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MOISIL ALGEBRAS

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INTRODUCTION.

Many-valued logics were introduced by J.Lukasiewicz, who defined a three-valued propositional calculus in 1921. Later, the same author considered propositional calculi with finitely many, and even denumerably many, truth-values [9] . We recall that if we denote by T_n the set of all fractions $k/n-1$, where $k = 0,1,\ldots,n-1$ and by $v(p) \in T_n$ the truth-value of a proposition p, then the truth-value of the n-valued implication defined by Lukasiewicz (n an integer ≥ 2) is given by:

$$v(p \supset q) = \min(1, 1-v(p)+v(q))$$

At the same time, E.L.Post [22] also studied many-valued propositional calculi, but different of those of

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Lukasiewicz.

In 1940 G.Moisil [11] introduced the notion of <u>three-valued Lukasiewicz algebra</u>, with the main purpose of obtaining an algebraic characterization of the matrices of the three-valued Lukasiewicz propositional calculus, and in 1942 P.C.Rosenbloom [23] introduced the <u>Post algebras of order n</u> with an analogous purpose with respect to the Post n-valued propositional calculus.

Later, Moisil [12] generalized the notion of three-valued Lukasiewicz algebra, defining the <u>n-valued Lukasiewicz algebras</u>. But A.Rose (in a personal comunication to the author) observed that these algebras are not matrices of the n-valued propositional calculus of Lukasiewicz if $n \geq 5$. For, if we consider the subset $A_n \subseteq T_n$ formed by the fractions $i/n-1$, where $i = 0,1, n-2,n-1$, it follows that:

$$\min (1, 1-(n-2/n-1)+(1/n-1)) = 2/n-1 \notin A_n \text{ if } n \geq 5.$$

Therefore A_n is not closed under the Lukasiewicz n-valued implication, although it is closed under all the operations of the n-valued Lukasiewicz algebras (see example 1.6 below). For this reason we have called <u>Moisil algebras of order n</u> the algebras that Moisil called n-valued Lukasiewicz algebras (see definition 1.2 below).

In any case, these algebras have an interest of their own. For example, among Moisil's motivations in studying them (see, for example, [13]) was their possible application to the study of switching circuits, and from this point of view the implication operator is of secondary importance, and certain modal operators of the Moisil algebra play a central role.

The aim of this paper is to study Moisil algebras of order n from an algebraic standpoint. Our main objective is the characterization of the free algebras with a finite set of generators, and to this end we need a detailed study of the structure of Moisil algebras.

In general, we have followed a path analogous to that of A.Monteiro in his lectures on three-valued Lukasiewicz algebras [19] .

The main new result of section 1 is an equational characterization of Moisil algebras of order n. In sections 2 and 3 we study the congruences and homomorphisms, showing that the Moisil algebras of order n are semi-simple (Corollary 3.14). In section 4 we analyze the structure of the ordered set of prime filters of Moisil algebras of order n and in section 5 we characterize the simple algebras, obtaining a new proof of a Moisil's representation theorem. In section 6 we consider some properties of finite algebras that we apply in section 7 to the determination of the free algebras. Finally, in section 8 we establish the connection between Moisil and Post algebras.

Some results concerning the representation of Moisil algebras, both by numerical continuous functions on a Boolean space and by topological spaces will be published elsewhere.

1.PRELIMINARIES.

1.1.DEFINITION. A De Morgan algebra is a system $\langle A,1,\vee,\wedge,-\rangle$ such that $\langle A,1,\vee,\wedge\rangle$ is a distributive lattice with unit 1 and $-$ is a unary operation defined on A fulfilling the conditions:

M 1) $- - x = x$ and M 2) $-(x \vee y) = -x \wedge -y$

If $-$ also satisfies the condition:

$$K) \quad x \wedge -x \leq y \vee -y$$

the system $\langle A,1,\vee,\wedge,-\rangle$ is called a Kleene algebra.

These notions have been studied by G.Moisil [10] , J. Kalman [8] , Bialinicky-Birula and Rasiowa [1] , [2] , [3] and A.Monteiro [17] , [18] . We follow the terminology introduced by the last author.

It is easy to check that in any De Morgan algebra the following properties hold:

M 3) $x \leq y$ if and only if $-x \leq -y$.

M 4) $-(x \wedge y) = -x \vee -y$.

M 5) $- 1 = 0$ is the zero of the lattice $\langle A,\vee,\wedge\rangle$.

1.2.DEFINITION. A Moisil algebra of order n (n an integer ≥ 2) is a system $\langle A,1,\vee,\wedge,-,s_1,\ldots,s_{n-1}\rangle$ such that $\langle A,1,\vee,\wedge,-\rangle$ is a De Morgan algebra and s_i $(1 \leq i \leq n-1)$ are unary operations defined on A fulfilling the conditions:

L 1) $s_i(x \vee y) = s_i x \vee s_i y$

L 2) $s_i x \vee -s_i x = 1$

L 3) $s_i s_j x = s_j x$

L 4) $s_i -x = -s_{n-i} x$

L 5) $s_1 x \leq s_2 x \leq \ldots \leq s_{n-1} x$

L 6) If $s_i x = s_i y$ for $i = 1,2,\ldots,n-1$, then $x = y$.

We shall deal with Moisil algebras of order n, where n is a fixed integer ≥ 2, and which we shall usually denote by the letters A,A',A".

This notion was introduceded by G.Moisil in [12] , developed later by the same author ([13] , [14] , [15]) and recently by C.Sicoe ([24] , [25] , [26] , [27]).

1.3.LEMMA. The following properties are true in any Moisil algebra of order n:

L 7) $s_i(x \wedge y) = s_i x \wedge s_i y$

L 8) $s_i x \wedge -s_i x = 0$

L 9) $x \leq y$ if and only if $s_i x \leq s_i y$ for $i = 1,\ldots,n-1$.

L 10) $x \leq s_{n-1} x$

L 11) $s_1 x \leq x$

L 12) $s_i 1 = 1$, $s_i 0 = 0$ for $i = 1,2,\ldots,n-1$.

L 13) $-x \vee s_{n-1}x = 1$

L 14) $x \wedge -s_i x \wedge s_{i+1}y \leq y$ $(1 \leq i \leq n-2)$.

PROOF. All these results, except L 14), are well known. To prove L 14), let $z = x \wedge -s_i x \wedge s_{i+1}y$. It is easy to see that $s_j z = 0 \leq s_j y$ if $1 \leq j \leq i$, and that $s_j z \leq s_{i+1}y \leq s_j y$ if $i+1 \leq j \leq n-1$. Using L 9) we obtain L 14).

1.4. THEOREM. A system $\langle A, 1, \vee, \wedge, -, s_1, \ldots, s_{n-1} \rangle$ is a Moisil algebra of order n if and only if $\langle A, 1, \vee, \wedge, - \rangle$ is a De Morgan algebra and s_i $(1 \leq i \leq n-1)$ are unary operations defined on A that fulfill the properties L 1) - L 5), L 10) and L 14).

PROOF. We need to prove that in a De Morgan algebra with operators s_i satisfying L 1) - L 5), L 10) and L 14) the condition L 6) holds.

First of all, we can prove by induction on k that:

(1) $$\bigvee_{i=1}^{k} ((-s_i y \vee y) \wedge (s_{i+1}y \vee y)) = y \vee s_{i+1}y ,$$

where $1 \leq k \leq n-2$.

Suppose now that x, y are such that $s_i x = s_i y$ for $i = 1,2,\ldots,n-1$. If we substitute in L 14) $s_i x$ by $s_i y$ for $i = 1,2,\ldots,n-3$, and $s_{n-1}y$ by $s_{n-1}x$ if $i = n-2$, we have:

(2) $$y = (x \vee y) \wedge (-s_i y \vee y) \wedge (s_{i+1}y \vee y)$$

for $i = 1,2,\ldots,n-3$, and

(3) $$x \wedge -s_{n-2}x \wedge s_{n-1}x \leq y$$

From L 10) and the equality $s_{n-2}x = s_{n-2}y$, (3) yields:

(4) $$y = (x \vee y) \wedge (-s_{n-2}y \vee y)$$

From (2) and (4) we obtain:

(5) $$y = (x \vee y) \wedge \Big(\bigvee_{i=1}^{n-3} ((-s_i y \vee y) \wedge (s_{i+1}y \vee y)) \vee (-s_{n-2}y \vee y)\Big)$$

and from (5) and (1) it follows that $y = x \vee y$, i.e., $y \geq x$. Interchanging x with y in the above proof, we obtain that $x \geq y$, and therefore $x = y$.

1.5.REMARK. Theorem 1.4 gives an equational characterization of Moisil algebras of order n.

For, De Morgan algebras are equationally definable, axioms L 1) - L 4) are equations and L 5), L 10) and L 14) may be written in the form:

L 5) $s_i x \vee s_{i+1}x = s_{i+1}x$ for $i = 1,2,\dots,n-2$.

L 10) $x \vee s_{n-1}x = s_{n-1}x$

L 14) $(x \wedge -s_i x \wedge s_{i+1}y) \vee y = y$, for $i = 1,\dots,n-2$.

We remark that C.Sicoe [26], [27] has given shorter characterizations of Moisil algebras, but using always L 6) as an axiom.

1.6.EXAMPLE. Let L_n be the set of the fractions $j/n-1$, with $j = 0,1,\dots,n-1$, considered as a sublattice of the real numbers, and with $-(j/n-1) = n-1-j/n-1$, and $s_i(j/n-1) = 0$ if $i+j < n$, and $s_i(j/n-1) = 1$ if $i+j \geq n$.

It is easy to check that L_n is a Moisil algebra of order n (cf [13]).

We shall denote by B(A) the set of all complemented elements of A. Since A is a distributive lattice, it follows that B(A) is a sublattice of A which is a Boolean algebra, and moreover from L 2) it follows that if $b \in B(A)$, $-b$ is the Boolean complement of b, and therefore, B(A) is a De Morgan subalgebra of A.

From L 12), L 10), L 3), L 7) and L 2) it follows that s_{n-1} is a Boolean multiplicative closure (in the sense of [5]) defined on A, and therefore we have that:

(1.7) $\quad s_{n-1}x = \bigwedge \{b \in B(A) : x \leq b\}$

(1.8) $\quad x \in B(A)$ <u>if and only if</u> $s_{n-1}x = x$

If $K_i = \{x \in A : x = s_i x\}$ then from L 3) and (1.8) we have the following result, proved by G.Moisil (13 ,p 123) in a different way:

<u>1.9.THEOREM.</u> $K_1 = K_2 = \dots = K_{n-1} = B(A)$.

It follows that any Boolean algebra is a Moisil algebra of order n, if we define $-x$ as the complement of x and $s_i x = x$ for all i. On the other hand, any Moisil algebra of order 2 is a Boolean algebra if we define the complement of x as $-x$.

2. WEAK IMPLICATION AND DEDUCTIVE SYSTEMS.

We define a new binary operation, called <u>weak implication</u> and denote it by $\longrightarrow$ as follows:

(2.1) $$x \longrightarrow y = s_{n-1}-x \vee y$$

The weak implication was defined and studied by A. Monteiro [19] (see also [20]) in the case $n = 3$, and we shall extend some of his results for $n > 3$.

2.2. THEOREM. *The weak implication has the following properties:*

W 1) $x \longrightarrow (y \longrightarrow x) = 1$

W 2) $x \longrightarrow (y \longrightarrow z) = (x \longrightarrow y) \longrightarrow (x \longrightarrow z)$

W 3) $x \longrightarrow (y \wedge z) = (x \longrightarrow y) \wedge (x \longrightarrow z)$

W 4) $x \longrightarrow (y \longrightarrow z) = (x \wedge y) \longrightarrow z$

W 5) $(x \vee y) \longrightarrow z = (x \longrightarrow y) \wedge (y \longrightarrow z)$

W 6) $(x \longrightarrow y) \longrightarrow x = x$

W 7) $1 \longrightarrow x = x$

W 8) *If* $x \leq y$, *then* $x \longrightarrow y = 1$

W 9) $x \leq y$ *if and only if* $s_i x \longrightarrow s_i y$ *for* $i = 1,\dots,n-1$,

W10) $x \longrightarrow s_1 x = 1$.

2.3. DEFINITIONS. *A set* $D \subseteq A$ *is a deductive system if* *1)* $1 \in D$ *and 2) (Modus Ponens) If* x *and* $x \longrightarrow y$ *belong to* D, *then* $y \in D$. D *is proper if* $D \neq A$. *The deductive system generated by a set* $H \subseteq A$, $D(H)$, *is the set theoretical intersection of all the deductive systems containing* H.

2.4.DEFINITIONS. If $H \subseteq A$ is non void, we say that t is a consequence of H if there is a finite set $\{h_1,\dots,h_2\} \subseteq H$ such that:

$$(h_1 \wedge h_2 \wedge \dots \wedge h_k) \longrightarrow t = 1$$

We denote by $C(H)$ the set of all consequences of H, and we set $C(\emptyset) = \{1\}$.

From W 4) it follows that:

2.5.LEMMA. t is a consequence of H $(\neq \emptyset)$ if and only if there exists a finite set $\{h_1,\dots,h_k\} \subseteq H$ such that:

$$h_1 \longrightarrow (h_2 \longrightarrow \dots (h_k \longrightarrow t) \dots) = 1$$

2.6.THEOREM. If $H \subseteq A$, then $D(H) = C(H)$.

PROOF. It is easy to see that $C(\emptyset) = \{1\}$, therefore $D(\emptyset) = C(\emptyset)$. If $H \neq \emptyset$, the proof can be done in the following steps:

1) $C(H)$ is a deductive system. Since $H \neq \emptyset$, from W 8) we obtain that $1 \in C(H)$. Suppose that x and $x \longrightarrow y$ belong to $C(H)$. There are elements $h_{i_1},\dots,h_{i_r},h_{j_1},\dots,h_{j_s}$ belonging to H such that:

(1) $$(h_{i_1} \wedge \dots \wedge h_{i_r}) \longrightarrow x = 1$$

and:

(2) $$(h_{j_1} \wedge \dots \wedge h_{j_s}) \longrightarrow (x \longrightarrow y) = 1$$

Setting $h_i = h_{i_1} \wedge \dots \wedge h_{i_r}$ and $h_j = h_{j_1} \wedge \dots \wedge h_{j_s}$, from (1), (2) and W 8) we obtain:

(3) $$h_j \longrightarrow (h_i \longrightarrow x) = 1$$

and

(4) $$h_i \longrightarrow (h_j \longrightarrow (x \longrightarrow y)) = 1$$

and applying W 4) to (3) and (4) we have:

(5) $$(h_j \wedge h_i) \longrightarrow x = 1$$

and

(6) $$(h_j \wedge h_i) \longrightarrow (x \longrightarrow y) = 1$$

Setting $h = h_j \wedge h_i$, from W 2), (5) and (6) we obtain:

(7) $$(h \longrightarrow x) \longrightarrow (h \longrightarrow y) = 1$$

and from (6), (7) and W 7) it follows that:

$$h \longrightarrow y = 1$$

which is an abbreviation of:

$$(h_{i_1} \wedge \dots \wedge h_{i_r}) \wedge (h_{j_1} \wedge \dots \wedge h_{j_s}) \longrightarrow y = 1$$

therefore $y \in C(H)$.

2) $H \subseteq C(H)$. For, if $h \in H$, by W 8) it follows that $h \longrightarrow h = 1$, so $h \in C(H)$.

3) <u>If</u> D <u>is a deductive system containing</u> H,<u>then</u> $C(H) \subseteq D$.

Let $x \in C(H)$. By lemma 2.5 there are elements $h_1, \dots, h_k$ belonging to $H \subseteq D$ such that:

(8) $$h_1 \longrightarrow (h_2 \longrightarrow \dots (h_k \longrightarrow x) \dots) = 1$$

Since $1 \in D$, we have by modus ponens that $x \in D$.

<u>2.7.DEDUCTION THEOREM</u>. <u>If</u> $H \subseteq A$ <u>and</u> $x \in A$, <u>then</u>:

$$D(H \cup \{x\}) = C(H \cup \{x\}) = \{y \in A : x \longrightarrow y \in C(H)\}$$

<u>PROOF</u>. Let $D' = \{y \in A : x \longrightarrow y \in C(H)\}$.

1) D' <u>is a deductive system.</u> From W 8) it follows that $x \longrightarrow 1 = 1$, hence $1 \in D'$. Suppose that y and $y \longrightarrow z$ belong to D', i.e. suppose that we have:

$$(1) \qquad x \longrightarrow y \in C(H)$$

and

$$(2) \qquad x \longrightarrow (y \longrightarrow z) \in C(H)$$

(2) and W 3) imply:

$$(3) \qquad (x \longrightarrow y) \longrightarrow (x \longrightarrow z) \in C(H)$$

and since $C(H) = D(H)$ is a deductive system, from (1) and (3) we obtain that $x \longrightarrow z \in C(H)$, so $z \in D'$.

2) $H \cup \{x\} \subseteq D'$. $x \in D'$ because by W 8), $x \longrightarrow x = 1$. On the other hand, if $h \in H$, from W 1) it follows that $x \longrightarrow h \in C(H)$, and therefore $h \in D'$.

3) <u>If</u> D <u>is a deductive system containing</u> $H \cup \{x\}$, <u>then</u> $D' \subseteq D$. By hypothesis we have:

$$(4) \qquad H \subseteq D \qquad \text{and} \qquad (5) \qquad x \in D$$

The former implies:

$$(4') \qquad C(H) = D(H) \subseteq D$$

Suppose now that $y \in D'$, i.e. that:

$$(6) \qquad x \longrightarrow y \in C(H)$$

From (4') and (6) we obtain:

$$(7) \qquad x \longrightarrow y \in D$$

and since D is a deductive system, (5) and (7) imply that $y \in D$.

From 1), 2) and 3) it follows that $D' = D(H \cup \{x\}) = C(H \cup \{x\})$.

2.8.COROLLARY. *If $x \in A$, then* $D(\{x\}) = \{y \in A : x \longrightarrow y = 1\}$

From properties W 1), W 3), W 7) and W 8) it is easy to prove that:

2.9.LEMMA. *Every deductive system of A is a filter.*

But the converse is not true if $n \geq 3$. We are going to characterize the filters that are deductive systems.

2.10.DEFINITION. *A filter F of a didtributive lattice L with 0 and 1 is called a Stone filter if for any $x \in F$, there exists $b \in F \cap B(L)$ such that $b \leq x$, where $B(L)$ is the set of all complemented elements of L.*

The notion of a Stone filter was introduced by A. Monteiro ([16] , p 152) and also studied in [5] ,under the name of B-filter.

It is easy to see ([5],p 1169) that if F is a Stone filter of L, then $F^* = F \cap B(L)$ is a filter of the Boolean algebra $B(L)$, and moreover:

(2.11) $F = \{x \in L : b \leq x \text{ for some } b \in F^*\}$

Conversely, if F^* is a filter of the Boolean algebra $B(L)$, then the set F defined by (2.11) is a Stone filter of L and $F^* = F \cap B(L)$. Therefore we have:

3.12.LEMMA. *Let L be a distributive lattice with 0 and 1. The map $F \longrightarrow F^*$ is an isomorphism between the set* of Stone filters of L ordered by inclussion and the set of filters of B(L), also ordered by inclussion.

From the theorem 1.9 and property L 11) we can prove that:

2.13.LEMMA. In order that a filter F of a Moisil algebra A of order n be a Stone filter it is necessary and sufficient that if $x \in F$, then $s_1 x \in F$.

The notion of a Stone filter in Moisil algebras was introduced by G.Moisil ([13] , p 127) under the name of strong dual ideal, in the form indicated in the above lemma.

The next theorem, establishing the equivalence between Stone filters and deductive systems, was first proved by A.Monteiro [19] for the case n = 3.

2.14.THEOREM. In order that $D \subseteq A$ be a deductive system it is necessary and sufficient that D be a Stone filter.

PROOF. The necessity of the condition follows at once from lemma 2.9 and property W 10). To prove the sufficience, let D be a Stone filterof A. By definition, $1 \in D$. Suppose that x and $x \longrightarrow y$ belong to D. Then $s_1 x$ and $s_1(x \longrightarrow y)$ belong to D. But L 1), L 3) and L 4) imply that:

$$s_1(x \longrightarrow y) = s_1(s_{n-1}-x \vee y) = -s_1 x \vee s_1 y$$

and from L 2) we obtain:

$$s_1x \wedge s_1(x \longrightarrow y) = s_1x \wedge s_1y \in D$$

Since by L 11) $s_1x \wedge s_1y \leq y$, it follows that $y \in D$.

2.15.COROLLARY. The map $D \longrightarrow D^* = D \cap B(A)$ is an isomorphism from the set of all deductive systems of A onto the set of all filters of $B(A)$, with both sets ordered by inclussion.

Applying a well known result of M.H.Stone (see,for example, [4], p 129) to the above corollary, we obtain:

2.16.COROLLARY. The set of all deductive systems of A, ordered by inclussion, is a complete Brouwerian algebra.

We remark that C.Sicoe ([24] ,Th.17) proved, by a direct calculation, that the set of deductive systems of A is a distributive lattice.

2.17.DEFINITION. A deductive system M is called maximal if 1) M is proper, and 2) If D is a prper deductive system of A such that $M \subseteq D$, then $M = D$.

From corollary 3.15 it follows at once that:

2.18.COROLLARY. A deductive system M of A is maximal if and only if M^* is a maximal (= prime) filter of the Boolean algebra $B(A)$.

Recall that a minimal prime filter of a lattice L is a minimal element of the set of all prime filters of L ordered by inclussion.

Since s_{n-1} is a Boolean multiplicative closure on A, and $s_1 = -s_{n-1}-$, we can apply to s_1 the dual results of those of [5], and therefore from lemma 2.5 and theorem 3.5 of [5] we obtain the following theorem, first proved by A.Monteiro [19] in case n = 3:

2.19.THEOREM. M _is a maximal deductive system of_ A _if and only if_ M _is a minimal prime filter of_ A.

The following result is a consequence of Corollary 2.15:

2.20.THEOREM. _The set theoretical intersection of all maximal deductive systems of_ A _is the deductive system formed by the element_ 1.

If we define _irreducible deductive system_ and _completely irreducible deductive system_ in the natural fashion, from corollary 2.15 and theorem 2.19 we also obtain:

2.21.THEOREM. _The notions of irreducible deductive system, completely irreducible deductive system, maximal deductive system, and minimal prime filter are all equivalent._

3. HOMOMORPHISMS AND QUOTIENT ALGEBRAS.

3.1.DEFINITION. _If_ A,A' _are Moisil algebras of order_ n, _an homomorphism_ h _from_ A _into_ A' _is a map_ $h: A \longrightarrow A'$

<u>fulfilling the conditions</u>:

H 1) $h(x \vee y) = h(x) \vee h(y)$

H 2) $h(x \wedge y) = h(x) \wedge h(y)$

H 3) $h(-x) = -h(x)$

H 4) $h(s_i x) = s_i h(x) \quad (i = 1,2,\ldots,n-1)$

H 5) $h(0) = 0$, $h(1) = 1$

<u>A one-to-one and onto homomorphism is called an isomorphism</u>.*)

<u>3.2.REMARK</u>. Some of the conditions H j) that appear in the above definition are redundant, since they can be obtained from the others. For instance, H 2) can be deduced from H 1) and H 3), using M 1) and M 2). Analogously, from H 3) and M 5) it follows that $h(1) = 1$ implies that $h(0) = 0$. Finally, from L 3), H 1), H 3) and H 4) we obtain:

$$h(1) = h(-x \vee s_{n-1}x) = -h(x) \vee s_{n-1}h(x) = 1$$

On the other hand, it is obvious that:

H 6) $h(x \longrightarrow y) = h(x) \longrightarrow h(y)$

<u>3.3.DEFINITION</u>. <u>The kernel of an homomorphism</u> $h:A \longrightarrow A'$ <u>is the set</u>:

$$\text{Ker } h = \{x \in A : h(x) = 1\}$$

---

*) It can be proved that the monomorphisms of the category of Moisil algebras of order n are just the one-to-one homomorphisms, but there are examples of epimorphisms that are not onto homomorphisms.

3.4.THEOREM. *If* $h:A \longrightarrow A'$ *is an homomorphism, then* Ker h *is a deductive system of* A *and* $h(x) \leq h(y)$ *if and only if* $s_i x \longrightarrow s_i y \in$ Ker h *for* $i = 1,\ldots,n-1$.

PROOF. From H 6) it follows that for any deductive system D' of A', $h^{-1}(D')$ is a deductive system of A. The second part follows from W 9) and H 6).

From the above theorem it is easy to prove that:

3.5.LEMMA. *Let* $h':A \longrightarrow A'$ *and* $h'':A \longrightarrow A''$ *be onto homomorphisms. If* Ker $h' \subseteq$ Ker h'', *then there exists a unique (onto) homomorphism* $h:A' \longrightarrow A''$ *such that* $h''=hh'$.

3.6.THEOREM. *Let* $h':A \longrightarrow A'$ *and* $h'':A \longrightarrow A''$ *be onto homomorphisms. If* Ker h' = Ker h'', *then* A' *and* A'' *are isomorphic*.

We are going to determine all the homomorphic images of A by mean of a construction on A. First of all we recall the following definition ([13] , p 127):

3.7.DEFINITION. *If* D *is a deductive system of* A, *we say that the elements* x *and* y *are congruent modulo* D, *and we write* $x \equiv y$ (D), *if there exists an element* $d \in D$ *such that* $x \wedge d = y \wedge d$.

It is easy to see that $\equiv$ is a congruence relation on the algebra A, and since the notion of Moisil algebra of order n is equationally definable, it follows ([4], Chapter VI) that the set A/D of all congruence classes, algebrized in the natural fashion, is a Moisil algebra of order n, called the quotient algebra of A by D. Moreover, if we denote by x/D the congruence class containing x, the map h(x) = x/D is an homomorphism from A onto A/D, called the natural homomorphism, and Ker h = = D.

From the above remarks, theorem 3.4 and theorem 3.6 we obtain:

3.8.THEOREM. *If* $h: A \longrightarrow A'$ *is an onto homomorphism, then* A' *is isomorphic with* A/Ker h.

3.9.COROLLARY. $x \equiv y$ (D) *if and only if* $s_i x \longrightarrow s_i y \in D$ *and* $s_i y \longrightarrow s_i x \in D$ *for* $i = 1,2,\ldots,n-1$.

Moreover, we have:

3.10.THEOREM. *The ordered set of all congruences of the algebra* A *is isomorphic with the set of all deductive systems of* A, *ordered by inclussion*.

We recall that an algebra S is *simple* if the only congruences on S are the trivial ones ([4],Chapter VI), therefore from the last theorem it follows that:

3.11.THEOREM. *A Moisil algebra* S *of order n is simple if and only if* $\{1\}$ *is the only proper deductive system of* S.

3.12.COROLLARY. *If* M *is a maximal deductive system of* A, *then* A/M *is simple*.

Analogously, from ([4],Chapter VI, p 138) we obtain:

3.13.THEOREM. *In order that* A *be isomorphic to a subdirect product of a family* $\{A_j\}_{j \in J}$ *of algebras, it is necessary and sufficient that there exists a family* $\{D_j\}_{j \in J}$ *of deductive systems of* A *such that: 1)* $\bigcap_{j \in J} D_j = \{1\}$, *and 2)* $A_j = A/D_j$ *for all* $j \in J$.

The next corollary was first proved by A.Monteiro [19] for the case $n = 3$:

3.14.COROLLARY. *Any Moisil algebra of order n (with more than one element) is a subdirect product of a family of simple algebras.*

PROOF. It is a consequence of theorems 3.13, 2.20 and corollary 3.12.

Our next objective will be the determination of simple Moisil algebras of order n. To this end we shall need to study the structure of the set of prime filters of Moisil algebras.

4. PRIME FILTERS.

4.1.LEMMA. *If* U *is an ultrafilter (= prime filter) of* B(A), *then the sets:*

$$U_i = \{x \in A : s_i x \in U\} , \quad i = 1,\dots,n-1$$

are prime filters of A, *and moreover the following relations hold:*

(4.2) $\qquad U = U_i \cap B(A) \qquad (1 \leq i \leq n-1)$

and

(4.3) $$U_1 \subseteq U_2 \subseteq \ldots \subseteq U_{n-1}$$

4.4.LEMMA. *Let* P *be a prime filter of* A *and* $P^* = P \cap B(A)$. *Then* P^* *is an ultrafilter of* $B(A)$, *and:*

(4.5) $$P^*_1 \subseteq P \subseteq P^*_{n-1}$$

4.6.LEMMA. *Let* P *and* P^* *be as in the above lemma. If* $1 \leq i \leq n-2$, *then either* $P \subseteq P^*_i$ *or* $P^*_{i+1} \subseteq P$.

PROOF. Assume that the thesis is not true, i.e.,assume that:

(1) $P \not\subseteq P^*_i$ and (2) $P^*_{i+1} \not\subseteq P$

(1) implies that there exists an $x \in A$ such that:

(3) $x \in P$ and (4) $s_i x \notin P^*$

and since P^* is an ultrafilter of $B(A)$, (4) is equivalent to:

(4') $$-s_i x \in P^* \subseteq P$$

(2) implies that there exists an $y \in A$ such that:

(5) $y \notin P$ and (6) $s_{i+1} y \in P^* \subseteq P$

From (3), (4') and (6) it follows that:

(7) $$x \wedge -s_i x \wedge s_{i+1} y \in P$$

Since P is a filter, from (7) and L 14) it follows that $y \in P$, in contradiction with (5).

4.7.THEOREM. *For any prime filter* P *of* A, *there exists a unique ultrafilter* P^* *of* $B(A)$ *and an* i, $1 \leq i \leq n-1$ *such that* $P = P^*_i$.

PROOF. Let $P^* = P \cap B(A)$. We know that P^* is a prime filter of B(A) and that $P^*_1 \subseteq P \subseteq P^*_{n-1}$. Let $i_0 = \max\{i : P^*_i \subseteq P\}$. If $i_0 = n-1$, then $P = P^*_{n-1}$. If $i_0 \leq n-2$, we have:

(1) $P^*_{i_0} \subseteq P$ and (2) $P^*_{i_0+1} \not\subseteq P$

Recalling lemma 4.6, (2) implies:

(3) $P \subseteq P^*_{i_0}$

and from (1) and (3) it follows that $P^*_{i_0} = P$.

The uniqueness of P^* follows from (4.2).

4.8.COROLLARY. Any prime filter P of A belongs to one and only one maximal chain of prime filters, and this chain has at most n-1 elements.

4.9.COROLLARY. The set of prime filters of a Moisil algebra of order n, ordered by inclussion, is the cardinal sum of totally ordered sets, each of them having at most n-1 elements.

From theorems 2.14 and 4.17 we also obtain:

4.10.COROLLARY. M is a maximal deductive system of A if and only if there exists a (unique) ultrafilter P^* of B(A) such that $M = P^*_1$.

Corollaries 4.8 and 4.9 have been proved by A. Monteiro [19] for the case n = 3, but using the properties of prime filters in Kleene algebras.

Since Moisil algebras are De Morgan algebras, we can define for any prime filter P of A the Bialynicki-

Birula and Rasiowa transformation [2] by mean of the formula:

$$g(P) = C\,-P \qquad (4.11)$$

where C designs the set theoretical complement and $-P = \{-x : x \in P\}$.

Since any prime filter P of A is of the form P^*_i, where P^* is an ultrafilter of B(A), the following theorem characterizes the transformation g in the case of Moisil algebras of order n:

4.12.THEOREM. *If U is an ultrafilter of B(A), then* $g(U_i) = U_{n-i}$.

PROOF. The following conditions are equivalent: 1) $x \in g(U_i)$, 2) $x \notin -U_i$, 3) $-x \notin U_i$, 4) $s_i -x \notin U$, 5) $-s_{n-i}x \notin U$, 6) $s_{n-i}x \in U$, and 7) $x \in U_{n-i}$.

4.13.COROLLARY. *Any Moisil algebra A of order n is a Kleene algebra*.

PROOF. If P is a prime filter of A, there exists an ultrafilter P^* of B(A) such that $P = P^*_i$ for some i ($1 \leq i \leq n-1$), and by the above theorem, $g(P) = g(P^*_i) = P^*_{n-i}$. Therefore by (4.3) it follows that either $g(P) \subseteq P$ or $P \subsetneq g(P)$, and the last condition is necessary and sufficient in order that a De Morgan algebra be a Kleene algebra (see [3] and [18]).

We remark that corollary 4.13 was proved by C.Sicoe [25], [26] by a direct calculation. It was proved earlier by A.Monteiro [19] for the case $n = 3$.

5. SIMPLE ALGEBRAS. MOISIL'S REPRESENTATION THEOREM.

Let us introduce the following notations: $s_0 x = 0$ and $s_n x = 1$ for all $x \in A$, and if U is an ultrafilter of $B(A)$, $U_0 = \emptyset$ and $U_n = A$.

Taking into account L 3) and the corresponding definitions, it follows that:

5.1. LEMMA. *Let* U *be an ultrafilter of* $B(A)$. *Then for* $j = 1,2,\ldots,n-1$ *we have*:

1) $x \in U_j$ *if and only if* $s_j x \equiv 1\ (U_1)$
2) $x \in U_j$ *if and only if* $s_j x \equiv 0\ (U_1)$

5.2. THEOREM. *If* M *is a maximal deductive system of* A, *then there exists a unique homomorphism* $h : A \to L_n$ *such that* $\operatorname{Ker} h = M$, *and moreover* h *is given by the formula*:

$$(5.3) \quad h(x) = n-j/n-1 \text{ if and only if } x \in M^*_j - M^*_{j-1}$$

where $j = 1,2,\ldots,n$ *and* $M^* = M \cap B(A)$.

PROOF. The proof that formula (5.3) defines a homomorphism from A into L_n is long but computational, so it will be omitted. On the other hand, if $h : A \longrightarrow L_n$ is an homomorphism, then, since h is in particular a lattice homomorphism into a chain, it follows that $M = \operatorname{Ker} h$ is a prime Stone filter, and therefore, a maximal deductive system of A. From the definition of the s_i in L_n and lemma 5.1, it follows that h is defined by (5.3).

5.4.COROLLARY. *There is a one-to-one correspondence between the maximal deductive systems of a Moisil algebra A of order n and the homomorphisms from A into L_n.*

From theorems 3.8 and 5.2 we obtain:

5.5.COROLLARY. *If M is a maximal deductive system of A, then A/M is isomorphic to a subalgebra of L_n.*

5.6.COROLLARY. *The simple Moisil algebras of order n are just the subalgebras of L_n.*

PROOF. From theorem 3.11 it follows that the subalgebras of L_n are simple. On the other hand, if S is simple, then $\{1\}$ is a maximal deductive system of S, hence $S = S/\{1\}$ is isomorphic to a subalgebra of L_n.

From corollaries 3.14 and 5.6 it follows that:

5.7.THEOREM. *Any Moisil algebra of order n (with more than one element) is isomorphic to a subdirect product of a family of subalgebras of L_n.*

The above theorem contains the following:

MOISIL'S REPRESENTATION THEOREM ([13], p 134). *Any Moisil algebra of order n (with more than one element) is isomorphic to a subalgebra of a direct product of of algebras L_n.*

We know that any maximal doductive system M of A is the first element of a chain of prime filters of A, with at most n-1 elements:

$$M = M_1^* \subseteq M_2^* \subseteq \ldots \subseteq M_{n-1}^*$$

We say that M is of order k $(2 \leq k \leq n)$ if the corresponding chain of prime filters has $k-1$ distinct elements.

For instance, M is of order 2 if and only if we have that:

$$M = M_1^* = M_2^* = \ldots = M_{n-1}^*$$

and M is of order n if and only if $M_i^* \neq M_j^*$ for $i \neq j$ $(1 \leq i,j \leq n-1)$.

From theorem 5.2 and corollary 5.5 we obtain:

5.8.THEOREM. *A maximal deductive system* M *of* A *is of order* k $(2 \leq k \leq n)$ *if and only if* A/M *is isomorphic to a* k-*element subalgebra of* L_n.

To end this section, we are going to point out some properties of the subalgebras of L_n that we shall need in section 7.

5.9.LEMMA. *If* A *and* A' *are subalgebras of* L_n *such that* $A \neq A'$ *as subsets of* L_n, *then* A *and* A' *cannot be isomorphic as Moisil algebras of order* n.

PROOF. The result is trivial if A and A' have a different number of elements. Suppose that A and A' have the same number m of elements $(2 < m < n)$. Then we have:

$$A = \{0, i_1/n-1, \ldots, i_{m-2}/n-1, 1\}$$

and

$$A' = \{0, j_1/n-1, \ldots, j_{m-2}/n-1, 1\}$$

with $0 = i_0 < i_1 < \ldots < i_{m-2} < i_{m-1} = n-1$ and $0 = j_0 <$ $< j_1 < \ldots < j_{m-2} < j_{m-1} = n-1$. The only lattice isomorphism that we can define from A onto A' is given by $h(i_s/n-1) = j_s/n-1$, $s = 0,1,\ldots,m-1$. Since $A \neq A'$, there exists s such that $i_s \neq j_s$. We can assume that $i_s < j_s$, therefore $s_{j_s}(j_s/n-1) = 1$ and $s_{j_s}(i_s/n-1) = 0$, hence $h(s_{j_s}(i_s/n-1)) \neq s_{j_s}(h(i_s/n-1))$ and h cannot be an homomorphism.

We are going to enumerate the number of subalgebras of L_n.

Assume first that n is an even number, and let A be a subalgebra of L_n. Since the conditions $i/n-1 \in A$ and $-(i/n-1) = (n-1-i)/n-1 \in A$ are equivalent, it follows that A is determined by the elements of the form $i/n-1$ with $0 \leq i \leq n-2/2$ belonging to A, and since 0 belongs to any subalgebra, it follows that the number of 2k-element subalgebras of L_n $(1 \leq k \leq n/2)$ is equal to the number of combinations that we can form with k-1 elements chosen among $1/n-1$, $2/n-1$, ..., $(n-2)/n-1$. Therefore we have:

5.10.THEOREM. *If* n *is even and if* $1 \leq k \leq n/2$, *then the number of 2k-element subalgebras of* L_n *is* $\binom{n-2/2}{k-1}$

5.11.COROLLARY. *If* n *is even, the number of subalgebras of* L_n *is* $2^{(n-2/2)}$.

Assume now that n is odd, and let $z = (n-1/2)/n-1$. Since z is the only element of L_n such that $-z = z$, it follows that A is a subalgebra of L_n with an odd number of elements if and only if $z \in A$. Therefore, we obtain a subalgebra with 2k+1 elements adding z to a subalgebra with 2k elements ($1 \leq k \leq n-1/2$). As in the case n even, it follows that the number of subalgebras with 2k elements is equal to the number of combinations that we can form with k-1 elements chosen among $1/n-1, 2/n-1, \ldots, (n-3/2)/n-1$. From these remarks we obtain:

5.12.THEOREM. *If n is odd and $1 \leq k \leq n-1/2$, then the number of 2k-element subalgebras of L_n is equal to the number of 2k+1-element subalgebras of L_n and equal to* $\binom{n-3/2}{k-1}$.

5.13.COROLLARY. *If n is odd, the number of subalgebras of L_n is $2^{(n-1/2)}$.*

6.FINITE ALGEBRAS.

The aim of this section is to improve some of the previous results in the case that A has a finite number of elements. We shall denote by A a Moisil algebra of order n with a finite number of elements, and by (x) the principal filter generated by the element x.

Since A is finite, all the filters of A are principal, and therefore the deductive systems of A are just the principal filters generated by the complemented elements of A. In particular, the maximal deductive systems are the principal filters generated by the atoms of the Boolean algebra B(A).

First of all, and following a suggestion of A.Monteiro, we are going to determine the structure of the quotient algebra A/(b) in terms of A and the element b B(A).

For any b in B(A), we set $A_b =$ x A : $x \in b$ .Since A_b is an ideal, it follows that A_b is a sublattice of A, the zero of A_b being 0 and the unit of A_b being b.Moreover, since $b \in B(A)$ it follows that A_b is closed under the operations $s_1, \ldots, s_{n-1}$, and if we define $\sim x =$ $= -x \wedge b$, it is easy to check that A_b is also closed under $\sim$.

<u>6.1.THEOREM.</u> $\langle A_b, b, \vee, \wedge, \sim, s_1, \ldots, s_{n-1} \rangle$ <u>is a Moisil algebra of order n isomorphic to the quotient algebra</u> A/(b).

<u>PROOF.</u> For any x in A we define $h(x) = x \wedge b$. It is well known that h is a lattice homomorphism from A onto A_b. Futhermore, $h(s_i x) = s_i x \wedge b = s_i x \wedge s_i b = s_i(x \wedge b) =$ $= s_i h(x)$ $(1 \leq i \leq n-1)$, and $\sim h(x) = -h(x) \wedge b =$ $-(x \wedge b) \wedge b = -x \wedge b = h(-x)$, so h is an homomorphism from A onto the similar algebra A_b, and, since Moisil algebras of order n are equationally definable, this

implies that A_b is a Moisil algebra of order n.

Moreover, $h(x) = b$ if and only if $x \leq b$, which shows that Ker $h = (b)$, and from theorem 3.8 it follows that A_b is isomorphic to $A/(b)$.

6.2.THEOREM. *If $b_1, b_2, \dots, b_r$ are complemented elements of A such that: 1) $b_1 \vee b_2 \vee \dots \vee b_r = 1$ and 2) $b_i \wedge b_j = 0$ if $i \neq j$; then A is isomorphic to the product algebra $A_{b_1} \times A_{b_2} \times \dots \times A_{b_r}$.*

PROOF. If x is in A and $1 \leq j \leq r$, set $x_j = x \wedge b_j$. It is well known that the map $x \longrightarrow (x_1, x_2, \dots, x_r)$ is a lattice isomorphism from A into $A_{b_1} \times \dots \times A_{b_r}$, and it is easy to check that it is also a Moisil algebra isomorphism.

6.3.COROLLARY. *Let $\{a_1, a_2, \dots, a_r\}$ be the set of atoms of the Boolean algebra B(A). Then A is isomorphic to the product algebra $A_{a_1} \times A_{a_2} \times \dots A_{a_r}$.*

Since the maximal deductive systems of A are just the principal filters generated by the atoms of B(A), we have the following improvement of theorem 5.7 (cf [13], p 132):

6.4.THEOREM. *Any finite Moisil algebra A of order n (with more than one element) is isomorphic to the direct product $A/M_1 \times \dots A/M_r$, where $\{M_1, M_2, \dots, M_r\}$ is the set of all maximal deductive systems of A, and r is* the number of atoms of $B(A)$.

We remark that the above theorem can also be obtained from theorem 5.7 (as was done in the author's doctoral thesis), but the above derivation has the advantage of avoiding transfinite induction.

7. FREE ALGEBRAS.

The aim of this section is to determine the structure of the Moisil algebras of order n with r free generators (r finite cardinal > 0). We shall apply a technique used by L.Monteiro ([21] , p 20) to determine the structure of the free three-valued Heyting algebras, and also used by A.Monteiro (unpublished) to determine the structure of the Moisil algebras of order 3 with a finite set of free generators. We begin by recalling the following:

7.1.DEFINITION. If c is a cardinal number > 0, then by a free Moisil algebra of order n with c free generators we mean any Moisil algebra $F_n(c)$ of order n such that: 1) $F_n(c)$ has a set of generators G of power c, and 2) any map f from G into a Moisil algebra A of order n can be extended to a homomorphism h_f from $F_n(c)$ into A.

Since the notion of Moisil algebra of order n is equationally definable, by a well known theorem of G. Birkhoff ([4],Chapter VI) it follows that:

<u>7.2.THEOREM. For any cardinal</u> $c > 0$ <u>there exists</u> $F_n(c)$ <u>and it is unique up to isomorphisms. Moreover, the homomorphism</u> h_f <u>that appears in definition 7.1 is unique.</u>

In particular, there is a one-to-one correspondence between the set of all functions from G into L_n and the set of all homomorphisms from $F_n(c)$ into L_n, and therefore by corollary 5.4 we obtain the following theorem, which allows us to apply the mentioned technique of L. Monteiro:

<u>7.3.THEOREM. If</u> G <u>is a set of free generators of</u> $F_n(c)$, <u>then the application that maps each function</u> $f:G \longrightarrow L_n$ <u>into the deductive system</u> $M_f = \mathrm{Ker}\, h_f$ <u>establishes a one-to-one correspondence between the set of all functions from</u> G <u>into</u> L_n <u>and the set of all maximal deductive systems of</u> $F_n(c)$.

<u>7.4.COROLLARY. If</u> r <u>is a finite cardinal</u> > 0, <u>then</u> $F_n(r)$ <u>is finite.</u>

<u>PROOF</u>. Let G be a set of free generators of $F_n(r)$. Since G has r elements and L_n n elements, there are n^r functions from G into L_n, and by the theorem, it follows that $F_n(r)$ has n^r maximal deductive systems. Therefore, from the proofs of theorem 3.13 and corollary 3.14 we obtain that $F_n(r)$ is isomorphic to a subdirect product of a finite family of subalgebras of L_n, so $F_n(r)$ is finite.

We shall need the following results:

7.5.LEMMA. *If* $h:A \longrightarrow A'$ *is an homomorphism and if* G *is a set of generators of* A, *then* $h(G)$ *is a set of generators of* $f(A)$.

7.6.THEOREM. *Let* G *be a set of free generators of* $F_n(c)$, $f:G \longrightarrow L_n$ *a function and* A *a subalgebra of* L_n. *In order that* $F_n(c)/M_f$ *be isomorphic to* A *it is necessary and sufficient that the following conditions hold: 1)* $f(G) \subseteq A$, *and 2)* $f(G) \not\subseteq A'$ *for any maximal proper subalgebra* A' *of* A.

From now on, we shall deal with free Moisil algebras of order n with r free generators, where r is a finite cardinal > 0. We shall denote by $G = \{g_1, g_2, \ldots, g_r\}$ a set of free generators of $F_n(r)$, and by $H(n,r)$ the set of all functions from G into L_n. Since $F_n(r)$ is finite, we can apply theorem 6.4 to obtain the following, where = indicates isomorphism:

$$(7.7) \qquad F_n(r) = \prod_{f \in H(n,r)} F_n(r)/M_f$$

Assume that n is an even number. If H_k denotes the set of all functions from G into L_n such that $F_n(r)/M_f$ is isomorphic to a 2k-element subalgebra of L_n (i.e. such that M_f is of order 2k) we have that:

$$H(n,r) = \bigcup_{n=1}^{n/2} H_k \qquad \text{and} \qquad H_k \cap H_{k'} = \emptyset \text{ if } k \neq k'.$$

Therefore, from (7.7) it follows that:

$$(7.8) \qquad F_n(r) = \prod_{k=1}^{n/2} \Big(\prod_{f \in H_k} F_n(r)/M_f \Big)$$

If A_{ki}, $i = 1,2,\ldots,s_k = \binom{n-2/2}{k-1}$ are the subalgebras of L_n with 2k elements (cf theorem 5.10), and $H_{ki} = \{ f \in H_k : F_n(r)/M_f = A_{ki} \}$, then it is clear that:

$$H_k = \bigcup_{i=1}^{s_k} H_{ki} \qquad \text{and} \qquad H_{ki} \cap H_{ki'} = \emptyset \text{ if } i \neq i'$$

Therefore, if N(X) denotes the number of elements of the finite set X, we have:

$$(7.9) \qquad N(H_k) = \sum_{i=1}^{s_k} N(H_{ki}) \quad \Big(s_k = \binom{n-2/2}{k-1}\Big)$$

We are going to calculate $N(H_{ki})$. To this end, we fix an i such that $1 \leq i \leq s_k$, and to simplify, we write A_k instead of A_{ki}. As in the proof of theorem 5.10 we can see that the number of subalgebras of A_k with 2k-2 elements (i.e. the number of proper maximal subalgebras of A_k) is $\binom{(2k-2)/2}{k-1} = k-1$. These subalgebras will be denoted by A'_{kj}, $j = 1,2,\ldots,k-1$.

Since the subalgebras A'_{kj} differ from A_k and among themselves in just two elements, the set theoretical intersection of h subalgebras A'_{kj} $(1 \leq h \leq k-1)$ is a (2k-2h)-element subalgebra of A_k.

Let $G_k^i = \{ f \in H(n,r) : f(G) \subseteq A_{ki} \}$, and $G_{kj}^i = \{ f \in H(n,r) : f(G) \subseteq A'_{kj} \}$. From theorem 7.6 it follows that:

$$H_{ki} = G_k^i - \bigcup_{j=1}^{k-1} G_{kj}^i$$

and therefore:

$$(7.10) \qquad N(H_{ki}) = N(G_k^i) - N(\bigcup_{j=1}^{k-1} G_{kj}^i)$$

Since we are supposing that i is fixed, we may write G_k and G_{kj} instead of G_k^i and G_{kj}^i. It is well known that:

$$N(\bigcup_{j=1}^{k-1} G_{kj}) = \sum_{i=1}^{k-1} (-1)^{i-1} \sum_{1 \leq j_1 < \ldots < j_i \leq k-1} N(G_{kj_1} \cap \ldots \cap G_{kj_i})$$

Since $G_{kj_1} \cap \ldots \cap G_{kj_i} = \{ f \in H(n,r) : f(G) \subseteq A_{kj_1} \cap \ldots \cap A_{kj_i} \}$ and since $A_{kj_1} \cap \ldots, \cap A_{kj_i}$ is a (2k-2i)-element subalgebra of A_k, it follows that $N(G_{kj_1} \cap \ldots \cap G_{kj_i}) = (2k-2i)^r$.

Therefore we have:

$$N(\bigcup_{j=1}^{k-1} G_{kj}) = 2^r \sum_{i=1}^{k-1} (-1)^{i-1} (k-i)^r$$

and from (7.10) we obtain:

$$(7.11) \quad N(H_{ki}) = 2^r \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} (k-j)^r = a(r,k)$$

Since $N(H_{ki})$ does not depend on i, from (7.9) it follows that:

$$(7.12) \qquad N(H_k) = \binom{(n-2)/2}{k-1} a(r,k)$$

In this way we have obtained the following:

<u>7.13.THEOREM</u>. <u>Let</u> n <u>be an even number and let</u> A_{kj}, $j = 1,2,\ldots,\binom{(n-2)/2}{k-1}$ <u>be the</u> 2k-<u>element subalgebras of</u> L_n, $k = 1,2,\ldots,n/2$. <u>If we set</u>:

$$a(r,k) = 2^r \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} (k-i)^r$$

then the following holds:

$$F_n(r) = \prod_{k=1}^{n/2} \Big(\prod_{j=1}^{\binom{(n-2)/2}{k-1}} A_{kj}^{a(r,k)} \Big)$$

where A^s denotes the product of s algebras isomorphic to A.

7.14.COROLLARY. If n is even, then:

$$N(F_n(r)) = \prod_{k=1}^{n/2} (2k)^{a(r,k) \binom{(n-2)/2}{k-1}}$$

Assume now that n is odd. We denote by A_s an s-element subalgebra of L_n, and by H_s the set of all functions from G into L_n such that M_f is of order s $(2 \leq s \leq n)$. As before, from (7.7) it follows that:

$$(7.15) \quad F_n(r) = \prod_{s=2}^{n} \Big(\prod_{f \in H_s} F_n(r)/M_f \Big)$$

If s is even, $s = 2k$ $(1 \leq k \leq (n-1)/2)$, there are (cf theorem 5.12) $\binom{(n-3)/2}{k-1}$ subalgebras of L_n with s elements. On the other hand, all the subalgebras of A_s have an even number of elements, and the number of subalgebras of A_s with $s-2 = 2k-2$ elements is $k-1$, and as before we can prove that:

$$(7.16) \quad N(H_{2k}) = \binom{(n-3)/2}{k-1} a(r,k)$$

If s is odd, $s = 2k+1$ $(1 \leq k \leq (n-1)/2)$, there are (theorem 5.12) $\binom{(n-3)/2}{k-1}$ subalgebras of L_n with s elements, which we denote by A_{si}, $i = 1,2,\ldots,t_s = \binom{(n-3)/2}{k-1}$.

If $H_{si} = \{ f \in H(n,r) : F_n(r)/M_f = A_{si} \}$, then we have:

$$(7.15) \qquad N(H_s) = \sum_{i=1}^{t_s} N(H_{si})$$

We fix an i such that $1 \leq i \leq t_s$, and we write A_k instead of $A_{si} = A_{(2k+1)i}$.

A_k has only one subalgebra with 2k elements (i.e. the subalgebra obtained by eliminating from A_k the only z such that $z = -z$), which we shall denote by A_{k0}. It is easy to check that the number of subalgebras of A_k with 2k-1 elements is $\binom{k-1}{k-2} = k-1$, and they will be denoted by A_{kj}, $j = 1,2,\ldots,k-1$. Therefore A_k contains k maximal proper subalgebras, namely, $A_{k0}, A_{k1}, \ldots, A_{k(k-1)}$.

Observe that if $1 \leq j_1 < \ldots < j_i \leq k-1$, $1 \leq i \leq k-1$, then the following formulae hold:

$$(7.18) \qquad N(A_{kj_1} \cap \ldots \cap A_{kj_i}) = 2k+1-2i$$

$$(7.19) \quad N(A_{k0} \cap A_{kj_1} \cap \ldots \cap A_{kj_i}) = 2k-2i$$

Set $G_k^i = \{ f \in H(n,r) : f(G) \subseteq A_k = A_{(2k+1)i} \}$ and

$G_{kj}^i = \{ f \in H(n,r) : f(G) \subseteq A_{kj} \}$, $j = 0,1,\ldots,k-1$.

From theorem 7.6 it follows that:

$$H_{si} = H_{(2k+1)i} = G_k^i - \bigcup_{j=0}^{k-1} G_{kj}^i$$

and hence we obtain:

$$(7.20) \qquad N(H_{(2k+1)i}) = N(G_k^i) - N(\bigcup_{j=0}^{k-1} G_{kj}^i)$$

To simplify the notation the index i will be omitted, because it was fixed. We have that:

$$N(\bigcup_{j=0}^{k-1} G_{kj}) = \sum_{i=1}^{k} (-1)^{i-1} \sum_{0 \le j_1 < \ldots < j_i \le k-1} N(G_{kj_1} \cap \ldots \cap G_{kj_i}) =$$

$$= \sum_{i=1}^{k-1} (-1)^{i-1} \sum_{1 \le j_1 < \ldots < j_i \le k-1} N(G_{kj_1} \cap \ldots \cap G_{kj_i}) + N(G_{k0}) +$$

$$+ \sum_{i=2}^{k} (-1)^{i-1} \sum_{1 \le j_1 < \ldots < j_i \le k-1} N(G_{k0}\ G_{kj_1} \cap \ldots \cap G_{kj_{i-1}})$$

and from (7.18) and (7.19) it follows that:

$$N(\bigcup_{j=0}^{k-1} G_{kj}) = \sum_{i=1}^{k-1} (-1)^{i-1} \binom{k-1}{i} (2k+1-2i)^r + (2k)^r +$$

$$+ \sum_{i=2}^{k-1} (-1)^{i-1} \binom{k-1}{i-1} (2k-2(i-1))^r =$$

$$= \sum_{i-1}^{k-1} (-1)^{i-1} \binom{k-1}{i} (2k+1-2i)^r + \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} (2k-2i)^r.$$

From (7.20) it then follows that:

$$N(H_{(2k+1)i}) = \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} ((2k+1-2i)^r - (2k-2i)^r)$$

Since $N(H_{(2k+1)i})$ does not depend on i, and since there are $\binom{(n-3)/2}{k-1}$ subalgebras with 2k+1 elements, from

(7.17) we obtain:

$$N(H_{(2k+1)}) = \binom{(n-3)/2}{k-1}\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}((2k+1-2i)^r-(2k-2i)^r).$$

So we have proven the following:

7.21.THEOREM. *Let* n *be an odd number and* $A_{(2k)j}$ *and* $A_{(2k+1)j}$, $j = 1,2,\dots,\binom{(n-3)/2}{k-1}$ *be the subalgebras of* L_n *with* $2k$ *and* $2k+1$ *elements respectively*, $k = 1,2,\dots$ $\dots, (n-1)/2$. *If we set*:

$$a(r,k) = \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i}(2k-2i)^r$$

$$b(r,k) = \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i}(2k+1-2i)^r$$

$$c(r,k) = b(r,k) - a(r,k)$$

then the following holds:

$$F_n(r) = \prod_{k=1}^{(n-1)/2} \left(\prod_{j=1}^{\binom{(n-3)/2}{k-1}} (A_{(2k)j}^{a(r,k)} \times A_{(2k+1)j}^{c(r,k)}) \right)$$

7.22.COROLLARY. *If* n *is odd, then*:

$$N(F_n(r)) = \prod_{k=1}^{(n-1)/2} (2k)^{\binom{(n-3)/2}{k-1}a(r,k)} (2k+1)^{\binom{(n-3)/2}{k-1}c(r,k)}$$

We remark that in case $n = 2$, theorem 7.13 reduces to the familiar formula for the free Boolean algebra with r generators, and when $n = 3$, theorem 7.21 gives a formula obtained earlier by A.Monteiro for the free three-valued Lukasiewicz algebra with r generators (unpublished).

8. POST ALGEBRAS.

The notion of Post algebra of order n was introduced by P.C.Rosenbloon [23] and developed later by G.Epstein [7], T.Traczyk [29], [30], [31] and P.Dwinger [6]. The following definition is due to Traczyk [29] :

8.1.DEFINITION. A Post algebra of order n (n an integer $\geqslant 2$) is a system $\langle A,0,1,\vee,\wedge,e_1,\ldots,e_{n-2}\rangle$ such that $\langle A,0,1,\vee,\wedge\rangle$ is a distributive lattice with zero 0 and unit 1, and $e_1,\ldots,e_{n-2}$ are n-2 elements of A that fulfill the conditions:

P 1) $0 = e_0 \leq e_1 \leq \ldots \leq e_{n-2} \leq e_{n-1} = 1$

P 2) For any x in A, there are elements $b_1,\ldots,b_{n-1}$ belonging to B(A) such that:

$$x = (b_1 \wedge e_1) \vee (b_2 \wedge e_2) \vee \ldots \vee b_{n-1}$$

P 3) If $b \in B(A)$ and $b \wedge e_j \leq e_{j-1}$ for some j $(1 \leq j \leq n-1)$, then $b = 0$.

We shall denote simply by A a Post algebra of order n.

It is well known ([7], [29]) that in a Post algebra of order n any element x admits a unique representation in the form:

$$(8.2) \quad x = (d_1 \wedge e_1) \vee \ldots \vee (d_{n-2} \wedge e_{n-2}) \vee d_{n-1}$$

where $d_i \in B(A)$ and $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_{n-1}$.

We denote by $D_1(x), D_2(x), \ldots, D_{n-1}(x)$ the uniquely determined Boolean coefficients of the representation (8.2) of x.

Epstein [7] proved that the operator:

$$(8.3) \qquad \beta(x) = \bigvee_{i=1}^{n-1} (e_i \wedge (D_{n-i}(x))')$$

where b' denotes the Boolean complement of an element $b \in B(A)$, satisfies M 1) and M 2), so any Post algebra is a De Morgan algebra if we define $-x = \beta(x)$.

Moreover, if we define the operators:

$$(8.4) \qquad s_i x = D_{n-i}(x) \quad (1 \leq i \leq n-1)$$

then the s_i satisfy L 1) - L 6) ([7], [29]), and we have also:

$$(8.5) \qquad s_i e_j = D_{n-i}(e_j) = \begin{cases} 0 \text{ if } i+j < n \\ 1 \text{ if } i+j \geq n \end{cases}$$

Therefore any Post algebra of order n is a Moisil algebra of the same order. The following theorem establishes the precise relation between Post and Moisil algebras:

<u>8.6.THEOREM.</u> A <u>is a Post algebra of order n if and only if: 1)</u> A <u>is a Moisil algebra of order n, and 2)</u> A <u>has</u> n-2 <u>elements</u> $e_1, \ldots, e_{n-2}$ <u>that satisfy the condition (8.5)</u>.

<u>PROOF.</u> From the above remarks it follows that any Post algebra of order n fulfills conditions 1) and 2).

Conversely, suppose that A is a Moisil algebra of order n with elements $e_1,\dots,e_{n-2}$ satisfying (8.5). We set $e_0=0$ and $e_{n-1}=1$. From (8.5) and L 9) it follows that $e_0 < e_1 < \dots < e_{n-2} < e_{n-1}$, and using L 6), that any x in A cab be written in the form:

$$x = (s_{n-1}x \wedge e_1) \vee \dots \vee (s_2 \wedge e_{n-2}) \vee s_1x$$

Finally, if $b \in B(A)$ and $b \wedge e_j \leq e_{j-1}$, then $b \wedge s_{n-j}e_j = s_{n-j}e_{j-1} = 0$, so $b \leq -s_{n-j}e_j = -1 = 0$.

8.7.COROLLARY. Let A,A' be Moisil algebras of order n and $h:A \longrightarrow A'$ an homomorphism. If A is a Post algebra of order n, then A' is also a Post algebra of order n.

If A is a Post algebra of order n and if U is an ultrafilter of B(A), from 2) of theorem 8.6 it follows that $e_{n-j} \in U_j$ and $e_{n-j} \notin U_{j-1}$ ($j = 1,2,\dots,n-1$). Therefore $U_{j-1} \neq U_j$, and from corollary 4.8 we obtain the following theorem, that has been proven earlier by Epstein [7] and Traczyk [29]:

8.8.THEOREM. Any prime filter of a Post algebra of order n belongs to one and only one maximal chain of prime filters, and this chain has exactly n-1 elements.

The following question arises naturally:

8.9.PROBLEM. If all the maximal chains of prime filters of a Moisil algebra A of order n have exactly n-1 elements, is then A a Post algebra of order n ?

We remark that L_n is a Post algebra of order n, and that L_n does not have proper Post subalgebras. Using the same technique as in the case of Moisil algebras, we can prove the following theorem, that does not seem to be in the literature:

8.10. THEOREM. *Let* r *be a finite cardinal* > 0. *Then the free Post algebra of order* n *with* r *free generators is the direct product of* n^r *copies of* L_n, *and in particular, it has* n^{n^r} *elements*.

We have obtained some other results concerning Post and Moisil algebras, for example, that *the injective Moisil algebras of order n are just the complete Post algebras of the same order n*. The details will be published elsewhere.

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REFERENCES.

1. A.Bialynicki-Birula, Remarks on Quasi-Boolean algebras, Bull.Acad.Pol.Sci.,Cl.III, 5 (1957), 615 - 619.

2. A.Bialynicki-Birula and H.Rasiowa, On the representation of Quasi-Boolean algebras, Bull.Acad.Pol. Sci.,Cl.III, 5 (1957), 259 - 261.

3. A.Bialynicki-Birula and H.Rasiowa, On constructible falsity in the constructive logic with strong negation, Colloq.Math., 6 (1958), 287 - 310.

4. G.Birkhoff, Lattice theory, Amer.Math.Soc.Colloq.Pub. 25, 3rd ed., Providence, 1967.

5. R.Cignoli, Boolean multiplicative closures, I & II, Proc.Japan Acad.,42 (1965), 1168 - 1174.

6. P.Dwinger, Notes on Post algebras, I & II, Indag.Math., 28 (1966), 462 - 478.

7. G.Epstein, The lattice theory of Post algebras, Trans. Amer.Math.Soc., 95 (1960), 300 - 317.

8. J.Kalman, Lattices with involution, Trans.Amer.Math. Soc., 87 (1958), 485 - 491.

9. J.Lukasiewicz and A.Tarski, Untersuchungen über den Aussagenkalkül, C.R.Acad.Sci.Lett.Vars., 23 (1930) 30 - 50 (The English translation of this paper is the Chapter IV of [28]).

10. G.C.Moisil, Recherches sur l'algèbre de la logique, Ann.Sci.Univ.Jassy, 22 (1935), 1 - 117.

11. G.C.Moisil, Recherches sur les logiques non-chrys-sippiennes, Ann.Sci.Univ.Jassy, 26 (1940), 431-436.

12. G.C.Moisil, Notes sur les logiques non-chryssippiennes, Ann.Sci.Univ.Jassy, 27 (1941), 86 - 98.

13. G.C.Moisil, Le algebre di Lukasiewicz, Acta Logica (Bucharest), 6 (1963), 97 - 135.

14. G.C.Moisil, Sur les logiques de Lukasiewicz a un nombre fini de valeurs, Rev.Roum.Math.Pures et Appl., 9 (1964), 905 - 920.

15. G.C.Moisil, Incercări vechi şi noi de logică neclasică, Edit.Ştiinţifică, Bucureşti,1965.

16. A.Monteiro, L'arithmétique des filtres et les espaces topologiques, Segundo Symposium Americano de Mat., Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, 1954, 129 - 162.

17. A.Monteiro, Matrices de Morgan caractéristiques pour le calcul propositionnel classique, An.Acad.Bras.Ci., 32 (1960), 1 - 7.

18. A.Monteiro, Algebra de la lógica III, lectures given at the Univ.Nac.del Sur, Bahía Blanca, 1962.

19. A.Monteiro, Algebras de Lukasiewicz trivalentes, lectures given at the Univ.Nac.del Sur,Bahía Blanca, 1963.

20. A.Monteiro, Construction des algèbres de Lukasiewicz

trivalentes dans les algèbres de Boole monadiques, Math.Japon., 12 (1967), 1 - 23.

21. L.Monteiro, Sur les algèbres de Heyting trivalentes, Notas de Lógica Matemática 19, Univ.Nac.del Sur, Bahía Blanca, 1964.

22. E.L.Post, Introduction to a general theory of elementary propositions, Amer.J.Math., 43(1921),163-185.

23. P.C.Rosenbloom, Post algebras.I.Postulates and general theory, Amer.J.Math., 64 (1942), 167 - 188.

24. C.O.Sicoe, Sur les idéaux des algèbres Lukasiewicziennes polivalentes, Rev.Roum.Math.Pures et Appl., 12 (1967), 391 - 401.

25. C.O.Sicoe, Note supra algebrelor Lukasiewicziene polivalente, Stud.si Cerc.Mat., 19 (1967), 1203-1207.

26. C.O.Sicoe, On many-valued Lukasiewicz algebras,Proc. Japan Acad., 43 (1967), 725 - 728.

27. C.O.Sicoe, Sur la définition des algèbres Lukasiewicziennes polyvalentes, Rev.Roum.Math.Pures et Appl., 13 (1968), 1027 - 1030.

28. A.Tarski, Logic,Semantics,Metamathematics,Clarendon Press, Oxford, 1956.

29. T.Traczyk, Axioms and some properties of Post algebras, Colloq.Math., 10 (1963), 193 - 209.

30. T.Traczyk, An equational definition of a class of

Post algebras, Bull.Acad.Pol.Sci.Séri sci.math.,astr. et phys., 12 (1964), 147 - 149.

31. T.Traczyk, _A generalization of the Loomis-Sikorski theorem_, Colloq.Math., 12 (1964), 155 - 161.