi:B_1\longrightarrow B_2$ which verifies $\varphi^{-1}(h_2(d))=h_1(d)$ for all $d\in Div(n)$.*

**Remark 2.3.** Let $\langle A,\to,\neg,\forall,1\rangle$ be a monadic $(n+1)$-valued Wajsberg algebra. Then $\langle B(A),\forall,h_A\rangle\in MB^{n+1}$, where, for each $d\in Div(n)$, $h_A(d)=P_d\cap B(A)$ being $P_d=\bigcap\{P\in\chi(A):A/P\subseteq L_{d+1}\}$.

**Theorem 2.3.** *Let $\langle B,\forall,h\rangle\in MB^{n+1}$ and let $A=M(B,h)$. Then $\langle B,\forall,h\rangle$ and $\langle B(A),\mathbb{V},h_A\rangle$ are isomorphic objects in $MB^{n+1}$.*

*Proof.* Let $\langle B,\forall,h\rangle\in MB^{n+1}$ and $A=M(B,h)$. By Corollary 2.2 we know that $\langle A,\mapsto,\neg,\mathbb{V},\mathbb{I}\rangle$ is a monadic $(n+1)$-valued Wajsberg algebra where $(\mathbb{V}f)(i)=\forall(f(i))$, for all $f\in A$ and every integer $1\leq i\leq n$.

It is easy to see that $B(A)$ is the subalgebra that consist of all constant functions. If $h_A$ is the function from the lattice of divisors of $n$ into the lattice of filters of $B(A)$ defined by $h_A(d) = P_d \cap B(A)$, being $P_d = \bigcap\{P \in \chi(A) : A/P \subseteq L_{d+1}\}$, then $\langle B(A), \mathbb{V}, h_A\rangle \in MB^{n+1}$ (because Remark 2.3).

Let $\mu : B \longrightarrow B(A)$ such that $\mu(a)$ is the constant function from $\{1,2,\dots,n\}$ into $B$ that takes the value $a$, for each $a \in B$. In [19, Theorem 3] it is prove that $\mu$ is a boolean isomorphism from $B$ onto $B(A)$ which verifies $\mu^{-1}(P_d \cap B(A)) = h(d)$, for each $d \in Div(n)$. Moreover, for each $x \in B$ and all $i \in \{1,2,\dots n\}$, it is $(\mu(\forall x))(i) = \forall x = \forall(\mu(x)(i)) = (\mathbb{V}\mu(x))(i)$. $\square$

Let $\mathscr{MW}^{n+1}$ be the category of monadic $(n+1)$-valued $W$-algebras and monadic $W$-homomorphisms. Let $\mathscr{MB}^{n+1}$ be the category whose objects are the 3-tuples in $MB^{n+1}$ and whose morphisms are defined in the following way: if $O_1 = \langle B_1, \forall_1, h_1\rangle$ and $O_2 = \langle B_2, \forall_2, h_2\rangle$ are objects in this category, $\theta$ is a morphism from $O_1$ into $O_2$ if it is a monadic boolean homomorphism from $B_1$ into $B_2$ which verifies $h_1(d) \subseteq \theta^{-1}(h_2(d))$ for any $d \in Div(n)$.

It is easy to see that $\theta$ is an isomorphism from $O_1$ onto $O_2$ if it is a monadic boolean isomorphism from $B_1$ onto $B_2$ which verifies $h_1(d) = \theta^{-1}(h_2(d))$ for each $d \in Div(n)$.

Let $B$ be defined from $\mathscr{MW}^{n+1}$ to $\mathscr{MB}^{n+1}$ as follows:

(i) For each object $\mathscr{A} = \langle A, \rightarrow, \neg, \forall, 1\rangle$ in the category $\mathscr{MW}^{n+1}$, $B(\mathscr{A}) = \langle B(A), \forall, h_A\rangle$, where $B(A)$ is the set of boolean elements of $A$ and for all $d$ divisor of $n$, $h_A(d) = P_d \cap B(A)$, being $P_d = \bigcap\{P \in \chi(A) : A/P \subseteq L_{d+1}\}$.

(ii) If $\mathscr{A}_1$ and $\mathscr{A}_2$ are objects in the category $\mathscr{MW}^{n+1}$ and $g : \mathscr{A}_1 \longrightarrow \mathscr{A}_2$ is an $\mathscr{MW}^{n+1}$-morphism, $B(g) : \langle B(A_1), \forall_1, h_{A_1}\rangle \longrightarrow \langle B(A_2), \forall_2 h_{A_2}\rangle$ is defined by $B(g) = g/_{B(A_1)}$.

It is immediate that $B(g)$ is a monadic boolean homomorphism. Moreover, $B(g)$ is an $\mathscr{MB}^{n+1}$-morphism. Indeed, let $a \in h_{A_1}(d)$. If $a \notin B(g)^{-1}(h_{A_2}(d))$ then $g(a) \notin h_{A_2}(d)$, hence there exists a prime implicative filter $P$ of $A_2$ such that $A_2/P \subseteq L_{d+1}$ and $g(a) \notin P$. Thus $a \notin g^{-1}(P) \cap B(A_1)$. The function $v : A_1/g^{-1}(P) \longrightarrow A_2/P$ defined by $v([x]_{g^{-1}(P)}) = [g(x)]_P$ is an embedding from $A_1/g^{-1}(P)$ into $A_2/P \subseteq L_{d+1}$, i.e., $A_1/g^{-1}(P) \subseteq L_{d+1}$ then $a \notin h_{A_1}(d)$ which is a contradiction. It is easy to verify that $B$ is a functor.

Let $M$ be defined from $\mathscr{MB}^{n+1}$ to $\mathscr{MW}^{n+1}$ as follows:

(i) For each object $\langle B, \forall, h\rangle$ in $\mathscr{MB}^{n+1}$, let $M(\langle B, \forall, h\rangle) = \langle M(B,h), \mapsto, \neg, \mathbb{V}, \mathbb{I}\rangle$, where $\mathbb{V}$ is defined pointwise.

(ii) If $\langle B_1, \forall_1, h_1\rangle$ and $\langle B_2, \forall_2, h_2\rangle$ are objects in $\mathscr{MB}^{n+1}$ and $g$ is an $\mathscr{MB}^{n+1}$-morphism from $\langle B_1, \forall_1, h_1\rangle$ into $\langle B_2, \forall_2, h_2\rangle$ let $M(g) : M(B_1, h_1) \longrightarrow M(B_2, h_2)$ where $M(g)(f) = g \circ f$, for any $f \in M(B_1, h_1)$.

It is clear that $M(g)$ is well defined because, if $f \in M(B_1, h_1)$ then for each $d \in Div(n)$ and all integer $q$, $1 \leq q \leq d$ we have $\xi_{d,q}(f) \in h_1(d)$; hence $\xi_{d,q}(g \circ f) = g(\xi_{d,q}(f)) \in g(h_1(d)) \subseteq gg^{-1}(h_2(d) \subseteq h_2(d)$. Therefore $g \circ f \in M(B_2, h_2)$. Besides $M(g)$ is a monadic $W$-homomorphism. It is easy to see that $M$ is a functor.

From Theorems 2.2 and 2.3 follows that the functors $B$ and $M$ define a natural equivalence between the categories $\mathscr{MW}^{n+1}$ and $\mathscr{MB}^{n+1}$.

## 3. REPRESENTATION BY RICH ALGEBRAS

Using the natural equivalence established in section 2 and the Representation Theorem by rich algebras for monadic Boolean algebras [14], we will prove that every monadic $(n+1)$-valued Wajsberg algebra can be represented by a rich algebra. Specifically, we will prove that every monadic $(n+1)$-valued $W$-algebra is isomorphic to a subalgebra $B$ of a functional algebra $A^I$ such that, for every $b \in B$ there exists $x_0 \in I$ such that $b(x_0) = \bigwedge_{x\in I} b(x)$.

Let $\langle A, \forall\rangle$ a monadic $(n+1)$-valued Wajsberg algebra.

**Claim 3.1** $\langle B(A), \forall, h_A\rangle \in MB^{n+1}$ (see Remark 2.3). Particularly, $\langle B(A), \forall\rangle$ is a monadic Boolean algebra, therefore it can be represented by a rich algebra as follows [14]. A *constant* of $B(A)$ is a boolean homomorphism $c : B(A) \to \forall(B(A))$ such that $c(x) = x$ for every $x \in \forall(B(A))$; the set of all constants of $B(A)$ is denoted by $I$. The functional algebra $\langle(\forall(B(A)))^I, V\rangle$ is a monadic boolean algebra where $(Vf)(c) = \bigwedge_{c\in I} f(c)$, for each $f \in (\forall(B(A)))^I$. Then $\eta : B(A) \to (\forall(B(A)))^I$ defined by $\eta(b)(c) = c(b)$ for each $b \in B(A)$ is a monadic boolean monomorphism such that $\eta(b)(c) = \bigwedge_{x\in I}(\eta(b))(x)$.

**Claim 3.2** The image of a filter in $B(A)$ under $\forall$ is a filter in $\forall(B(A))$. Let $h^1$ be the function from the lattice of divisors of $n$ into the lattice of filters of $\forall(B(A))$ defined by $h^1(d) = \forall(h_A(d))$. It is easy to show that $\langle\forall(B(A)), \forall, h^1\rangle \in MB^{n+1}$; then, by Corollary 2.2, $\langle M(\forall(B(A)), h^1), \mathbb{V}\rangle$ is a monadic $(n+1)$-valued Wajsberg algebra.

**Claim 3.3** If $F$ is a filter in $\forall(B(A))$, then $F^I$ is a filter in $(\forall(B(A)))^I$. Let $h^2$ be the function from the lattice of divisors of $n$ into the lattice of filters of $(\forall(B(A)))^I$ defined by $h^2(d) = (\forall(h_A(d)))^I$. It is easy to show that $\langle(\forall(B(A)))^I, V, h^2\rangle \in MB^{n+1}$. Therefore, $\langle M((\forall(B(A)))^I, h^2), \mathbb{V}\rangle$ is a monadic $(n+1)$-valued Wajsberg algebra, follows from Corollary 2.2.

**Claim 3.4** $\langle M((\forall(B(A)))^I, h^2), \mathbb{V}\rangle$ and $\langle(M(\forall(B(A)), h^1))^I, V\rangle$ are isomorphic algebras.

Let $\Psi : M((\forall(B(A)))^I, h^2) \to (M(\forall(B(A)), h^1))^I$ be the function defined by $((\Psi(g))(c))(i) = g(i)(c)$, for each $g \in M((\forall(B(A)))^I, h^2)$, $c \in I$ and $i \in \{1, 2, \ldots, n\}$.

The function $\Psi$ is well defined and it is a monadic $W$-isomorphism. Indeed, let $g \in M((\forall(B(A)))^I, h^2)$, $d \in Div(n)$ and $1 \le q \le d$ be an integer. For short let $i_0 = (q-1)\frac{n}{d} + 1$ and $i_1 = q\frac{n}{d}$; then $\xi_{d,q}(g) = g(i_1) \to g(i_0) \in h^2(d) = (\forall(h_A(d)))^I$. Therefore for each $c \in I$ we have $\xi_{d,q}((\Psi(g))(c)) = ((\Psi(g))(c))(i_1) \to ((\Psi(g))(c))(i_0) = g(i_1)(c) \to g(i_0)(c) = (g(i_1) \to g(i_0))(c) \in h^1(d) = \forall(h_A(d))$.

On the other hand, let $f, g \in M((\forall(B(A)))^I, h^2)$, $c \in I$ and $i \in \{1, 2, \ldots, n\}$; then:

(i) $\Psi(f \mapsto g) = \Psi(f) \to \Psi(g)$, indeed:

$(\Psi(f \mapsto g)(c))(i) = (f \mapsto g)(i)(c) = \bigwedge_{k=1}^{n-i+1}(f(k)(c) \to g(k+i-1)(c))$
$= \bigwedge_{c\in I}(\Psi(g)(c))(i) = \bigwedge_{k=1}^{n-i+1}((\Psi(f)(c))(k) \to (\Psi(g)(c))(k+i-1))$
$= (\Psi(f)(c) \mapsto \Psi(g)(c))(i)$.

(ii) $\Psi(\neg f) = \neg\Psi(f)$, and

(iii) $\Psi(\mathbb{V}g) = V\Psi(g)$, indeed:
$(\Psi(\mathbb{V}g)(c))(i) = ((\mathbb{V}g)(i))(c) = (V(g(i)))(c) = \bigwedge_{c\in I} g(i)(c) = \bigwedge_{c\in I} (\Psi(g)(c))(i)$
$= \left(\bigwedge_{c\in I} (\Psi(g)(c))\right)(i) = ((V\Psi(g))(c))(i)$.

(iv) $\Psi$ is bijective.

**Claim 3.5** From Theorem 2.2 $\langle A,\forall\rangle$ and $\langle M(B(A),h_A),\mathbb{V}\rangle$ are isomorphic monadic $(n+1)$-valued Wajsberg algebras; the isomorphism is $\varphi : A \longrightarrow M(B(A),h_A)$ defined by $\varphi(x)(i) = \sigma_i(x)$ for all $x \in A$ and every integer $1 \leq i \leq n$.

**Claim 3.6** The monomorphism $\eta$ is a morphism between the objects $\langle B(A),\forall,h_A\rangle$ and $\langle(\forall(B(A)))^I,V,h^2\rangle$ in $\mathscr{MB}^{n+1}$. Thus, $M(\eta)$ is a monadic $W$-monomorphism from $\langle M(B(A),h_A),\mathbb{V}\rangle$ into $\langle M((\forall(B(A)))^I,h^2),\mathbb{V}\rangle$.

From Claim 3.1 we only have to show $h_A(d) \subseteq \eta^{-1}\left((\forall h_A(d))^I\right)$, for every $d \in Div(n)$. If $x \in h_A(d)$ then $\forall x \in \forall(h_A(d))$, on the other hand, $\forall x = c(\forall x) \leq c(x)$, for each $c \in I$. Therefore $c(x) = \eta(x)(c) \in \forall(h_A(d))$ for every $c \in I$, i.e., $\eta(x) \in (h_A(d))^I$, so $x \in \eta^{-1}\left((\forall h_A(d))^I\right)$.

**Claim 3.7** From Claims 3.1 to 3.6 we have the situation that is shown in the following diagram. The function $\gamma = \Psi \circ M(\eta) \circ \varphi$ from $A$ into $(M(\forall(B(A)),h^1))^I$ is a monadic $W$-monomorphism such that for every $a \in A$ there exists $x_0 \in I$ such that $(\gamma(a))(x_0) = \bigwedge_{c\in I} (\gamma(a))(c)$.

$$
\begin{array}{ccc}
\langle A,\forall\rangle & & \\
B \downarrow & & \\
\langle B(A),\forall,h_A\rangle & \overset{\eta}{\longrightarrow} & \langle(\forall(B(A)))^I,V,h^2\rangle \\
M \downarrow & & \downarrow M \\
\langle M(B(A),h_A),\mathbb{V}\rangle & \underset{M(\eta)}{\longrightarrow} & \langle M((\forall(B(A)))^I,h^2),\mathbb{V}\rangle \\
 & & \downarrow \Psi \\
 & & \langle(M(\forall(B(A)),h^1))^I,V\rangle
\end{array}
$$

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