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THREE-VALUED ŁUKASIEWICZ ALGEBRAS

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Preface to this English translation

This is an edition of lecture notes for a course on three-valued Łukasiewicz algebras given by Dr. Antonio A. R. Monteiro in 1963 in the Universidad Nacional del Sur, translated into English for the first time for the series *Notas de Lógica Matemática* of the Instituto de Matemática de Bahía Blanca (INMABB, CONICET-UNS).

This translation is a group effort. We thank in particular Andrés Gallardo for his careful checking of the proofs and many improvements to the text. Luiz Monteiro supervised and approved this edition. We also thank Rosana Entizne and Fernando Gómez for their help.

Ignacio Viglizzo, translator
Bahía Blanca
December 2018

Preface

These lecture notes follow the course on three-valued Łukasiewicz algebras [36] *given by Dr. Antonio A. R. Monteiro in 1963 in the Universidad Nacional del Sur, as well as the seminars* [37], [44] *about this topic, where he presented original results.*

The first parts of the course included some background on partially ordered sets, distributive lattices, De Morgan algebras, and monadic boolean algebras, needed to follow the later parts. This background material can be found in the publications [66], [51], and [50].

Dr. Antonio Monteiro usually posed problems during his lectures, and some of them led to the following works, among others:

- *The doctoral dissertation of Roberto Cignoli,* Álgebras de Moisil de orden n *(1969)* [12], [13]*, which presents important results on n-valued Moisil algebras, which have three-valued Łukasiewicz algebras as a particular case. In the bibliography we include 15 works by this author.*
- *The doctoral dissertation of Luiz Monteiro,* Álgebras de Łukasiewicz trivalentes monádicas *(1971)* [61], [62]*, which generalizes the concept of three-valued Łukasiewicz algebras.*
- *The doctoral dissertation of Manuel Abad,* Estructuras cíclica y monádica de un álgebra de Łukasiewicz n-valente *(1986)* [36]*, which generalizes the concept of monadic three-valued Łukasiewicz algebras.*
- *The doctoral dissertations of Luisa Iturrioz* [23] *and Aldo V. Figallo* [18].

The book Łukasiewicz-Moisil algebras*, by V. Boicescu, A. Filipoiu, G. Georgescu, and S. Rudeanu* [9] *studies more general structures. Published in 1991, it compiles bibliography (up to 1989) related to three-valued Łukasiewicz algebras. In our bibliography we include more works, some of them published after 1989, undoubtedly originated in the topics developed in the course and seminars cited above, and also in the work meetings of Professor A. Monteiro with his disciples. One of these disciples was also his son, Luiz Monteiro, who took upon the task of editing all this material to make it widely available. In doing so, he added many results of his own, so when preparing the translation for this English edition, it was apparent that Luiz should be fully credited as author as well.*

To this day, the results by A. Monteiro presented in this lecture notes continue to be cited in the literature.

Contents

CHAPTER 1

Basic definitions and constructions

1.1. Definition and calculation rules

The concept of three-valued Łukasiewicz algebra, was introduced and developed by Gr. Moisil [25], [26], [30].

These algebras play to the Łukasiewicz three-valued propositional calculus, an analogue role to that of boolean algebras to the classical propositional calculus.

The following definition [1] ([36], [39]) is equivalent to those indicated by Moisil.

Definition 1.1.1. *A three-valued Łukasiewicz algebra is a system* $(L, 1, \sim, \nabla, \vee, \wedge)$ *formed by 1) a non-empty set* L*; 2) an element* $1 \in L$*; 3) two unary operations* $\sim$ *and* ∇ *defined on* L*; 4) two binary operations* $\vee$ *and* $\wedge$*, defined on* L *so that the following conditions are verified:*

L1) $1 \vee x = 1$, *for all* $x \in L$,
L2) $x \wedge (x \vee y) = x$, *for all* $x, y \in L$,
L3) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$, *for all* $x, y, z \in L$,
L4) $\sim\sim x = x$, *for all* $x \in L$,
L5) $\sim (x \wedge y) = \sim x \vee \sim y$, *for all* $x, y \in L$,
L6) $\sim x \vee \nabla x = 1$, *for all* $x \in L$,
L7) $\sim x \wedge x = \sim x \wedge \nabla x$, *for all* $x \in L$,
L8) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$, *for all* $x, y \in L$.

The operation $\sim$ *is denominated* negation *and* ∇ *is denominated the* possibility *operation. We will also say that* L *is a Łukasiewicz algebra.*

One of the first problems posed by Professor Antonio Monteiro in his class was to determine whether these axioms were independent. This was solved by L. Monteiro [54], who proved that axioms L2) to L8) are independent and L1) is a consequence of some of the axioms L2) to L8). One of the examples indicated by L. Monteiro in [54] led A. Monteiro to introduce in 1978 the notion of four-valued modal algebra. He then posed to I. Loureiro the task of developing the theory of these algebras, which she did in her doctoral dissertation [24], defended in Lisbon in 1983.

To establish the equivalence of the definition above with those indicated by Moisil we must put $\sim x = Nx$ and $\nabla x = \mu x$.

Another set of axioms for Łukasiewicz algebra was indicated by A. Monteiro and L. Monteiro before 1967, but only published in 1996, [48].

Axioms L1), L2) and L3) are the ones posed by M. Sholander [75] so the system $(L, 1, \wedge, \vee)$ is a distributive lattice with a top element 1. From axioms L1), L2),

[1] *The goal of A. Monteiro in presenting this definition was to use less axioms than those present in the different definitions by Moisil, while keeping the same operations as primitive.*

L3), L4) and L5) it follows that the system $(L, 1, \sim, \wedge, \vee)$ is a De Morgan algebra [6], [7], [51] and therefore $0 =\sim 1$ is the bottom element of the lattice L.

Therefore we can define a Łukasiewicz algebra as a De Morgan algebra on which an unary operation verifying L6), L7) and L8) is defined.

We shall assume known and use freely the calculation rules valid in De Morgan algebras. We now indicate some calculation rules involving the operator ∇.

L9) $x \leq \nabla x$.
$x = x \wedge 1 =$ (by L6)) $= x \wedge (\sim x \vee \nabla x) = (x \wedge \sim x) \vee (x \wedge \nabla x) =$ (by L7)) $= (\sim x \wedge \nabla x) \vee (x \wedge \nabla x) = (\sim x \vee x) \wedge \nabla x \leq \nabla x$.

L10) $\nabla 1 = 1$.
Immediate from L9).

L11) $\nabla 0 = 0$.
$0 = 0 \wedge 1 = 0 \wedge \sim 0 =$ (by L7)) $=\sim 0 \wedge \nabla 0 = 1 \wedge \nabla 0 = \nabla 0$.

L12) If $x \leq y$ then $\nabla x \leq \nabla y$.
We know that $x \leq y \iff x = x \wedge y$, so $\nabla x = \nabla(x \wedge y) =$ (by L8)) $= \nabla x \wedge \nabla y$ and this is equivalent to $\nabla x \leq \nabla y$.

L13) $x \vee \nabla \sim x = 1$.
$x \vee \nabla \sim x =$ (by L4)) $=\sim (\sim x) \vee \nabla \sim x =$ (by L6)) $= 1$.

L14) $\nabla \sim \nabla \sim x \leq x$.
From L13) it follows that $\sim (x \vee \nabla \sim x) =\sim 1$, this is $\sim x \wedge \sim \nabla \sim x = 0$. Then $\nabla(\sim x \wedge \sim \nabla \sim x) = \nabla 0 =$ (by L11)) $= 0$, and using L8) we obtain (1) $\nabla \sim x \wedge \nabla \sim \nabla \sim x = 0$.
Finally $x = x \vee 0 =$ (by (1)) $= x \vee (\nabla \sim x \wedge \nabla \sim \nabla \sim x) = (x \vee \nabla \sim x) \wedge (x \vee \nabla \sim \nabla \sim x) =$ (by L13)) $= 1 \wedge (x \vee \nabla \sim \nabla \sim x) = x \vee \nabla \sim \nabla \sim x$ from which L14) follows.

L15) $\nabla \sim \nabla \sim \nabla \sim x = \nabla \sim x$.
From L14), $\sim x \leq\sim \nabla \sim \nabla \sim x$, so by L12) we get:
(1) $\nabla \sim x \leq \nabla \sim \nabla \sim \nabla \sim x$. On the other hand, replacing in L14) x by $\nabla \sim x$ we get:
(2) $\nabla \sim \nabla \sim \nabla \sim x \leq \nabla \sim x$. From (1) and (2), L15) follows.

L16) $\nabla \sim \nabla x$ is the boolean complement of ∇x.
If in L13) we replace x by ∇x, we get (1) $\nabla x \vee \nabla \sim \nabla x = 1$. From L6) it follows that $\sim (\sim x \vee \nabla x) =\sim 1 = 0$, this is $x \wedge \sim \nabla x = 0$ and therefore (2) $0 = \nabla 0 = \nabla(x \wedge \sim \nabla x) =$ (by L8) $= \nabla x \wedge \nabla \sim \nabla x$. From (1) and (2), L16) obtains.

L17) $\sim \nabla x$ is the boolean complement of ∇x.
From $x \leq \nabla x$, replacing x by $\sim \nabla x$ we get $\sim \nabla x \leq \nabla \sim \nabla x$, so $\nabla x \wedge \sim \nabla x \leq \nabla x \wedge \nabla \sim \nabla x =$ (by L16) $= 0$, and therefore (1) $\nabla x \wedge \sim \nabla x = 0$, so $\sim (\nabla x \wedge \sim \nabla x) =\sim 0 = 1$, this is (2) $\nabla x \vee \sim \nabla x = 1$. From (1) and (2), L17) obtains.

L18) $\nabla \sim \nabla x =\sim \nabla x$.
This is an immediate consequence of L16) and L17), since in a distributive lattice if an element has a boolean complement, it is a unique one.

L19) $\nabla \sim \nabla \sim x =\sim \nabla \sim x$.
Follows from L18) replacing x by $\sim x$.

L20) $\nabla\nabla x = \nabla x$.

From L18) it follows that $\sim \nabla \sim \nabla x = \sim\sim \nabla x = \nabla x$, and then (1) $\nabla \sim \nabla \sim \nabla x = \nabla\nabla x$. If we replace x by $\sim x$ in L15) we obtain (2) $\nabla \sim \nabla \sim \nabla x = \nabla x$. From (1) and (2), L20) follows.

An element $x \in L$ of a Łukasiewicz algebra L is called a *constant* or *invariant* if $\nabla x = x$. We shall represent by $B(L)$ or just B the set of all the invariant elements of L. By L20) it follows that $\nabla x \in B(L)$ for all $x \in L$, so in particular B0) $0, 1 \in B(L)$. The set $B(L)$ has also the following properties:

B1) If $x, y \in B(L)$ then $x \wedge y \in B(L)$.

Indeed, by hypothesis $\nabla x = x, \nabla y = y$ so $\nabla(x \wedge y) = (\text{byL8})) = \nabla x \wedge \nabla y = x \wedge y$.

B2) If $x \in B(L)$ then $\sim x \in B(L)$.

From L18) we know that $\nabla \sim \nabla x = \sim \nabla x$, so if $\nabla x = x$, we have that $\nabla \sim x = \sim x$.

B3) If $x, y \in B(L)$ then $x \vee y \in B(L)$.

Follows immediately from B1), B2) and axiom L5).

Lemma 1.1.2. (Gr. [25])*If L is an Łukasiewicz algebra and $x \in L$ then x is invariant if and only if x is a boolean element.*

Proof. It is clear that 0 and 1 are boolean elements of L.

Let x be a boolean of L and denote with $-x$ its boolean complement, so $x \wedge -x = 0$ and $x \vee -x = 1$. By $L8$) we have (1) $\nabla x \wedge \nabla - x = 0$ and by $L9$) $1 = x \vee -x \leq \nabla x \vee \nabla - x$ so (2) $\nabla x \vee \nabla - x = 1$. Therefore ∇x is a boolean element with complement $\nabla - x$. On the other hand, (3) $\nabla x = \nabla x \wedge 1 = \nabla x \wedge (x \vee -x) = (\nabla x \wedge x) \vee (\nabla x \wedge -x) = x \vee (\nabla x \wedge -x)$. From $-x \leq \nabla - x$ it follows, using (1) that $-x \wedge \nabla x \leq \nabla - x \wedge \nabla x = 0$ and therefore (4) $-x \wedge \nabla x = 0$. From (3) and (4) it follows that $\nabla x = x$.

Assume now that $x \in B(L)$, so $\nabla x = x$. By L17) we know that ∇x is a boolean element with complement $\sim \nabla x$, so since $\sim \nabla x = \sim x$ we can derive $\sim x \wedge x = 0$ and $\sim x \vee x = 1$, then x is a boolean element with complement $\sim x$. □

We have proven that $B(L)$ is a boolean algebra.

L21) $\nabla(x \vee y) = \nabla x \vee \nabla y$.

From $x \leq x \vee y$ it follows by L12) that (1) $\nabla x \leq \nabla(x \vee y)$ and from $y \leq x \vee y$ it follows by L12) that (2) $\nabla y \leq \nabla(x \vee y)$. From (1) and (2) we have:

$$(3) \quad \nabla x \vee \nabla y \leq \nabla(x \vee y).$$

By L9) we know that $x \leq \nabla x$ and $y \leq \nabla y$, thus $x \vee y \leq \nabla x \vee \nabla y$ and so using L12) we have $\nabla(x \vee y) \leq \nabla(\nabla x \vee \nabla y)$, and since $\nabla x, \nabla y \in B(L)$ it follows by B3) that $\nabla x \vee \nabla y \in B(L)$, this is $\nabla(\nabla x \vee \nabla y) = \nabla x \vee \nabla y$, so (4) $\nabla(x \vee y) \leq \nabla x \vee \nabla y$. From (3) and (4), L21) follows.

L22) $\sim \nabla \sim x \vee (\nabla x \wedge \nabla \sim x) \vee \sim \nabla x = 1$.

$\sim \nabla \sim x \vee (\nabla x \wedge \nabla \sim x) \vee \sim \nabla x =$

$(\sim \nabla \sim x \vee \sim \nabla x) \vee (\nabla x \wedge \nabla \sim x) =$

$(\sim\nabla\sim x\vee\sim\nabla x\vee\nabla x)\wedge(\sim\nabla\sim x\vee\sim\nabla x\vee\nabla\sim x)=$ (by L17)) $=$ $1\wedge 1=1$.

L23) $\nabla(x\wedge\nabla y)=\nabla x\wedge\nabla y$.
$\nabla(x\wedge\nabla y)=$ (by L8)) $=\nabla x\wedge\nabla\nabla y=$ (by L20)) $=\nabla x\wedge\nabla y$.

We define a new unary operator on L as follows:

$$\Delta x=\sim\nabla\sim x$$

and we call it dual operator of ∇ or *necessity* operator. This terminology is justified by the following calculation rules:

L6$'$) $\sim x\wedge\Delta x=0$.
$\sim x\wedge\Delta x=\sim x\wedge\sim\nabla\sim x=\sim(x\vee\nabla\sim x)=$ (by L6)) $=\sim 1=0$.

L7$'$) $x\vee\sim x=\sim x\vee\Delta x$.
$\sim x\vee\Delta x=\sim x\vee\sim\nabla\sim x=\sim(x\wedge\nabla\sim x)=$ (by L7)) $=$ $\sim(x\wedge\sim x)=$ (by L5) and L4)) $=\sim x\vee x$.

L8$'$) $\Delta(x\vee y)=\Delta x\vee\Delta y$, $\Delta(x\wedge y)=\Delta x\wedge\Delta y$.
$\Delta(x\vee y)=\sim\nabla\sim(x\vee y)=$ (by L5)) $=\sim\nabla(\sim x\wedge\sim y)=$ (by L8)) $=$ $\sim(\nabla\sim x\wedge\nabla\sim y)=\sim\nabla\sim x\vee\sim\nabla\sim y=\Delta x\vee\Delta y$.
By applying the negation, L5) and L21), the other equality holds.

L9$'$) $\Delta x\leq x$.
By L9) we know that $\sim x\leq\nabla\sim x$, so $\Delta x=\sim\nabla\sim x\leq\sim\sim x=x$.

We can also prove promptly:

L24) $\nabla\Delta x=\Delta x$ and $\Delta\nabla x=\nabla x$.
Follows from L18).

Clearly every boolean algebra is a Łukasiewicz algebra in which for all x, $\nabla x=x$ and $\sim x=-x$.

Given a property P valid in all Łukasiewicz algebra, we call dual of P the property P' obtained by interchanging the elements 0 and 1 and the operations $\nabla,\vee,\wedge$ by $\Delta,\wedge,\vee$ respectively. We know that the duals of each of the axioms L1) to L8) used to define Łukasiewicz algebra are also valid in these algebras, so we can state the following result:

"If a property P is valid in a Łukasiewicz algebra then the dual property P' is also valid".

Since the ∇ operator has the properties of a closure operator (because of $L9$), $L12$) and $L20$)) it is natural to define the following operators:

$$Ext\ x=\sim\nabla x, Int\ x=\Delta x,$$

$$\partial x=\nabla x\wedge\nabla\sim x=\nabla(x\wedge\sim x),$$

called *exterior interior* and *frontier* respectively. The calculation rule L22) can now be written as:

$$\text{L22)}\qquad Int\ x\vee\partial x\vee Ext\ x=1.$$

This formula was called by Moisil the *principle of the excluded fourth.* Notice that:

- $Int\ x\wedge\partial x=\Delta x\wedge\nabla x\wedge\nabla\sim x=\nabla x\wedge\sim\nabla\sim x\wedge\nabla\sim x=$ (by L17)) $=$ $\nabla x\wedge 0=0$.

- $Int\ x \wedge Ext\ x = \Delta x \wedge \sim \nabla x = \Delta x \wedge \Delta \sim x =$ (by L8′)) $= \Delta(x \wedge \sim x) =$ $\sim \nabla \sim (x \wedge \sim x) =\sim \nabla(\sim x \vee x) =\sim (\nabla \sim x \vee \nabla x) =$ $\sim (\nabla(\sim x \vee \nabla x)) =$ (by L6)) $=\sim \nabla 1 =$ (by L10)) $=\sim 1 = 0$.
- $\partial x \wedge Ext\ x = \nabla x \wedge \nabla \sim x \wedge \sim \nabla x =$ (by L17)) $= 0$.

Lemma 1.1.3. *If (1) $x \wedge \sim x = 0$ then a) $\partial x = 0$, b) $\nabla x = x$, c) $Int\ x = x$ and d) $Ext\ x =\sim x$.*

Proof. a) $\partial x = \nabla(x \wedge \sim x) = \nabla 0 = 0$.

From (1) it follows that (2) $x \vee \sim x = 1$, so $\nabla x = \nabla x \wedge 1 = \nabla x \wedge (x \vee \sim x) =$ $(\nabla x \wedge x) \vee (\nabla x \wedge \sim x) =$ (by L9) and L7)) $= x \vee (x \wedge \sim x) = x$, and therefore $Ext\ x =\sim \nabla x =\sim x$. We already saw that (3) $0 = \partial x = \nabla x \wedge \nabla \sim x$, then $\nabla \sim x = \nabla \sim x \wedge 1 =$(by (2)) $= \nabla \sim x \wedge (x \vee \sim x) = (\nabla \sim x \wedge x) \vee (\nabla \sim x \wedge \sim x) =$ (by (3) and L9)) $= 0 \vee \sim x =\sim x$, so $Int\ x = \Delta x =\sim \nabla \sim x =\sim\sim x = x$. □

1.2. Centered Łukasiewicz algebras

An element c of a Łukasiewicz algebra L is called a *center* of L, if $\sim c = c$ (Moisil, [25]). This notion coincides with the corresponding one for order 3 Post algebras. For more on this see G. Epstein, *The lattice theory of Post algebras*, Trans. Amer. Math. Soc., 95 (1960), 300-317, and T. Traczyk, *Axioms and some properties of Post algebras*, Colloq. Math., 10 (1963), 193-209.

In case a Łukasiewicz algebra has a center, it will be called a centered Łukasiewicz algebra, or a Łukasiewicz algebra with center.

Lemma 1.2.1. *For an element c in a Łukasiewicz algebra L to be a center of L, it is necessary and sufficient that $\Delta c = 0$ and $\nabla c = 1$.*

Proof. Assume that $\sim c = c$. Then by axiom L6) and property L9) we get

$$1 =\sim c \vee \nabla c = c \vee \nabla c = \nabla c.$$

By L6’) and L9’) we have

$$0 =\sim c \wedge \Delta c = c \wedge \Delta c = \Delta c = 0.$$

Assume now $\Delta c = 0$ and $\nabla c = 1$. By L7 we get $c \wedge \sim c =\sim c \wedge \nabla c =$ $\sim c \wedge 1 =\sim c$, so $\sim c \leq c$. By L7’), $c \vee \sim c =\sim c \vee \Delta c =\sim c \vee 0 =\sim c$, this is $c \leq\sim c$. □

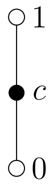
Lemma 1.2.2. *If a Lukasiewicz algebra L has a center, it is unique.*

Proof. Assume that c_1 and c_2 are centers of L this is $\nabla c_1 = \nabla c_2 = 1$ and $\Delta c_1 = \Delta c_2 = 0$. Then $\nabla(c_1 \wedge c_2) = \nabla c_1 \wedge \nabla c_2 = 1 \wedge 1 = 1$ and $\Delta(c_1 \wedge c_2) =$ $\Delta c_1 \wedge \Delta c_2 = 0 \wedge 0 = 0$. Consequently $c = c_1 \wedge c_2$ is a center of L so, by Lemma 1.2.1 $c =\sim c$ this is $c_1 \wedge c_2 =\sim (c_1 \wedge c_2) =\sim c_1 \vee \sim c_2 = c_1 \vee c_2$ and therefore $c_1 = c_2$. □

Note that to prove the preceding lemma, G. Moisil used Lemma 1.2.1 and the determination principle which we present next. Further on, (Lemma 1.4.4) we will indicate a different proof for Lemma 1.2.2.

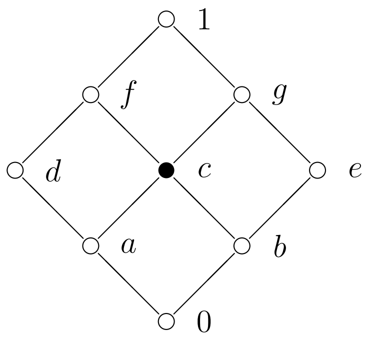
Let us see some examples of centered Łukasiewicz algebras.

Example 1.2.3. *Let* $\mathbf{T} = \{0, c, 1\}$ *be a partially ordered set (from now on, poset) with* $0 < c < 1$*, so since* $\mathbf{T}$ *is a finite chain,* $\mathbf{T}$ *is a bounded distributive lattice. The operations* $\sim, \nabla, \Delta$ *are defined by the following tables:*



x	$\sim x$	∇x	Δx
0	1	0	0
c	c	1	0
1	0	1	1

Example 1.2.4. *Consider the distributive lattice* L *with Hasse diagram indicated below in which the operations* $\sim, \nabla, \Delta$ *are given by the following table:*



x	$\sim x$	∇x	Δx
0	1	0	0
a	g	d	0
b	f	e	0
c	c	1	0
d	e	d	d
e	d	e	e
f	b	1	d
g	a	1	e
1	0	1	1

Lemma 1.2.5. *If* L *is a Łukasiewicz algebra with center* c *and* $b \in B(L)$ *then:*

$$c \wedge b = 0 \quad \text{if and only if} \quad b = 0.$$

PROOF. The condition is obviously sufficient. If $c \wedge b = 0$ then, since b is boolean:

$$b = 1 \wedge b = \nabla c \wedge \nabla b = \nabla(c \wedge b) = 0.$$

□

1.3. Axled Łukasiewicz algebras

An element e of a Łukasiewicz algebra L is called an *axis* of L ([27], p.88) if:

E1) $\Delta e = 0$,

E2) $\nabla x \leq \Delta x \vee \nabla e$, for all $x \in L$.

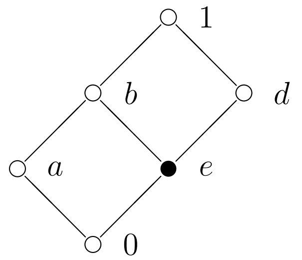
In case a Łukasiewicz algebra has an axis, it will be called an axled Łukasiewicz algebra, or a Łukasiewicz algebra with axis.

Note that condition E2) is equivalent to any of the two following ones ([62]):

E3) $\nabla x = \nabla x \wedge (\Delta x \vee \nabla e)$,

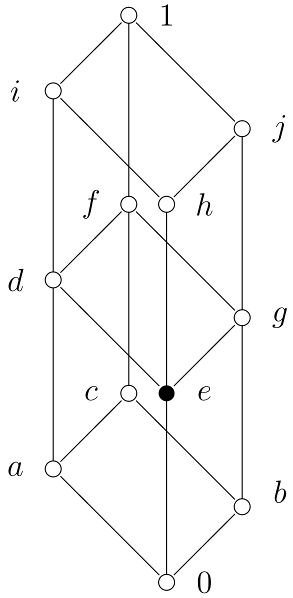
E4) $\nabla x \vee \nabla e = \Delta x \vee \nabla e$.

Example 1.3.1. *Consider the distributive lattice* A *with Hasse diagram indicated below and in which the operators* $\sim, \nabla, \Delta$ *are defined by the following table:*



x	$\sim x$	∇x	Δx	$\Delta x \vee \nabla e$
0	1	0	0	d
a	d	a	a	1
b	e	1	a	1
d	a	d	d	d
e	b	d	0	d
1	0	1	1	1

Example 1.3.2. *Consider the distributive lattice with Hasse diagram indicated below with the operators $\sim, \nabla, \Delta$ defined by the following table:*



x	$\sim x$	∇x	Δx	$\Delta x \vee \nabla e$
0	1	0	0	h
a	j	a	a	i
b	i	b	b	j
c	h	c	c	1
d	g	i	a	i
e	f	h	0	h
f	e	1	c	1
g	d	j	b	j
h	c	h	h	h
i	b	i	i	i
j	a	j	j	j
1	0	1	1	1

1.4. Moisil's determination principle

We indicate now the proof of the so called *Moisil's determination principle*:

> If L is a Łukasiewicz algebra and $a, b \in L$ are such that $\nabla a = \nabla b$ and $\Delta a = \Delta b$ then $a = b$.

Note that in the initial definitions of Łukasiewicz algebra given by Moisil, this principle was taken as an axiom. Later Moisil [26] defined the Łukasiewicz algebras using equalities, from which he proved this principle.

Let L be a Łukasiewicz algebra, $p, u \in B(L)$ such that $p \leq u$, and

$$L' = [p, u] = \{x \in L : p \leq x \leq u\}.$$

Given $x \in L$ define:

$$\approx x = p \vee (u \wedge \sim x).$$

Observe that $p \leq p \vee (u \wedge \sim x) = \approx x$. On the other hand, $\approx x = p \vee (u \wedge \sim x) = (p \vee u) \wedge (p \vee \sim x) = u \wedge (p \vee \sim x) \leq u$. Then $\approx x \in L'$, for all $x \in L$. If $x \in L'$ then $p \leq x \leq u$ and since $p, u \in B(L)$ we have by $L9)$ that $p = \nabla p \leq \nabla x \leq \nabla u = u$.

Therefore if $x \in L'$ then $\approx x, \nabla x \in L'$. Thus we have two unary operations defined on L'.

Theorem 1.4.1. *The system $(L', u, \approx, \nabla, \wedge, \vee)$ is a Łukasiewicz algebra.*

Proof. It is well known that $(L', p, u, \wedge, \vee)$ is a distributive lattice with top element u and bottom element p.

Since $p, u \in B(L)$, then $p \wedge \sim p = 0, p \vee \sim p = 1, u \wedge \sim u = 0, u \vee \sim u = 1$.

L4) Let $x \in L'$ then $\approx\approx x = p \vee (u \wedge \sim (\approx x)) =$
$p \vee (u \wedge \sim (p \vee (u \wedge \sim x))) = p \vee (u \wedge \sim p \wedge (\sim u \vee x)) =$
$(p \vee u) \wedge (p \vee \sim p) \wedge (p \vee \sim u \vee x) = u \wedge 1 \wedge (\sim u \vee x) =$
$u \wedge (\sim u \vee x) = (u \wedge \sim u) \vee (u \wedge x) = 0 \vee (u \wedge x) = u \wedge x = x$.

L5) Let $x, y \in L'$, so $x \wedge y \in L'$ and $\approx (x \wedge y) = p \vee (u \wedge \sim (x \wedge y)) =$
$p \vee (u \wedge (\sim x \vee \sim y)) = p \vee (u \wedge \sim x) \vee (u \wedge \sim y) =$
$p \vee (u \wedge \sim x) \vee p \vee (u \wedge \sim y) = \approx x \vee \approx y$.

L6) If $x \in L'$ then $\nabla x \leq u$ and therefore $\approx x \vee \nabla x = p \vee (u \wedge \sim x) \vee \nabla x =$
$p \vee ((u \vee \nabla x) \wedge (\sim x \vee \nabla x)) = p \vee (u \wedge 1) = p \vee u = u$.

L7) If $x \in L'$ then $\approx x \wedge \nabla x = (p \vee (u \wedge \sim x)) \wedge \nabla x = (p \wedge \nabla x) \vee (u \wedge \sim x \wedge \nabla x) =$
$p \vee (u \wedge \sim x \wedge \nabla x) = p \vee (u \wedge \sim x \wedge x) = (p \vee (u \wedge \sim x)) \wedge (p \vee x) =$
$\approx x \wedge x$.

L8) Let $x, y \in L'$, so $x \wedge y \in L'$ and $\nabla(x \wedge y) = \nabla x \wedge \nabla y$.

Note that in L' the necessity operator is defined by $\approx \nabla \approx x$, with $x \in L'$. Using the fact that $p, u \in B(L), p \leq u$ and that $p \leq \Delta x \leq u$, then if $x \in L'$ we have:

$$\approx \nabla \approx x = \approx \nabla(p \vee (u \wedge \sim x)) = \approx (\nabla p \vee (\nabla u \wedge \nabla \sim x)) =$$
$$\approx (p \vee (u \wedge \nabla \sim x)) = p \vee (u \wedge \sim (p \vee (u \wedge \nabla \sim x))) =$$
$$p \vee (u \wedge (\sim p \wedge (\sim u \vee \sim \nabla \sim x))) = (p \vee u) \wedge (p \vee \sim p) \wedge (p \vee \sim u \vee \Delta x)) =$$
$$u \wedge 1 \wedge (p \vee \sim u \vee \Delta x) = u \wedge (p \vee \sim u \vee \Delta x) =$$
$$(u \wedge p) \vee (u \wedge \sim u) \vee (u \wedge \Delta x) = p \vee 0 \vee \Delta x = \Delta x$$

□

We prove now Moisil's determination principle. Assume that $a, b \in L$ verify $\nabla a = \nabla b$ and $\Delta a = \Delta b$. Since $p = \Delta a \leq \nabla a = u$ we can consider the interval $L' = [\Delta a, \nabla a] = [\Delta b, \nabla b]$. Since $\Delta a, \nabla a \in B(L)$ then by the previous theorem $(L', u = \nabla a, \approx, \nabla, \wedge, \vee)$ is a Łukasiewicz algebra, in which if $x \in L'$, then $\approx x = \Delta a \vee (\nabla a \wedge \sim x)$. We shall prove that the element a in L', is a center of L', this is, that $\approx a = a$. Indeed $\approx a = \Delta a \vee (\nabla a \wedge \sim a) = \Delta a \vee (a \wedge \sim a) = (\Delta a \vee a) \wedge (\Delta a \vee \sim a) = a \wedge (a \vee \sim a) = a$.

In analogous way one can prove that b is a center of L', then by Lemma 1.2.2, $a = b$.

Immediately after proving this, Professor A. Monteiro posed his students the problem of finding a proof of Moisil's determination principle starting from the axioms for three-valued Łukasiewicz algebras. This was solved in 1965 by L. Monteiro and published in 1969 in [58]. Since that publication has several typos, we reproduce the proof here.

Note that in every Łukasiewicz algebra L it holds that:

$$(1) \quad x = (\Delta x \vee \sim x) \wedge \nabla x = (\nabla x \wedge \sim x) \vee \Delta x.$$

Indeed, using L7′) and L7):

$$(\Delta x \vee \sim x) \wedge \nabla x = (x \vee \sim x) \wedge \nabla x = (x \wedge \nabla x) \vee (\sim x \wedge \nabla x) = x \vee (x \wedge \sim x) = x.$$

$$x = (\Delta x \vee \sim x) \wedge \nabla x = (\Delta x \wedge \nabla x) \vee (\sim x \wedge \nabla x) = \Delta x \vee (\sim x \wedge \nabla x).$$

Assume now that $a, b \in L$ verify (2) $\nabla a = \nabla b$ and (3) $\Delta a = \Delta b$, then:

$$a \vee b = (\text{by } (1)) = (\Delta(a \vee b) \vee \sim (a \vee b)) \wedge \nabla(a \vee b) =$$
$$(\Delta a \vee \Delta b) \vee (\sim a \wedge \sim b)) \wedge (\nabla a \vee \nabla b) = (\text{by } (2) \text{ and } (3)) =$$
$$(\Delta a \vee (\sim a \wedge \sim b)) \wedge \nabla a = (\Delta a \vee \sim a) \wedge (\Delta a \vee \sim b) \wedge \nabla a =$$
$$((\Delta a \vee \sim a) \wedge \nabla a) \wedge ((\Delta a \vee \sim b) \wedge \nabla a) = (\text{by } (2) \text{ and } (3)) =$$
$$((\Delta a \vee \sim a) \wedge \nabla a) \wedge ((\Delta b \vee \sim b) \wedge \nabla b) = (\text{by } (1)) = a \wedge b.$$

Then $a \vee b = a \wedge b$ and therefore $a = a \wedge (a \vee b) = a \wedge (a \wedge b) = a \wedge b = b \wedge (a \wedge b) = b \wedge (a \vee b) = b$.

As a corollary to Moisil's determination principle we have:

Corollary 1.4.2. *$x \leq y$ if and only if $\Delta x \leq \Delta y$ and $\nabla x \leq \nabla y$.*

Lemma 1.4.3. *Every Łukasiewicz algebra L is a Kleene algebra, this is,*

$$(\mathrm{K}) \qquad x \wedge \sim x \leq y \vee \sim y, \text{ for all } \ x, y \in L.$$

Proof. We want to prove that (1) $x \wedge \sim x = (x \wedge \sim x) \wedge (y \vee \sim y)$. Using L6), L9) and L21) respectively we have

$$1 = \sim y \vee \nabla y \leq \nabla \sim y \vee \nabla y = \nabla(\sim y \vee y)$$

and therefore (2) $\nabla(\sim y \vee y) = 1$. Using the duality principle we also get (3) $\Delta(\sim x \wedge x) = 0$. Now applying L8) and (2) we get:

$$(4) \ \nabla((x \wedge \sim x) \wedge (y \vee \sim y)) = \nabla(x \wedge \sim x) \wedge \nabla(y \vee \sim y) =$$
$$\nabla(x \wedge \sim x) \wedge 1 = \nabla(x \wedge \sim x).$$

From L8′) and (3) we deduce:

$$(5) \ \Delta((x \wedge \sim x) \wedge (y \vee \sim y)) = \Delta(x \wedge \sim x) \wedge \Delta(y \vee \sim y) =$$
$$0 \wedge \Delta(y \vee \sim y) = 0 = \Delta(x \wedge \sim x).$$

From (4) and (5), using Moisil's determination principle it follows that (1) is verified. □

A proof of the previous result, not using Moisil's determination principle was indicated by L. Monteiro:

Let $p = (x \wedge \sim x) \wedge (y \vee \sim y)$, so (1) $p \leq \sim x \wedge x$. We just saw that in every Łukasiewicz algebra $a = (\Delta a \vee \sim a) \wedge \nabla a$ holds for every $a \in L$, then $p = (\Delta p \vee \sim p) \wedge \nabla p$.

Since by the proof of Lemma 1.4.3, $\Delta p = 0$ and $\nabla p = \nabla x \wedge \nabla \sim x$, then:

$$(2) \ p = \sim p \wedge \nabla p = ((\sim x \vee x) \vee (\sim y \wedge y)) \wedge \nabla x \wedge \nabla \sim x =$$
$$(\sim x \wedge \nabla x \wedge \nabla \sim x) \vee (x \wedge \nabla x \wedge \nabla \sim x) \vee (\sim y \wedge y \wedge \nabla x \wedge \nabla \sim x) =$$

$$(\sim x \wedge \nabla x) \vee (x \wedge \nabla \sim x) \vee (\sim y \wedge y \wedge \nabla x \wedge \nabla \sim x) =$$
$$(\sim x \wedge x) \vee (x \wedge \sim x) \vee (\sim y \wedge y \wedge \nabla x \wedge \nabla \sim x) =$$
$$(\sim x \wedge x) \vee (\sim y \wedge y \wedge \nabla x \wedge \nabla \sim x) \geq \sim x \wedge x.$$

From (1) and (2) it follows that $x \wedge \sim x = p = (x \wedge \sim x) \wedge (y \vee \sim y)$.

Lemma 1.4.4. *If A is a Kleene algebra and there exists an element $w \in A$ such that $\sim w = w$, then it is the unique one with this property.*

Proof. If $z = \sim z$ then by condition (K) we have

$$z = z \wedge \sim z \leq w \vee \sim w = w \quad \text{and} \quad w = w \wedge \sim w \leq z \vee \sim z = z.$$

□

Lemma 1.4.5. *If L is a Łukasiewicz algebra with axis e, then for all $x \in L$,*

a) $x = \Delta x \vee (e \wedge \nabla x \wedge \nabla \sim x)$. (G. Moisil, [27])
b) $x = (\Delta x \vee e) \wedge \nabla x = (\nabla x \wedge e) \vee \Delta x$. (L. Monteiro, [61])
c) $x = (\Delta x \vee \sim e) \wedge \nabla x = (\nabla x \wedge \sim e) \vee \Delta x$. (L. Monteiro, [61])

Proof. a) Let $a = \Delta x \vee (e \wedge \nabla x \wedge \nabla \sim x)$ then

$$(1) \Delta a = \Delta x \vee (\Delta e \wedge \nabla x \wedge \nabla \sim x) = \Delta x \vee (0 \wedge \nabla x \wedge \nabla \sim x) = \Delta x,$$

and

$$\nabla a = \Delta x \vee (\nabla e \wedge \nabla x \wedge \nabla \sim x) = (\Delta x \vee \nabla e) \wedge (\Delta x \vee \nabla x) \wedge (\Delta x \vee \nabla \sim x) =$$
$$(\Delta x \vee \nabla e) \wedge \nabla x \wedge 1 = (\Delta x \vee \nabla e) \wedge \nabla x.$$

By condition E3) we know that $\nabla x = (\Delta x \vee \nabla e) \wedge \nabla x$, so (2) $\nabla a = \nabla x$. From (1) and (2), by Moisil's determination principle, it follows property a).

b) From a) it follows that $x = (\Delta x \vee e) \wedge (\Delta x \vee \nabla x) \wedge (\Delta x \vee \nabla \sim x) = (\Delta x \vee e) \wedge \nabla x \wedge 1 = (\Delta x \vee e) \wedge \nabla x = (\Delta x \wedge \nabla x) \vee (e \wedge \nabla x) = \Delta x \vee (e \wedge \nabla x)$.

c) By b) $\sim x = (\Delta \sim x \vee e) \wedge \nabla \sim x$ so $x = \sim\sim x = (\nabla x \wedge \sim e) \vee \Delta x = (\nabla x \vee \Delta x) \wedge (\Delta x \vee \sim e) = \nabla x \wedge (\Delta x \vee \sim e)$.

□

Lemma 1.4.6. *If L is a Łukasiewicz algebra such that there exists an element $e \in L$ verifying:*

E1) $\Delta e = 0$,
E2′) $x = (\Delta x \vee e) \wedge \nabla x$, *for all* $x \in L$

then e is an axis of L (L. Monteiro, [61]).

Proof. By hypothesis E1) holds , and from E2′) it follows that $\nabla x = (\Delta x \vee \nabla e) \wedge \nabla x$ this is, E3) holds, which is equivalent to E2). Then e is an axis of L. □

Lemma 1.4.7. *If c is a center of a Łukasiewicz algebra L, then c is an axis of L.*

Proof. By hypothesis $\Delta c = 0$ and $\nabla c = 1$ so $\nabla x \leq 1 = \Delta x \vee \nabla c$. This proves that c verifies conditions E1) and E2), so c is an axis of L. □

Lemma 1.4.8. *If c is a center of a Łukasiewicz algebra L then:*

$$x = (\Delta x \vee c) \wedge \nabla x = (\nabla x \wedge c) \vee \Delta x, \text{for all } x \in L.$$

Proof. Follows from Lemma 1.4.7 and Lemma 1.4.5, b). □

It is clear to see that every boolean algebra is a Łukasiewicz algebra with the bottom element 0 as its axis.

Lemma 1.4.9. *Let L be a Łukasiewicz algebra that is not a boolean algebra. If e is an axis of L then, $e \neq 0$ and $e \neq 1$.*

Proof. If $e = 1$ then $0 = \Delta e = 1$ and L is a boolean algebra with a single element. If $e = 0$ then by E2): $\nabla x \leq \Delta x \vee \nabla e = \Delta x \vee 0 = \Delta x$, this is $\nabla x = \Delta x$ for all $x \in L$ and therefore $\nabla x = x$ for all $x \in L$, so L would be a boolean algebra. □

Lemma 1.4.10. $\Delta x = 0$ *if and only if* $x \leq \sim x$.

Proof. If $\Delta x = 0$, then since $x = (\Delta x \vee \sim x) \wedge \nabla x$, we have that $x = \sim x \wedge \nabla x \leq \sim x$.
If $x \leq \sim x$ then $\Delta x = \Delta x \wedge x \leq \Delta x \wedge \sim x = 0$. □

1.5. Implications

In these algebras, several implication operations may be defined. Among them we have:

$$a \to b = \nabla \sim a \vee b, \tag{1.5.1}$$

$$a \rightarrowtail b = (a \to b) \wedge (\sim b \to \sim a), \tag{1.5.2}$$

$$a \Rightarrow b = \Delta \sim a \vee \Delta b \vee (\nabla \sim a \wedge \nabla b) \vee (\Delta a \wedge b \wedge \sim b). \tag{1.5.3}$$

These operations are called *weak implication*, *contraposed implication* or *Łukasiewicz implication*, and *intuitionistic implication* [28, 29, 31].

Lemma 1.5.1. $a \rightarrowtail b = \sim a \vee b \vee (\nabla \sim a \wedge \nabla b)$.

Proof. $a \rightarrowtail b = (\nabla \sim a \vee b) \wedge (\nabla b \vee \sim a) =$
$(\nabla \sim a \wedge \nabla b) \vee (\nabla \sim a \wedge \sim a) \vee (b \wedge \nabla b) \vee (b \wedge \sim a) =$
$(\nabla \sim a \wedge \nabla b) \vee \sim a \vee b \vee (b \wedge \sim a) = (\nabla \sim a \wedge \nabla b) \vee \sim a \vee b =$
$\sim a \vee b \vee (\nabla \sim a \wedge \nabla b)$. □

Lemma 1.5.2. $a \Rightarrow b = \Delta \sim a \vee b \vee (\nabla \sim a \wedge \nabla b)$. (A. Monteiro [44])

Proof. From (1.5.3) it follows that:
(i) $\Delta(a \Rightarrow b) = \Delta \sim a \vee \Delta b \vee (\nabla \sim a \wedge \nabla b) \vee (\Delta a \wedge \Delta b \wedge \Delta \sim b) =$
$\Delta \sim a \vee \Delta b \vee (\nabla \sim a \wedge \nabla b) = \Delta(\Delta \sim a \vee b \vee (\nabla \sim a \wedge \nabla b))$.
From (1.5.3) we obtain
(ii) $\nabla(a \Rightarrow b) = \Delta \sim a \vee \Delta b \vee (\nabla \sim a \wedge \nabla b) \vee (\Delta a \wedge \nabla b \wedge \nabla \sim b) =$
$\Delta \sim a \vee [(\Delta b \vee \Delta a) \wedge (\Delta b \vee \nabla b) \wedge (\Delta b \vee \nabla \sim b)] \vee (\nabla \sim a \wedge \nabla b) =$
$\Delta \sim a \vee [(\Delta b \vee \Delta a) \wedge \nabla b] \vee (\nabla \sim a \wedge \nabla b) =$
$\Delta \sim a \vee (\Delta b \wedge \nabla b) \vee (\Delta a \wedge \nabla b) \vee (\nabla \sim a \wedge \nabla b) =$
$\Delta \sim a \vee \Delta b \vee (\Delta a \wedge \nabla b) \vee (\nabla \sim a \wedge \nabla b) =$

$\Delta \sim a \vee \Delta b \vee (\nabla b \wedge (\Delta a \vee \nabla \sim a)) = \Delta \sim a \vee \Delta b \vee \nabla b = \Delta \sim a \vee \nabla b =$
$\Delta \sim a \vee \nabla b \vee (\nabla \sim a \wedge \nabla b) = \nabla(\Delta \sim a \vee b \vee (\nabla \sim a \wedge \nabla b))$.
Then, by Moisil's determination principle, from (i) and (ii) the lemma follows.
□

Note that $a \Rightarrow b = \Delta \sim a \vee b \vee (\nabla \sim a \wedge \nabla b) =$
$(\Delta \sim a \vee b \vee \nabla \sim a) \wedge (\Delta \sim a \vee b \vee \nabla b) = (\nabla \sim a \vee b) \wedge (\Delta \sim a \vee \nabla b) =$
$(a \to b) \wedge (\nabla a \to \nabla b)$.

From (1.5.1), Lemma 1.5.2, and Lemma 1.5.1 it follows that:

$$a \Rightarrow b \leq a \rightarrowtail b \leq a \to b. \tag{1.5.4}$$

If $b = 0$ then from (1.5.1) to (1.5.3) we have that $a \Rightarrow 0 = \Delta \sim a = \sim \nabla a$, $a \rightarrowtail 0 = \sim a$ and $a \to 0 = \nabla \sim a$. Then by (1.5.4) we have that:

$$\sim \nabla a \leq \sim a \leq \nabla \sim a. \tag{1.5.5}$$

Thus we are led to consider the following operations:

$$\neg a = a \Rightarrow 0 = \sim \nabla a, \text{(Strong negation)}, \tag{1.5.6}$$

$$\sim a = a \rightarrowtail 0, \text{(Negation)}, \tag{1.5.7}$$

$$\ulcorner a = a \to 0 = \nabla \sim a, \text{(Weak negation)}. \tag{1.5.8}$$

This terminology is due to Moisil. We can interpret the elements of a Łukasiewicz algebra as a set of propositions, the symbols $\wedge$, $\vee$ and $\sim$ representing the logical connectives *and*, *or*, and *not*, respectively, and ∇ and Δ representing *it is possible* and *it is necessary* respectively. Moisil indicated the following example to justify his terminology. Assume that "a" represents the proposition "*I write*" then we have:

- $\sim a =$ *I do not write*,
- $\neg a = \sim \nabla a =$ *It is not possible that I write* $=$ *It is impossible that I write,*
- $\ulcorner a = \nabla \sim a =$ *It is possible that I don't write.*

then by (1.5.5) we have:

$$\neg a \leq \sim a \leq \ulcorner a. \tag{1.5.9}$$

therefore $\neg$ is the strongest negation, $\ulcorner$ the weakest negation, and $\sim$ an intermediate negation between the two preceding ones.

Note that from the following inequality:

$$\Delta a \leq a \leq \nabla a, \tag{1.5.10}$$

it follows that

$$\neg a = \sim \nabla a \leq \sim a \leq \sim \Delta a = \nabla \sim a = \ulcorner a. \tag{1.5.11}$$

Inequality (1.5.10) tells us that Δa is stronger proposition than a and that a is a stronger proposition than ∇a, which agrees with the intuitive interpretation. By (1.5.11) we see that the negation of the stronger proposition becomes the weaker one and the negation of the weaker proposition becomes the stronger one.

The tables for $\rightarrow, \rightarrowtail$ and $\Rightarrow$ for the Łukasiewicz algebra T indicated in Example 1.2.3 are:

$\rightarrow$	0	c	1
0	1	1	1
c	1	1	1
1	0	c	1

$\rightarrowtail$	0	c	1
0	1	1	1
c	c	1	1
1	0	c	1

$\Rightarrow$	0	c	1
0	1	1	1
c	0	1	1
1	0	c	1

Łukasiewicz studied a propositional calculus having as characteristic matrix the system $(T, \sim, \rightarrowtail)$.

Lemma 1.5.3. $(a \Rightarrow b) \vee (\sim b \Rightarrow \sim a) = a \rightarrowtail b$. (Moisil)

PROOF. Using Lemma 1.5.2 we have:
$(a \Rightarrow b) \vee (\sim b \Rightarrow \sim a) =$
$\Delta \sim a \vee b \vee (\nabla \sim a \wedge \nabla b) \vee \Delta b \vee \sim a \vee (\nabla b \wedge \nabla \sim a) = \sim a \vee b \vee (\nabla \sim a \wedge \nabla b) =$ (by Lemma 1.5.1)$= a \rightarrowtail b$.

$\square$

Lemma 1.5.4. $(a \rightarrowtail b) \rightarrowtail b = a \vee b$.

PROOF. $(a \rightarrowtail b) \rightarrowtail b = \sim (a \rightarrowtail b) \vee b \vee (\nabla \sim (a \rightarrowtail b) \wedge \nabla b)$.
Since $\sim (a \rightarrowtail b) \vee b = (a \wedge \sim b \wedge (\Delta a \vee \Delta \sim b)) \vee b =$
$(a \wedge \sim b \wedge \Delta a) \vee (a \wedge \sim b \wedge \Delta \sim b) \vee b = (\sim b \wedge \Delta a) \vee (a \wedge \Delta \sim b) \vee b$, and
$\nabla \sim (a \rightarrowtail b) \wedge \nabla b = \nabla a \wedge \nabla \sim b \wedge (\Delta a \vee \Delta \sim b) \wedge \nabla b =$
$(\nabla a \wedge \nabla \sim b \wedge \Delta a \wedge \nabla b) \vee (\nabla a \wedge \nabla \sim b \wedge \Delta \sim b \wedge \nabla b) = \nabla \sim b \wedge \Delta a \wedge \nabla b$,
then:
$(a \rightarrowtail b) \rightarrowtail b = (\sim b \wedge \Delta a) \vee (a \wedge \Delta \sim b) \vee b \vee (\nabla \sim b \wedge \Delta a \wedge \nabla b) =$
$(\Delta a \wedge (\sim b \vee (\nabla \sim b \wedge \nabla b)) \vee (a \wedge \Delta \sim b) \vee b =$
$(\Delta a \wedge (\sim b \vee \nabla \sim b) \wedge (\sim b \vee \nabla b)) \vee (a \wedge \Delta \sim b) \vee b =$
$(\Delta a \wedge \nabla \sim b) \vee (a \wedge \Delta \sim b) \vee b =$
$((\Delta a \vee b) \wedge (\nabla \sim b \vee b)) \vee (a \wedge \Delta \sim b) = \Delta a \vee b \vee (a \wedge \Delta \sim b)$. So:
(1) $\Delta((a \rightarrowtail b) \rightarrowtail b) = \Delta a \vee \Delta b \vee (\Delta a \wedge \Delta \sim b) = \Delta a \vee \Delta b = \Delta(a \vee b)$ and
(2) $\nabla((a \rightarrowtail b) \rightarrowtail b) = \Delta a \vee \nabla b \vee (\nabla a \wedge \Delta \sim b) =$
$(\Delta a \vee \nabla b \vee \nabla a) \wedge (\Delta a \vee \nabla b \vee \Delta \sim b) = (\nabla b \vee \nabla a) \wedge (\Delta a \vee \nabla b \vee \sim \nabla b) =$
$(\nabla a \vee \nabla b) \wedge 1 = \nabla a \vee \nabla b = \nabla(a \vee b)$.
From (1) and (2) using Moisil's determination principle it follows that
$(a \rightarrowtail b) \rightarrowtail b = a \vee b$. $\square$

Corollary 1.5.5. $(a \rightarrowtail b) \rightarrowtail b = (b \rightarrowtail a) \rightarrowtail a$.

Lemma 1.5.6. $a \rightarrow b = a \rightarrowtail (a \rightarrowtail b)$.

PROOF. By Lemma 1.5.1

$$a \rightarrowtail (a \rightarrowtail b) = \sim a \vee (a \rightarrowtail b) \vee (\nabla \sim a \wedge \nabla(a \rightarrowtail b)),$$

and using again Lemma 1.5.1,
$a \rightarrowtail (a \rightarrowtail b) =$

$$\sim a \vee \sim a \vee b \vee (\nabla \sim a \wedge \nabla b) \vee (\nabla \sim a \wedge (\nabla \sim a \vee \nabla b \vee (\nabla \sim a \wedge \nabla b))) =$$

$$\sim a \vee b \vee (\nabla \sim a \wedge \nabla b) \vee \nabla \sim a = \sim a \vee b \vee \nabla \sim a = \nabla \sim a \vee b = a \rightarrow b.$$

$\square$

This result allows us to go from the contraposed implication to the weak implication.

We shall present some properties of the weak and contraposed implications that we will use later.

Lemma 1.5.7. *The operation $\to$ has the following properties:*

ID1) *If* $a \leq b$ *then* $a \to b = 1$,
ID2) $a \to 1 = 1$,
ID3) $a \to a = 1$,
ID4) $1 \to a = a$,
ID5) $a \to (b \to a) = 1$,
ID6) *If* $a \leq b$ *then* $c \to a \leq c \to b$,
ID7) *If* $a \leq b$ *then* $b \to c \leq a \to c$,
ID8) $a \to (a \wedge b) = a \to b$,
ID9) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$,
ID10) $(a \wedge b) \to c = a \to (b \to c)$,
ID11) $a \to (b \to c) = b \to (a \to c)$,
ID12) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$,
ID13) $a \to \Delta a = 1$,
ID14) $a \wedge (a \to b) = a \wedge (\sim a \vee b)$,
ID15) $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1$,
ID16) $a \to (b \vee c) = (a \to b) \vee (a \to c)$.

PROOF. ID1) If $a \leq b$ then by L13), $1 = \nabla \sim a \vee a \leq \nabla \sim a \vee b = a \to b$.
ID2) Follows from $a \leq 1$ and ID1).
ID3) Follows immediately from $a \leq a$ and ID1).
ID4) $1 \to a = \nabla \sim 1 \vee a = 0 \vee a = a$.
ID5) Since $a \leq \nabla \sim b \vee a = b \to a$, then by ID1), ID5) obtains.
ID6) If $a \leq b$ then $c \to a = \nabla \sim c \vee a \leq \nabla \sim c \vee b = c \to b$.
ID7) If $a \leq b$ then $\sim b \leq \sim a$ and therefore $\nabla \sim b \vee c \leq \nabla \sim a \vee c$, this is $b \to c \leq a \to c$.
ID8) $a \to (a \wedge b) = \nabla \sim a \vee (a \wedge b) = (\nabla \sim a \vee a) \wedge (\nabla \sim a \vee b) = 1 \wedge (a \to b) = a \to b$.
ID9) $a \to (b \wedge c) = \nabla \sim a \vee (b \wedge c) = (\nabla \sim a \vee b) \wedge (\nabla \sim a \vee c) = (a \to b) \wedge (a \to c)$.
ID10) $(a \wedge b) \to c = \nabla \sim (a \wedge b) \vee c = \nabla \sim a \vee \nabla \sim b \vee c = \nabla \sim a \vee (b \to c) = a \to (b \to c)$.
ID11) $a \to (b \to c) =$ (by ID10)$= (a \wedge b) \to c = (b \wedge a) \to c =$ (by ID10)$= b \to (a \to c)$.
ID12) $(a \vee b) \to c = \nabla \sim (a \vee b) \vee c = (\nabla \sim a \wedge \nabla \sim b) \vee c = (\nabla \sim a \vee c) \wedge (\nabla \sim b \vee c) = (a \to c) \wedge (b \to c)$.
ID13) $a \to \Delta a = \nabla \sim a \vee \Delta a = \sim \Delta a \vee \Delta a = 1$.
ID14) $a \wedge (a \to b) = a \wedge (\nabla \sim a \vee b) = (a \wedge \nabla \sim a) \vee (a \wedge b) = (a \wedge \sim a) \vee (a \wedge b) = a \wedge (\sim a \vee b)$.
ID15) $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) =$
$$\nabla \sim (a \to (b \to c)) \vee ((a \to b) \to (a \to c)) =$$
$$\nabla \sim (\nabla \sim a \vee (b \to c)) \vee (\nabla \sim (a \to b) \vee (a \to c)) =$$
$$(\Delta a \wedge \nabla \sim (\nabla \sim b \vee c)) \vee (\Delta a \wedge \nabla \sim b) \vee \nabla \sim a \ \vee c =$$

$(\Delta a \wedge \Delta b \wedge \nabla \sim c) \vee (\Delta a \wedge \nabla \sim b) \vee \nabla \sim a \ \vee c =$
$(\Delta a \wedge ((\Delta b \wedge \nabla \sim c) \vee \nabla \sim b)) \vee \nabla \sim a \ \vee c =$
$(\Delta a \wedge ((\Delta b \vee \nabla \sim b) \wedge (\nabla \sim c \vee \nabla \sim b)) \vee \nabla \sim a \ \vee c =$
$(\Delta a \wedge (\nabla \sim c \vee \nabla \sim b)) \vee \nabla \sim a \ \vee c =$
$(\Delta a \vee \nabla \sim a \ \vee c) \wedge (\nabla \sim c \vee \nabla \sim b \vee \nabla \sim a \ \vee c) = 1 \wedge 1 = 1.$

ID16) $a \to (b \vee c) = \nabla \sim a \vee b \vee c = \nabla \sim a \vee b \vee \nabla \sim a \vee c = (a \to b) \vee (a \to c).$ □

Lemma 1.5.8. *The operation $\rightarrowtail$ has the following properties:*

IC1) *If $a \leq b$ then $a \rightarrowtail b = 1$,*
IC2) $a \rightarrowtail 1 = 1$,
IC3) $a \rightarrowtail a = 1$,
IC4) $1 \rightarrowtail a = a$,
IC5) $a \rightarrowtail (b \rightarrowtail a) = 1$,
IC6) *If $a \leq b$ then $c \rightarrowtail a \leq c \rightarrowtail b$,*
IC7) *If $a \leq b$ then $b \rightarrowtail c \leq a \rightarrowtail c$,*
IC8) $a \rightarrowtail (a \wedge b) = a \rightarrowtail b$,
IC9) $a \rightarrowtail (b \wedge c) = (a \rightarrowtail b) \wedge (a \rightarrowtail c)$,
IC10) $(a \rightarrowtail b) \rightarrowtail ((b \rightarrowtail c) \rightarrowtail (a \rightarrowtail c)) = 1$,
IC11) $\Delta(a \rightarrowtail b) \rightarrowtail (\nabla a \rightarrowtail \nabla b) = 1$,
IC12) *If $a \rightarrowtail b = 1$ then $a \leq b$,*
IC13) *$a = b$ if and only if $a \rightarrowtail b = 1$ and $b \rightarrowtail a = 1$,*
IC14) $a \rightarrowtail c \leq (a \vee b) \rightarrowtail (c \vee b)$,
IC15) $\sim a \rightarrowtail a = \nabla a$,
IC16) $\sim a \rightarrowtail \sim b = b \rightarrowtail a$.

Proof. IC1) If (1) $a \leq b$ then (2) $\sim b \leq \sim a$. From (1) it follows by ID1) that $a \to b = 1$ and from (2) it follows by ID1) that $\sim b \to \sim a = 1$. Then:

$a \rightarrowtail b = (a \to b) \wedge (\sim b \to \sim a) = 1 \wedge 1 = 1.$

IC2) Is an immediate consequence of $a \leq 1$ and IC1).
IC3) Is an immediate consequence of $a \leq a$ and IC1).
IC4) $1 \rightarrowtail a = (1 \to a) \wedge (\sim a \to \sim 1) =$ (by ID4) $= a \wedge (\nabla a \vee 0) = a \wedge \nabla a = a$.
IC5) $b \rightarrowtail a = (\nabla \sim b \vee a) \wedge (\nabla a \vee \sim b) \geq a \wedge \nabla a = a$, then by IC1, IC5 follows.
IC6) If (1) $a \leq b$ then (2) $\sim b \leq \sim a$. From (1) it follows by ID6 that (3) $c \to a \leq c \to b$ and from (2) it follows by ID7 that (4) $\sim a \to \sim c \leq \sim b \to \sim c$. From (3) and (4):

$$(c \to a) \wedge (\sim a \to \sim c) \leq (c \to b) \wedge (\sim b \to \sim c)$$

this is $c \rightarrowtail a \leq c \rightarrowtail b$.

IC7) If (1) $a \leq b$ then (2) $\sim b \leq \sim a$. From (1) it follows by ID7) that (3) $b \to c \leq a \to c$ and from (2) it follows by ID6 that (4) $\sim c \to \sim b \leq \sim c \to \sim a$. From (3) and (4):

$$(b \to c) \wedge (\sim c \to \sim b) \leq (a \to c) \wedge (\sim c \to \sim a)$$

this is $b \rightarrowtail c \leq a \rightarrowtail c$.

IC8) $a \rightarrowtail (a \wedge b) = (a \to (a \wedge b)) \wedge (\sim (a \wedge b) \to \sim a) = $ (by ID8))=
$(a \to b) \wedge ((\sim a \vee \sim b) \to \sim a) = $ (by ID12))=
$(a \to b) \wedge (\sim a \to \sim a) \wedge (\sim b \to \sim a) = $ (by ID3)) $=$
$(a \to b) \wedge 1 \wedge (\sim b \to \sim a) = (a \to b) \wedge (\sim b \to \sim a) = a \rightarrowtail b.$

IC9) $a \rightarrowtail (b \wedge c) = (a \to (b \wedge c)) \wedge (\sim (b \wedge c) \to \sim a) =$
$(\nabla \sim a \vee (b \wedge c)) \wedge ((\nabla b \wedge \nabla c) \vee \sim a) =$
$(\nabla \sim a \vee b) \wedge (\nabla \sim a \vee c) \wedge (\nabla b \vee \sim a) \wedge (\nabla c \vee \sim a) =$
$(\nabla \sim a \vee b) \wedge (\nabla b \vee \sim a) \wedge (\nabla \sim a \vee c) \wedge (\nabla c \vee \sim a) =$
$(a \rightarrowtail b) \wedge (a \rightarrowtail c).$

IC10) Since $x \rightarrowtail y = \sim x \vee y \vee (\nabla \sim x \wedge \nabla y)$ then:

$$(1) \quad \nabla(x \rightarrowtail y) = \nabla \sim x \vee \nabla y \vee (\nabla \sim x \wedge \nabla y) = \nabla \sim x \vee \nabla y,$$

and

$$(2) \quad \Delta(x \rightarrowtail y) = \Delta \sim x \vee \Delta y \vee (\nabla \sim x \wedge \nabla y) =$$
$$(\nabla \sim x \vee \Delta y) \wedge (\Delta \sim x \vee \nabla y).$$

Since

$$\sim (x \rightarrowtail y) = x \wedge \sim y \wedge (\Delta x \vee \Delta \sim y),$$

then

$$(3) \quad \nabla \sim (x \rightarrowtail y) = \nabla x \wedge \nabla \sim y \wedge (\Delta x \vee \Delta \sim y) =$$
$$(\Delta x \wedge \nabla \sim y) \vee (\nabla x \wedge \Delta \sim y),$$

and

$$(4) \quad \Delta \sim (x \rightarrowtail y) = \Delta x \wedge \Delta \sim y \wedge (\Delta x \vee \Delta \sim y) = \Delta x \wedge \Delta \sim y.$$

Let $\alpha = a \rightarrowtail b$ and $\beta = (b \rightarrowtail c) \rightarrowtail (a \rightarrowtail c)$, then by (1)

$$(5) \quad \nabla\beta = \nabla \sim (b \rightarrowtail c) \vee \nabla(a \rightarrowtail c).$$

So by (3) and (1)
$\nabla\beta = (\nabla b \wedge \nabla \sim c \wedge (\Delta b \vee \Delta \sim c)) \vee \nabla \sim a \vee \nabla c =$
$(\nabla \sim a \vee \nabla c \vee \nabla b) \wedge (\nabla \sim a \vee \nabla c \vee \nabla \sim c) \wedge (\nabla \sim a \vee \nabla c \vee \Delta b \vee \Delta \sim c) =$
$(\nabla \sim a \vee \nabla c \vee \nabla b) \wedge 1 \wedge 1 = \nabla \sim a \vee \nabla c \vee \nabla b.$
By (1) $\nabla\alpha = \nabla(a \rightarrowtail b) = \nabla \sim a \vee \nabla b$, then

$$(6) \quad \nabla\alpha \leq \nabla\beta.$$

By (2)

$$\Delta\beta = (\nabla \sim (b \rightarrowtail c) \vee \Delta(a \rightarrowtail c)) \wedge (\Delta \sim (b \rightarrowtail c) \vee \nabla(a \rightarrowtail c))$$

Let $\gamma = \nabla \sim (b \rightarrowtail c) \vee \Delta(a \rightarrowtail c) =$ and $\delta = \Delta \sim (b \rightarrowtail c) \vee \nabla(a \rightarrowtail c)$, so that $\Delta\beta = \gamma \wedge \delta$. By (3) and (1),

$$\gamma = (\Delta b \wedge \nabla \sim c) \vee (\nabla b \wedge \Delta \sim c) \vee \Delta \sim a \vee \Delta c \vee (\nabla \sim a \wedge \nabla c).$$

By (4) and (1),

$$\delta = \Delta \sim (b \rightarrowtail c) \vee \nabla(a \rightarrowtail c) = (\Delta b \wedge \Delta \sim c) \vee \nabla \sim a \vee \nabla c =$$

$$(\Delta b \vee \nabla \sim a \vee \nabla c) \wedge (\Delta \sim c \vee \nabla \sim a \vee \nabla c) = \nabla \sim a \vee \Delta b \vee \nabla c.$$

By (2)

$$\Delta\alpha = \Delta(a \rightarrowtail b) = (\nabla \sim a \vee \Delta b) \wedge (\Delta \sim a \vee \nabla b) \leq \nabla \sim a \vee \Delta b \leq \nabla \sim a \vee \Delta b \vee \nabla c = \delta,$$

and also by Lemma 1.5.1

$$\Delta\alpha = \Delta \sim a \vee \Delta b \vee (\nabla \sim a \wedge \nabla b) =$$
$$\Delta \sim a \vee (\Delta b \wedge (\nabla \sim c \vee \Delta c)) \vee (\nabla \sim a \wedge \nabla b) =$$
$$\Delta \sim a \vee (\Delta b \wedge \nabla \sim c) \vee (\Delta b \wedge \Delta c) \vee (\nabla \sim a \wedge \nabla b) \leq$$
$$\Delta \sim a \vee (\Delta b \wedge \nabla \sim c) \vee (\Delta b \wedge \Delta c) \vee \Delta c \vee (\nabla \sim a \wedge \nabla b) =$$
$$\Delta \sim a \vee (\Delta b \wedge \nabla \sim c) \vee \Delta c \vee (\nabla \sim a \wedge \nabla b) =$$
$$\Delta \sim a \vee (\Delta b \wedge \nabla \sim c) \vee \Delta c \vee (\nabla \sim a \wedge \nabla b \wedge \nabla c) \vee (\nabla \sim a \wedge \nabla b \wedge \Delta \sim c) \leq$$
$$\Delta \sim a \vee (\Delta b \wedge \nabla \sim c) \vee \Delta c \vee (\nabla \sim a \wedge \nabla c) \vee (\nabla b \wedge \Delta \sim c) = \gamma,$$

so

$$(7) \quad \Delta\alpha \leq \gamma \wedge \delta = \Delta\beta.$$

From (6) and (7) it follows by Moisil's determination principle that $\alpha \wedge \beta = \alpha$ this is $\alpha \leq \beta$, so by IC2) we have that $\alpha \rightarrowtail \beta = 1$.

IC11) $\Delta(a \rightarrowtail b) = \Delta \sim a \vee \Delta b \vee (\nabla \sim a \wedge \nabla b)$, so
(1) $\sim \Delta(a \rightarrowtail b) = \nabla a \wedge \nabla \sim b \wedge (\Delta a \vee \Delta \sim b)$. We also have
(2) $\nabla a \rightarrowtail \nabla b = \Delta \sim a \vee \nabla b \vee (\Delta \sim a \wedge \nabla b) = \Delta \sim a \vee \nabla b$.
Therefore (3) $\nabla \sim \Delta(a \rightarrowtail b) = \sim \Delta(a \rightarrowtail b)$ and
(4) $\nabla(\nabla a \rightarrowtail \nabla b) = \nabla a \rightarrowtail \nabla b$. Then by (1), (2), (3) and (4) we have:

$$\Delta(a \rightarrowtail b) \rightarrowtail (\nabla a \rightarrowtail \nabla b) =$$
$$\sim \Delta(a \rightarrowtail b) \vee (\nabla a \rightarrowtail \nabla b) \vee (\nabla \sim \Delta(a \rightarrowtail b) \wedge \nabla(\nabla a \rightarrowtail \nabla b)) =$$
$$\sim \Delta(a \rightarrowtail b) \vee (\nabla a \rightarrowtail \nabla b) \vee (\sim \Delta(a \rightarrowtail b) \wedge (\nabla a \rightarrowtail \nabla b)) =$$
$$\sim \Delta(a \rightarrowtail b) \vee (\nabla a \rightarrowtail \nabla b) =$$
$$(\nabla a \wedge \nabla \sim b \wedge (\Delta a \vee \Delta \sim b)) \vee (\Delta \sim a \vee \nabla b) =$$
$$(\nabla a \wedge \nabla \sim b \wedge \Delta a) \vee (\nabla a \wedge \nabla \sim b \wedge \Delta \sim b) \vee \Delta \sim a \vee \nabla b =$$
$$(\nabla \sim b \wedge \Delta a) \vee (\nabla a \wedge \Delta \sim b) \vee \Delta \sim a \vee \nabla b =$$
$$(\nabla \sim b \wedge \Delta a) \vee (\nabla a \wedge \Delta \sim b) \vee \sim (\nabla a \wedge \Delta \sim b) =$$
$$(\nabla \sim b \wedge \Delta a) \vee 1 = 1$$

IC12) If $a \rightarrowtail b = 1$, then by Lemma 1.5.4, $a \vee b = (a \rightarrowtail b) \rightarrowtail b = 1 \rightarrowtail b =$ (by IC4)) $= b$, so $a \leq b$.

IC13) If $a = b$ then by IC3) we have that $a \rightarrowtail b = a \rightarrowtail a = 1$ and $b \rightarrowtail a = b \rightarrowtail b = 1$. Assume that $a \rightarrowtail b = 1$ and $b \rightarrowtail a = 1$, so by IC12) $a \leq b$ and $b \leq a$, therefore $a = b$.

IC14) $(a \vee b) \rightarrowtail (c \vee b) = \sim (a \vee b) \vee c \vee b \vee (\nabla \sim (a \vee b) \wedge \nabla(c \vee b)) =$

$$\sim (a \vee b) \vee c \vee b \vee (\nabla \sim a \wedge \nabla \sim b \wedge \nabla(c \vee b)) =$$
$$\sim (a \vee b) \vee c \vee b \vee (\nabla \sim a \wedge \nabla \sim b \wedge \nabla c) \vee (\nabla \sim a \wedge \nabla \sim b \wedge \nabla b) =$$
$$\sim (a \vee b) \vee c \vee ((\nabla \sim a \wedge \nabla c) \vee b) \wedge (b \vee \nabla \sim b)) \vee ((\nabla \sim a \wedge \nabla b) \vee b)) \wedge (b \vee \nabla \sim b)) =$$
$$\sim (a \vee b) \vee c \vee (\nabla \sim a \wedge \nabla c) \vee (\nabla \sim a \wedge \nabla b) \vee b =$$
$$(\sim a \wedge \sim b) \vee (\nabla \sim a \wedge \nabla b) \vee c \vee b \vee (\nabla \sim a \wedge \nabla c) =$$
$$((\sim a \vee \nabla \sim a) \wedge (\sim a \vee \nabla b) \wedge (\sim b \vee \nabla \sim a) \wedge (\sim b \vee \nabla b)) \vee c \vee b \vee (\nabla \sim a \wedge \nabla c) =$$
$$(\nabla \sim a \wedge (\sim a \vee \nabla b) \wedge (\sim b \vee \nabla \sim a)) \vee c \vee b \vee (\nabla \sim a \wedge \nabla c) =$$
$$(\nabla \sim a \wedge (\sim a \vee \nabla b)) \vee c \vee b \vee (\nabla \sim a \wedge \nabla c) =$$

$\sim a \vee (\nabla \sim a \wedge \nabla b) \vee c \vee b \vee (\nabla \sim a \wedge \nabla c) \geq$
$\sim a \vee c \vee (\nabla \sim a \wedge \nabla c) = a \rightarrowtail c.$
IC15) $\sim a \rightarrowtail a = a \vee a \vee (\nabla a \wedge \nabla a) = a \vee \nabla a = \nabla a.$
IC16) $\sim a \rightarrowtail \sim b = (\sim a \rightarrow \sim b) \wedge (b \rightarrow a) = b \rightarrowtail a.$

□

Note that from the formulas:

- $a \vee b = (a \rightarrowtail b) \rightarrowtail b,$
- $a \wedge b = \sim (\sim a \vee \sim b),$
- $\nabla a = \sim a \rightarrowtail a,$
- $a \rightarrowtail a = 1,$

it follows that in a Łukasiewicz algebra, from the operations $\rightarrowtail$ and $\sim$, the constant 1, and the operations $\vee, \wedge$, and ∇ can be determined, so it is possible to define a Łukasiewicz algebra as a system formed by a non-empty set A, a unary operation $\sim$ and a binary operation $\rightarrowtail$, as long as these two connectives fulfill certain conditions. At the moment this course was taught, this was an open problem.

- In 1984 A. Figallo and J. Tolosa [19] characterized the Łukasiewicz algebras as a system $(L, 1, \rightarrow, \wedge, \neg)$, using Moisil's representation Theorem (see section 3.5).
- Also in 1984 M. Abad and A. Figallo [1], gave a different proof of the same result.
- In 1992, A. Figallo and A. Ziliani charactered Łukasiewicz's three valued propositional calculus in terms of $\rightarrow, \wedge, \neg$, modus ponens and the substitution law. This was published in 1992, [20].

These three problems were posed by Professor A. Monteiro during the courses and seminars about Łukasiewicz algebras.

- In 1986 D. Díaz and A. Figallo [17] characterized Łukasiewicz algebras as systems $(L, 1, \neg, \rightarrowtail)$.

1.6. Heyting algebras

Definition 1.6.1. *A Heyting algebra is a lattice H with bottom element 0 and top element 1, in which for each pair (a, b) of elements in H there exists an element $c \in H$ such that:*
HA1) $a \wedge c \leq b$,
HA2) *If $a \wedge x \leq b$ then $x \leq c$.*
See, for instance, [46] and [73].

We shall denote the element c with $c = a \Rightarrow b$ and say that c is the intuitionistic implication of a and b.
In 1963 Gr. Moisil [29] proved the following result:

Theorem 1.6.2. *Every Łukasiewicz algebra is a Heyting algebra.*

PROOF. This proof is due to A. Monteiro. Moisil defines the intuitionistic implication by

$$a \Rightarrow b = \Delta \sim a \vee \Delta b \vee (\nabla \sim a \wedge \nabla b) \vee (\Delta a \wedge b \wedge \sim b).$$

But we saw in Lemma 1.5.2 that

$$a \Rightarrow b = \Delta \sim a \vee b \vee (\nabla \sim a \wedge \nabla b).$$

HA1) $a \wedge (a \Rightarrow b) \leq b$.

Indeed,

$$a \wedge (a \Rightarrow b) = a \wedge (\Delta \sim a \vee b \vee (\nabla \sim a \wedge \nabla b)) =$$
$$(a \wedge \Delta \sim a) \vee (a \wedge b) \vee (a \wedge \nabla \sim a \wedge \nabla b)$$

and since $a \wedge \Delta \sim a = 0$ we have that

$$(1) \quad a \wedge (a \Rightarrow b) = (a \wedge b) \vee (a \wedge \nabla \sim a \wedge \nabla b).$$

By Moisil's determination principle, to prove HA1) is equivalent to prove that (2) $\nabla(a \wedge (a \Rightarrow b)) \leq \nabla b$ and (3) $\Delta(a \wedge (a \Rightarrow b)) \leq \Delta b$.

From (1) it follows that

$$\nabla(a \wedge (a \Rightarrow b)) = \nabla((a \wedge b) \vee (a \wedge \nabla \sim a \wedge \nabla b)) =$$
$$(\nabla a \wedge \nabla b) \vee (\nabla a \wedge \nabla \sim a \wedge \nabla b) = \nabla a \wedge \nabla b \leq \nabla b.$$

Also from (1) we deduce

$$\Delta(a \wedge (a \Rightarrow b)) = \Delta((a \wedge b) \vee (a \wedge \nabla \sim a \wedge \nabla b)) =$$
$$(\Delta a \wedge \Delta b) \vee (\Delta a \wedge \nabla \sim a \wedge \nabla b) = \Delta a \wedge \Delta b \leq \Delta b.$$

Let us prove now that:

HA2) If (4) $a \wedge x \leq b$ then (5) $x \leq a \Rightarrow b$.

It follows from (4) that

$$(6) \quad \nabla a \wedge \nabla x = \nabla(a \wedge x) \leq \nabla b,$$

and

$$(7) \quad \Delta a \wedge \Delta x = \Delta(a \wedge x) \leq \Delta b$$

and to prove (5) is equivalent, by Moisil's determination principle, to prove that

$$(8) \quad \nabla x \leq \nabla(a \Rightarrow b)$$

and

$$(9) \quad \Delta x \leq \Delta(a \Rightarrow b).$$

But

$$\nabla(a \Rightarrow b) = \nabla(\Delta \sim a \vee b \vee (\nabla \sim a \wedge \nabla b)) =$$
$$\Delta \sim a \vee \nabla b \vee (\nabla \sim a \wedge \nabla b) = \Delta \sim a \vee \nabla b.$$

and

$$\Delta(a \Rightarrow b) = \Delta(\Delta \sim a \vee b \vee (\nabla \sim a \wedge \nabla b)) = \Delta \sim a \vee \Delta b \vee (\nabla \sim a \wedge \nabla b)) =$$
$$(\Delta \sim a \vee \Delta b \vee \nabla \sim a) \wedge (\Delta \sim a \vee \Delta b \vee \nabla b) =$$
$$(\nabla \sim a \vee \Delta b) \wedge (\Delta \sim a \vee \nabla b).$$

Therefore conditions (8) and (9) can be written as

$$(10) \quad \nabla x \leq \Delta \sim a \vee \nabla b,$$

$$(11)\quad \Delta x \leq (\nabla \sim a \vee \Delta b) \wedge (\Delta \sim a \vee \nabla b).$$

To prove this last condition is equivalent to proving

$$(12)\quad \Delta x \leq \nabla \sim a \vee \Delta b,$$

and

$$(13)\quad \Delta x \leq \Delta \sim a \vee \nabla b.$$

We shall prove that (10), (12), and (13) are deduced from (6) and (7).
From (6) it follows that

$$\sim \nabla a \vee (\nabla a \wedge \nabla x) \leq \sim \nabla a \vee \nabla b$$

this is

$$\sim \nabla a \vee \nabla x = (\sim \nabla a \vee \nabla a) \wedge (\sim \nabla a \vee \nabla x) \leq \sim \nabla a \vee \nabla b = \Delta \sim a \vee \nabla b$$

and therefore

$$(14)\quad \sim \nabla a \vee \nabla x \leq \Delta \sim a \vee \nabla b.$$

Then, since

$$(15)\quad \Delta x \leq \nabla x \leq \sim \nabla a \vee \nabla x$$

from (14) and (15) it follows that
$\Delta x \leq \Delta \sim a \vee \nabla b$ and $\nabla x \leq \Delta \sim a \vee \nabla b$ which proves (13) and (10).
From (7) it follows that

$$\sim \Delta a \vee (\Delta a \wedge \Delta x) \leq \sim \Delta a \vee \Delta b = \nabla \sim a \vee \Delta b$$

then

$$\Delta x \leq \sim \Delta a \vee \Delta x \leq \nabla \sim a \vee \Delta b,$$

which proves (12). □

We saw in the previous section that the intuitionistic negation of an element x of a Łukasiewicz algebra is

$$\neg x = x \Rightarrow 0 = \Delta \sim x = \sim \nabla x$$

and therefore

$$\neg\neg x = \sim \nabla(\neg x) = \sim \nabla(\sim \nabla x) = \Delta \nabla x = \nabla x.$$

Therefore the operator ∇ can be obtained from the intuitionistic implication.
In every Łukasiewicz algebra $\nabla(x \wedge y) = \nabla x \wedge \nabla y$ holds so

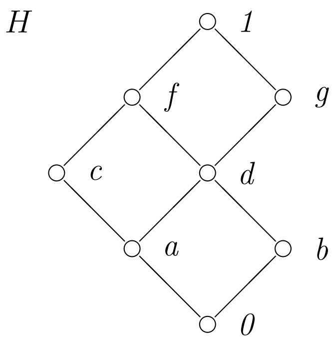
$$\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y \tag{1.6.1}$$

which is a valid formula in every Heyting algebra.
Since in every Łukasiewicz algebra $\nabla(x \vee y) = \nabla x \vee \nabla y$ holds, then

$$\neg\neg(x \vee y) = \neg\neg x \vee \neg\neg y \tag{1.6.2}$$

but this formula is not valid in every Heyting algebra. Indeed:

Example 1.6.3. *Let H be the Heyting algebra H indicated in the next figure,* [46]*. Then $\neg\neg(a \vee b) = \neg\neg d = \neg 0 = 1$ y $\neg\neg a \vee \neg\neg b = \neg b \vee \neg c = c \vee b = f$, which proves that* (1.6.2) *does not hold in every Heyting algebra.*



$\Rightarrow$	0	a	b	c	d	f	g	1
0	1	1	1	1	1	1	1	1
a	b	1	b	1	1	1	1	1
b	c	c	1	c	1	1	1	1
c	b	g	b	1	g	1	g	1
d	0	c	b	c	1	1	1	1
f	0	a	b	c	g	1	g	1
g	0	c	b	c	f	f	1	1
1	0	a	b	c	d	f	g	1

Therefore Łukasiewicz algebras are particular Heyting algebras.

Definition 1.6.4. *A Heyting algebra is said to be three valued if*

$$\text{T)} \quad (((a \Rightarrow c) \Rightarrow b) \Rightarrow (((b \Rightarrow a) \Rightarrow b) \Rightarrow b) = 1$$

holds.

Lemma 1.6.5. *In a Heyting algebra the condition* T) *is equivalent to each of the following conditions:* (L. Monteiro, [55], [60])

T1) $(\neg a \Rightarrow b) \Rightarrow (((b \Rightarrow a) \Rightarrow b) \Rightarrow b) = 1$,
T2) $((b \Rightarrow a) \Rightarrow b) \Rightarrow ((\neg a \Rightarrow b) \Rightarrow b) = 1$,
T3) $b = (\neg a \Rightarrow b) \wedge ((b \Rightarrow a) \Rightarrow b)$,
T4) $b = ((a \Rightarrow c) \Rightarrow b) \wedge ((b \Rightarrow a) \Rightarrow b)$.

The next result was obtained by L. Monteiro in 1963, and presented that same year in the seminar conducted by A. Monteiro [37], but only published in 1970, [59].

Theorem 1.6.6. *Every Łukasiewicz algebra is a three valued Heyting algebra.*

Proof. Since $\neg a \Rightarrow b = \Delta \sim a \Rightarrow b = \nabla a \vee b \vee (\nabla a \wedge \nabla b) = \nabla a \vee b$ and

$$(b \Rightarrow a) \Rightarrow b = (\Delta \sim b \vee a \vee (\nabla \sim b \wedge \nabla a)) \Rightarrow b =$$

$$\Delta \sim (\Delta \sim b \vee a \vee (\nabla \sim b \wedge \nabla a)) \vee b \vee (\nabla \sim (\Delta \sim b \vee a \vee (\nabla \sim b \wedge \nabla a)) \wedge \nabla b) =$$

$$(\nabla b \wedge \Delta \sim a \wedge (\Delta b \vee \Delta \sim a)) \vee b \vee (\nabla b \wedge \nabla \sim a \wedge (\Delta b \vee \Delta \sim a) \wedge \nabla b) =$$

$$(\nabla b \wedge \Delta \sim a) \vee b \vee (\nabla b \wedge \nabla \sim a \wedge (\Delta b \vee \Delta \sim a)) =$$

$$(\nabla b \wedge \Delta \sim a) \vee b \vee (\nabla b \wedge \nabla \sim a \wedge \Delta b) \vee (\nabla b \wedge \nabla \sim a \wedge \Delta \sim a) =$$

$$(\nabla b \wedge \Delta \sim a) \vee b \vee (\Delta b \wedge \nabla \sim a) \vee (\nabla b \wedge \Delta \sim a) =$$

$$(\nabla b \wedge \Delta \sim a) \vee b \vee (\Delta b \wedge \nabla \sim a) =$$

and since $\Delta b \wedge \nabla \sim a \leq \Delta b \leq b$ we have that

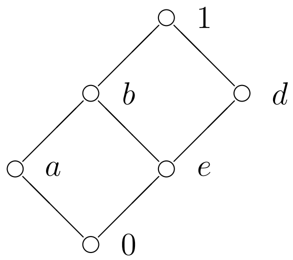
$$(b \Rightarrow a) \Rightarrow b = (\nabla b \wedge \Delta \sim a) \vee b,$$

then

$$(\neg a \Rightarrow b) \wedge ((b \Rightarrow a) \Rightarrow b) = (\nabla a \vee b) \wedge ((\nabla b \wedge \Delta \sim a) \vee b) =$$
$$(\nabla a \wedge \nabla b \wedge \Delta \sim a) \vee b = (\nabla a \wedge \sim \nabla a \wedge \nabla b) \vee b = 0 \vee b = b,$$

which proves that T3) holds. □

Consider the Łukasiewicz algebra from Example 1.3.1, namely:



x	$\sim x$	∇x
0	1	0
a	d	a
b	e	1
d	a	d
e	b	d
1	0	1

$\Rightarrow$	0	a	b	d	e	1
0	1	1	1	1	1	1
a	d	1	1	d	d	1
b	0	a	1	d	d	1
d	a	a	b	1	b	1
e	a	a	1	1	1	1
1	0	a	b	d	e	1

The subset $L' = \{0, b, 1\}$ is closed with respect to the operations $\wedge, \vee, \Rightarrow, \neg$, but not with respect to $\sim$, therefore in general, the operation $\sim$ cannot be expressed in terms of the operations $\wedge, \vee, \Rightarrow$, and $\neg$. This example was pointed out by A. Monteiro in his 1963 seminar. During the same, he posed the problem of characterizing the Łukasiewicz algebras by means of the connectives $\wedge, \vee, \Rightarrow, \sim$. This problem was solved that same year by L. Monteiro, who used the theory of prime filters. Then Professor A. Monteiro (see [45]), posed the problem of obtaining the same result in a purely algebraic way and L. Monteiro obtained the following result in 1969, [59].

Theorem 1.6.7. *If in a system $(L, 1, \sim, \vee, \wedge, \Rightarrow)$ verifying the axioms:*

1) $x \Rightarrow x = 1$,
2) $(x \Rightarrow y) \wedge y = y$,
3) $x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z)$,
4) $x \wedge (x \Rightarrow y) = x \wedge y$,
5) $(x \vee y) \Rightarrow z = (x \Rightarrow z) \wedge (y \Rightarrow z)$,
6) $(((x \Rightarrow z) \Rightarrow y) \Rightarrow (((y \Rightarrow x) \Rightarrow y) \Rightarrow y) = 1$,
7) $\sim\sim x = x$,
8) $\sim (x \wedge y) = \sim x \vee \sim y$,
9) $(x \wedge \sim x) \wedge (y \vee \sim y) = x \wedge \sim x$,

we define D) $\nabla x = \sim x \Rightarrow x$, *then the system $(L, 1, \sim, \nabla, \vee, \wedge)$ is a Łukasiewicz algebra and furthermore*

$$a \Rightarrow b = \Delta \sim a \vee b \vee (\nabla \sim a \wedge \nabla b).$$

Since all Łukasiewicz algebras are Heyting algebras, (see for instance H. Rasiowa and R. Sikorski [73], p. 55), we can claim that

Theorem 1.6.8. *If in a Łukasiewicz algebra L there exists $\bigvee_{i \in I} y_i$, then there exists $\bigvee_{i \in I} (x \wedge y_i)$ and*

$$\text{(D)} \qquad x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i).$$

We shall present a proof of (D) that does not require knowledge of the theory of Heyting algebras was indicated by L. Monteiro, [62].

Let $y = \bigvee_{i \in I} y_i$, then from $y_i \le y$ for all $i \in I$ it follows that

$$\text{S1)} \quad x \wedge y_i \le x \wedge y \text{ for all } i \in I.$$

Let us show that:

$$\text{S2)} \quad \text{If } t \text{ verifies (1) } x \wedge y_i \le t \text{ for all } i \in I, \text{ then } x \wedge y \le t.$$

For this will be enough, using Moisil's determination principle, that

$$\Delta(x \wedge y) \le \Delta t \quad \text{y} \quad \nabla(x \wedge y) \le \nabla t.$$

this is that

$$(2) \quad \Delta x \wedge \Delta y \le \Delta t$$

and

$$(3) \quad \nabla x \wedge \nabla y \le \nabla t.$$

From (1) it follows that $\nabla \sim x \vee y_i = \nabla \sim x \vee (x \wedge y_i) \le \nabla \sim x \vee t$, for all $i \in I$. Then $y_i \le \nabla \sim x \vee y_i \le \nabla \sim x \vee t$ for all $i \in I$ and in consequence

$$y = \bigvee_{i \in I} y_i \le \nabla \sim x \vee t.$$

Therefore

$$\Delta x \wedge y \le \Delta x \wedge (\nabla \sim x \vee t) = \Delta x \wedge t \le t,$$

then $\Delta x \wedge \Delta y = \Delta(\Delta x \wedge y) \le \Delta t$, which proves (2).

From the assumption (1) it follows that

$$\nabla x \wedge \nabla y_i \le \nabla t, \text{ for all } i \in I,$$

so

$$\sim \nabla x \vee \nabla y_i = \sim \nabla x \vee (\nabla x \wedge \nabla y_i) \le \sim \nabla x \vee \nabla t, \text{ for all } i \in I$$

and therefore

$$y_i \le \nabla y_i \le \sim \nabla x \vee \nabla y_i \le \sim \nabla x \vee \nabla t, \text{ for all } i \in I,$$

so $y = \bigvee_{i \in I} y_i \le \sim \nabla x \vee \nabla t$ and therefore $\nabla x \wedge y \le \nabla x \wedge (\sim \nabla x \vee \nabla t) = \nabla x \wedge \nabla t \le \nabla t$ thus $\nabla x \wedge \nabla y = \nabla(\nabla x \wedge y) \le \nabla t$, which proves (3).

From S1) and S2) it follows:

$$\bigvee_{i \in I} (x \wedge y_i) = x \wedge \bigvee_{i \in I} y_i.$$

1.7. Moisil's definition

As we mentioned before, the concept of Łukasiewicz algebra was introduced by Gr. Moisil in his 1940 [25], and 1941 [27] articles, and gave a simplified definition in 1960 [30] which we present next.

Definition 1.7.1. *A three-valued Łukasiewicz algebra is a system* $(L, 1, \sim, \nabla, \vee, \wedge)$ *composed by 1) a non-empty set L; 2) an element* $1 \in L$*; 3) two unary operations* $\sim$ *and* ∇ *defined over L; 4) two binary operations* $\vee$ *and* $\wedge$*, defined over L such that the following conditions are satisfied:*

I) $(L, 0, 1, \wedge, \vee)$ *is a bounded distributive lattice, this is*
 M0) $0 \wedge x = 0$, for all $x \in L$,
 M1) $1 \vee x = 1$, for all $x \in L$,
 M2) $x \wedge (x \vee y) = x$, for all $x, y \in L$,
 M3) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$, for all $x, y, z \in L$,
II) *Furthermore*
 M4) $\sim (x \vee y) = \sim x \wedge \sim y$, for all $x, y \in L$,
 M5) $\sim (x \wedge y) = \sim x \vee \sim y$, for all $x, y \in L$,
 M6) $\sim\sim x = x$, for all $x \in L$,
 M7) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$, for all $x, y \in L$.
 M8) $\nabla(x \vee y) = \nabla x \vee \nabla y$, for all $x, y \in L$.
 M9) $x \wedge \nabla x = x$, for all $x \in L$.
 M10) $\nabla\nabla x = \nabla x$, for all $x \in L$.
 M11) $\sim \nabla \sim \nabla x = \nabla x$, for all $x \in L$.
 M12) $\sim x \wedge x = \sim x \wedge \nabla x$, for all $x \in L$,
 M13) $\sim \nabla \sim x \vee (\nabla x \wedge \nabla \sim x) \vee \sim \nabla x = 1$, for all $x \in L$.

In section 1.1 we proved that if we adopt definition 1.1.1 then all the axioms from M0) to M13) are verified. To prove the equivalence of the two definitions, since axioms L1) to L5) and L7), L8) appear in Moisil's definition, it will be enough to show that axiom L6) is verified, this is that $\sim x \vee \nabla x = 1$. From M4) if follows immediately that:

$$(1) \quad \text{If } x \le y \ \text{ then } \ \sim y \le \sim x.$$

Let us prove that

$$(2) \quad \sim \nabla \sim x \le x.$$

Indeed, by M9) we have that $y \le \nabla y$ for all $y \in L$, then $\sim x \le \nabla \sim x$ so by (1) and M6): $\sim \nabla \sim x \le \sim\sim x = x$.

By (2) and M9) it follows that

$$(3) \quad \sim \nabla \sim x \le x \le \nabla x.$$

Considering (3), from M13) it follows that

$$\nabla x \vee (\nabla x \wedge \nabla \sim x) \vee \sim \nabla x = 1,$$

so

$$(4) \quad \nabla x \vee \sim \nabla x = 1.$$

From (2) and M13) we deduce

$$x \vee (\nabla x \wedge \nabla \sim x) \vee \sim \nabla x = 1$$

this is

$$(x \vee \sim \nabla x \vee \nabla x) \wedge (x \vee \sim \nabla x \vee \nabla \sim x) = 1$$

so by (4)

$$(5) \quad x \vee \sim \nabla x \vee \nabla \sim x = 1.$$

Replacing x by $\sim x$ in (3) we have that $\sim \nabla x \leq \nabla \sim x$ and therefore we have finally that $\sim x \vee \nabla x = 1$, which concludes the proof of the equivalence of the two definitions.

1.8. New examples

Let us recall the following definition: A pair $(B, \exists)$ formed by a boolean algebra B and a unary operator $\exists$, called an *existential quantifier*, defined over B is said to be a monadic boolean algebra P. R. Halmos [21, 22], A. and L. Monteiro, [50] if the following hold:

EQ0) $\exists 0 = 0$,
EQ1) $x \wedge \exists x = x$,
EQ2) $\exists (x \wedge \exists y) = \exists x \wedge \exists y$.

It is well known that in every monadic boolean algebra the following identities hold:

EQ3) $\exists 1 = 1$,
EQ4) $\exists\exists x = \exists x$,
EQ5) If $x \leq y$ then $\exists x \leq \exists y$.

The *discrete existential quantifier* on a boolean algebra is given by $\exists x = x$ for every $x \in B$. The *simple existential quantifier* is given by $\exists x = 1$ for every $x \neq 0$ and $\exists 0 = 0$.

In a monadic boolean algebra we denominate *universal quantifier* the operator defined by $\forall x = -\exists - x$, where $-x$ is the boolean complement of x.

The problem is posed of determining whether there exist Łukasiewicz algebras whose elements are subsets of a given set.

Example 1.8.1. *Let I be a non-empty set and φ an involution on I, this is a function from I to I such that $\varphi(\varphi(x)) = x$ for all $x \in I$. Clearly φ is a bijection with $\varphi^{-1} = \varphi$. We know that $(\mathcal{P}(I), I, \complement, \cap, \cup)$ is a boolean algebra.*

We define an operator $\boldsymbol{\nabla}$ over $\mathcal{P}(I)$ as follows:

St1) $\boldsymbol{\nabla}\emptyset = \emptyset$,
St2) *If $i \in I$ then $\boldsymbol{\nabla}\{i\} = \{i, \varphi(i)\}$,*
St3) *If $X \in \mathcal{P}(I)$, $X \neq \emptyset$ then $\boldsymbol{\nabla} X = \bigcup_{i \in X} \boldsymbol{\nabla}\{i\}$.*

Note that $\boldsymbol{\nabla}\{i\} = \boldsymbol{\nabla}\{\varphi(i)\}$ and that if $X \in \mathcal{P}(I)$ then

$$\boldsymbol{\nabla} X = \bigcup_{i \in X} \{i, \varphi(i)\} = \bigcup_{i \in X} \{i\} \cup \bigcup_{i \in X} \{\varphi(i)\} = X \cup \varphi(X).$$

The operator $\boldsymbol{\nabla}$ has the following properties:

St4) $\underline{X \subseteq \nabla X, \textit{ for all } X \in \mathcal{P}(I).}$
$\nabla X = X \cup \varphi(X) \supseteq X.$

St5) $\underline{\nabla(X \cap \nabla Y) = \nabla X \cap \nabla Y, \textit{ for all } X, Y \in \mathcal{P}(I).}$
$\nabla(X \cap \nabla Y) = (X \cap \nabla Y) \cup \varphi(X \cap \nabla Y) = (X \cap \nabla Y) \cup \varphi(X \cap (Y \cup \varphi(Y)) =$
$(X \cap \nabla Y) \cup (\varphi(X) \cap (\varphi(Y) \cup Y)) = (X \cap \nabla Y) \cup (\varphi(X) \cap \nabla Y) = (X \cup \varphi(X)) \cap \nabla Y =$
$\nabla X \cap \nabla Y.$

From St1), St4) *and* St5) *it follows that* $(\mathcal{P}(I), \nabla)$ *is a monadic boolean algebra.*

For each $X \in \mathcal{P}(I) = 2^I$ *we put* $\sim X = \complement\varphi(X)$. *Since* φ *is a bijection on* E, *then:*

St6) $\varphi(\complement X) = \complement\varphi(X)$, *for all* $X \in 2^I$.
St7) $\varphi(X \cap Y) = \varphi(X) \cap \varphi(Y)$, *for all* $X, Y \in 2^I$.
St8) $\varphi(X \cup Y) = \varphi(X) \cup \varphi(Y)$, *for all* $X, Y \in 2^I$.

Then, (see for instance [7], [51]*), it is well known that:*

St9) $\sim\sim X = X$, *for all* $X \in 2^I$.
St10) $\sim (X \cap Y) = \sim X \cup \sim Y$, *for all* $X, Y \in 2^I$.
St11) $\sim I = \emptyset$.

Therefore the system $(2^I, \cap, \cup, \sim, I)$ *is a De Morgan algebra.*

We shall prove next that the operator ∇ *also verifies axioms L6) and L7).*
<u>L6)</u> $\underline{\sim X \cup \nabla X = I, \textit{ for all } X \in \mathcal{P}(I).}$
Indeed $\nabla X = \bigcup_{i \in X} \nabla\{i\} = \bigcup_{i \in X} (\{i\} \cup \{\varphi(i)\}) = \bigcup_{i \in X} \{i\} \cup \bigcup_{i \in X} \{\varphi(i)\} = X \cup \varphi(X)$.
Then $\sim X \cup \nabla X = \complement\varphi(X) \cup X \cup \varphi(X) = I$.

<u>L7)</u> $\underline{X \cap \sim X = \sim X \cap \nabla X, \textit{ for all } X \in \mathcal{P}(I).}$
$\sim X \cap \nabla X = \complement\varphi(X) \cap (X \cup \varphi(X)) = (\complement\varphi(X) \cap X) \cup (\complement\varphi(X) \cap \varphi(X)) = (\complement\varphi(X) \cap X) \cup \emptyset = \complement\varphi(X) \cap X = \sim X \cap X.$

For the system $(\mathcal{P}(I), I, \sim, \nabla, \cap, \cup)$ *to be a Łukasiewicz algebra it is necessary and sufficient that :*

L8) $\nabla(X \cap Y) = \nabla X \cap \nabla Y$, *for all* $X, Y \in \mathcal{P}(I)$.

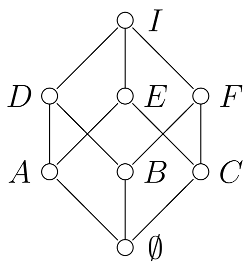
We show now that L8) holds if and only if $\varphi(i) = i$ *for all* $i \in I$, *and therefore* $\sim X = \complement X$ *and* $\nabla X = X \cup \varphi(X) = X$ *for all* $X \in \mathcal{P}(I)$, *so in fact* $(\mathcal{P}(I), \nabla)$ *is a monadic boolean algebra with the discrete quantifier.*

Assume that there exists $i \in I$ *such that* $j = \varphi(i) \neq i$. *Then* $\nabla(\{i\} \cap \{j\}) = \nabla\emptyset = \emptyset$, $\nabla\{i\} = \{i\} \cup \{\varphi(i)\} = \{i\} \cup \{j\} = \{i, j\}$ *and* $\nabla\{j\} = \{j\} \cup \{\varphi(j)\} = \{j\} \cup \{i\} = \{i, j\}$. *Then* $\nabla\{i\} \cap \nabla\{j\} = \{i, j\} \neq \emptyset$ *and therefore L8) does not hold. Assume now that* $\varphi(i) = i$ *for all* $i \in I$ *then it clear that L8) holds.*

Therefore the system $(\mathcal{P}(I), I, \sim, \nabla, \cap, \cup)$ *is not in general a Łukasiewicz algebra, but there may exist subsets* S *of* $\mathcal{P}(I)$ *such that* $(S, I, \sim, \nabla, \cap, \cup)$ *is a*

Łukasiewicz algebra. As an example let $I = \{a, b, c\}$ *and* $\varphi : I \to I$ *be defined by* $\varphi(a) = a, \varphi(b) = c, \varphi(c) = b$.

$\mathcal{P}(I)$ *is a boolean algebra with 3 atoms with the diagram shown below, where* $A = \{a\}, B = \{b\}, C = \{c\}, D = \{a, b\}, E = \{a, c\}, F = \{b, c\}$.



X	$\sim X$	∇X
$\emptyset$	I	$\emptyset$
A	F	A
B	D	F
C	E	F
D	B	I
E	C	I
F	A	F
I	$\emptyset$	I

Since φ *is not the identity on* I, $\mathcal{P}(I)$ *is not a Łukasiewicz algebra but it is easy to check that the subset* $S = \{\emptyset, A, C, E, F, I\}$ *is a Łukasiewicz algebra.*

Definition 1.8.2. *If* I *is a non-empty set,* φ *an involution on* I*, and for each* $X \in \mathcal{P}(I)$ *we define the operators* $\sim$ *and* ∇ *as before, then every subset* S *of* $\mathcal{P}(I)$ *such that* $(S, I, \sim, \nabla, \cap, \cup)$ *is a Łukasiewicz algebra will be called a* Łukasiewicz algebra of sets determined by φ *or just a* Łukasiewicz algebra of sets.

This example of Łukasiewicz algebra is the most general one, since we will prove later on that every Łukasiewicz algebra is isomorphic to a Łukasiewicz algebra of sets.

Remark 1.8.3. *To consider the boolean algebra* $\mathcal{P}(I)$ *is equivalent to considering the set* B^I *of all the functions from* I *to the boolean algebra* $B = \{0, 1\}$*, which algebrized coordinatewise is a boolean algebra with top element the function* $\mathbf{1}(i) = 1$ *for all* $i \in I$*. Given* $f \in B^I$*, if we define* $(\nabla f)(i) = f(i) \vee f(\varphi(i))$*, for all* $i \in I$*, then it is easy to prove that* (B^I, ∇) *is a monadic boolean algebra.*

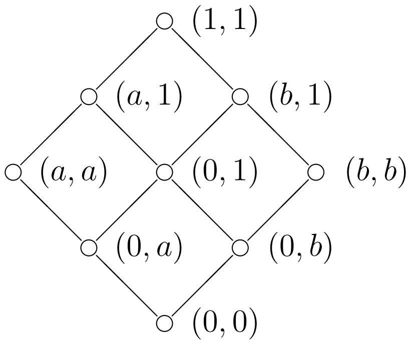
Given $f \in B^I$*, and defining* $(\sim f)(i) = -(f(\varphi(i)))$ *for all* $i \in I$ *then* $(B^I, \mathbf{1}, \sim, \wedge, \vee)$ *is a De Morgan algebra. It is easy to see that axioms L6 and L7 hold. If* $\varphi(i) = j \neq i$*, consider the following elements of* B^I*:*

$$f(x) = \begin{cases} 1 & \text{if } x = i \\ 0 & \text{if } x \neq i \end{cases} \qquad g(x) = \begin{cases} 1 & \text{if } x = \varphi(i) = j \\ 0 & \text{if } x \neq \varphi(i). \end{cases}$$

Then $(f \wedge g)(x) = 0$ *for all* $x \in I$ *and therefore* $(\nabla(f \wedge g))(x) = 0$ *for all* $x \in I$*. In the other hand* $(\nabla f \wedge \nabla g)(x) = 1$ *for* $x = i$ *or* $x = j$*, so in general it doesn't hold that* $\nabla(f \wedge g) = \nabla f \wedge \nabla g$.

Example 1.8.4. *This example is due to Gr. C. Moisil. Let* K *and* $B = \{0, 1\}$ *be boolean algebras and let* $F = K^{[B]}$ *be the set of all the isotone functions from* B *to* K*. Each element* $f \in K^{[B]}$ *can be represented as follows:* $f = (f(0), f(1))$*. We put by definition* $\sim f = (-f(1), -f(0))$ *and* $\nabla f = (f(1), f(1))$*. It is easy to check that* $(K^{[B]}, \mathbf{1}, \sim, \nabla, \wedge, \vee)$ *is a Łukasiewicz algebra.*

Let $K = \{0, a, b, 1\}$ be a boolean algebra with two atoms a and b. The elements of $K^{[B]}$, the operators $\sim, \nabla$ and the Hasse diagram are indicated below:



f	$\sim f$	∇f
$(0,0)$	$(1,1)$	$(0,0)$
$(0,a)$	$(b,1)$	(a,a)
$(0,b)$	$(a,1)$	(b,b)
$(0,1)$	$(0,1)$	$(1,1)$
(a,a)	(b,b)	(a,a)
(b,b)	(a,a)	(b,b)
$(a,1)$	$(0,b)$	$(1,1)$
$(b,1)$	$(0,a)$	$(1,1)$
$(1,1)$	$(0,0)$	$(1,1)$

This same construction was considered by A. Rose in [74].

1.9. 3-rings

Let $A = \{0, 1, 2\}$ be the ring of the integers modulo 3, so the table of the operations $+$ and $\cdot$ are

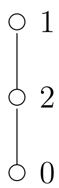
$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Therefore this ring verifies (1) $3x = 0$, (2) $x^3 = x$ and (3) $xy = yx$. Because of condition (2) it follows that every polynomial in two variables is of the form

$$P(x, y) = a + bx + cy + dx^2 + ey^2 + fxy + gx^2y + hxy^2 + ix^2y^2.$$

We want now to define meet and join operations on the set A so that A becomes the chain in the figure below, thus $\wedge$ and $\vee$ are given by the tables that follow:



$\wedge$	0	2	1
0	0	0	0
2	0	2	2
1	0	2	1

$\vee$	0	2	1
0	0	2	1
2	2	2	1
1	1	1	1

To whether there exists a polynomial function that yields the meet operation in this ring, we must solve the following system of equations:

$$(1) \qquad 0 \wedge 0 = 0 = P(0,0) = a,$$

$$(2) \qquad 0 \wedge 2 = 0 = P(0,2) = a + 2c + e,$$

$$(3) \qquad 0 \wedge 1 = 0 = P(0,1) = a + c + e,$$

$$(4) \qquad 2 \wedge 0 = 0 = P(2,0) = a + 2b + d,$$

$$(5) \qquad 2 \wedge 2 = 2 = P(2,2) = a + 2b + 2c + d + e + f + 2g + 2h + i,$$

$$(6) \qquad 2 \wedge 1 = 2 = P(2,1) = a + 2b + c + d + e + 2f + g + 2h + i,$$
$$(7) \qquad 1 \wedge 0 = 0 = P(1,0) = a + b + d,$$
$$(8) \qquad 1 \wedge 2 = 2 = P(1,2) = a + b + 2c + d + e + 2f + 2g + h + i,$$
$$(9) \qquad 1 \wedge 1 = 1 = P(1,1) = a + b + c + d + e + f + g + h + i.$$

By (1) we have $a = 0$ so replacing in the equations (2)-(9) we have:

$$(10) \qquad 0 = 2c + e,$$
$$(11) \qquad 0 = c + e,$$
$$(12) \qquad 0 = 2b + d,$$
$$(13) \qquad 2 = 2b + 2c + d + e + f + 2g + 2h + i,$$
$$(14) \qquad 2 = 2b + c + d + e + 2f + g + 2h + i,$$
$$(15) \qquad 0 = b + d,$$
$$(16) \qquad 2 = b + 2c + d + e + 2f + 2g + h + i,$$
$$(17) \qquad 1 = b + c + d + e + f + g + h + i.$$

Adding (10) and (11) we have $3c + 2e = 0$ this is $2e = 0$ and therefore $e = 0$. Then from (11) it follows that $c = 0$. Analogously, adding (12) and (15) we have $3b + 2d = 0$ and therefore $2d = 0$, so in consequence $d = 0$, and by (15) it follows that $b = 0$.

Replacing these values in (13), (14),(16) and (17) we have

$$(18) \qquad 2 = f + 2g + 2h + i,$$
$$(19) \qquad 2 = 2f + g + 2h + i,$$
$$(20) \qquad 2 = 2f + 2g + h + i,$$
$$(21) \qquad 1 = f + g + h + i.$$

Adding (19) and (21) we obtain $2g + 2i = 0$ and therefore:

$$(22) \qquad 0 = g + i.$$

Adding (18) and (21) we get $2f + 2i = 0$ so:

$$(23) \qquad 0 = f + i.$$

Adding (20) and (21) we get $2h + 2i = 0$ and therefore:

$$(24) \qquad 0 = h + i.$$

From (24) and (21) we have:

$$(25) \qquad 1 = f + g.$$

From the equations (22), (23) and (24) it follows that $i = -g$, $i = -f$ and $i = -h$. Therefore $g = f = h$. Thus from (25) it follows that $1 = f + g = f + f = 2f$ and therefore $f = 2$, so in consequence $h = g = 2$.

Replacing in (21) we have $1 = 2 + 2 + 2 + i$ and therefore $i = 1$. We thus obtain

$$(1.9.1) \qquad x \wedge y = 2xy + 2x^2y + 2xy^2 + x^2y^2 = 2xy + 2xy(x + y) + x^2y^2.$$

Analogously we have:

$$x \vee y = x + y + xy + x^2y + xy^2 + 2x^2y^2. \tag{1.9.2}$$

Performing the corresponding calculations we have that:

$$(x \wedge y)^2 = x^2y^2 = (xy)^2$$

and

$$(x \vee y)^2 = (x^2 + y^2)^2 = x^2 + 2x^2y^2 + y^2.$$

From the definition of the operations $\wedge$ and $\vee$ on A we know A is a distributive lattice given that A is a chain.

We shall define two unary operations on A so that A is *equal* to the Łukasiewicz algebra from Example 1.2.3. Every polynomial in one variable is of the form $P(x) = a + bx + cx^2$. Then if Q is the polynomial corresponding to $\sim$, the following equations must hold:

$$1 = Q(0) = a \tag{26}$$

$$2 = Q(2) = a + 2b + 2^2c = a + 2b + c = 1 + 2b + c \tag{27}$$

$$0 = Q(1) = a + b + c = 1 + b + c \tag{28}$$

Adding (27) and (28) we obtain $2 = 2 + 3b + 2c$ and therefore $3b + 2c = 0$, this is $2c = 0$, so in consequence $c = 0$. Then, replacing in (28) we have $0 = 1+b+0 = 1 + b$ and therefore $b = -1 = 2$, so

$$\sim x = 2x + 1 \tag{1.9.3}$$

If R is the polynomial corresponding to ∇, R must verify:

$$0 = R(0) = a \tag{29}$$

$$1 = R(2) = a + 2b + 2^2c = a + 2b + c = 2b + c \tag{30}$$

$$1 = R(1) = a + b + c = b + c \tag{31}$$

Adding (30) and (31) we obtain $2 = 3b+2c$, this is $2 = 2c$ and therefore $c = 1$, so from (31) it follows that $1 = 1 + b$ and therefore $b = 0$. Thus we obtain

$$\nabla x = x^2. \tag{1.9.4}$$

We now show that $\sim$ and ∇ verify the Łukasiewicz algebra axioms:

L4) $\sim\sim x = x$.

$\sim\sim x = \sim (1 + 2x) = 1 + 2(1 + 2x) = 1 + 2 + x = x$.

L5) $\sim x \vee \sim y = \sim (x \wedge y)$.

$\sim (x \wedge y) = 1 + 2(x \wedge y) = 1 + 2(2xy + 2x^2y + 2xy^2 + x^2y^2) = 1 + xy + x^2y + xy^2 + 2x^2y^2$.

$\sim x \vee \sim y = (1+2x) \vee (1+2y) = 1+2x+1+2y+(1+2x)(1+2y)+(1+x+x^2)(1+2y)+(1+2x)(1+y+y^2)+2(1+x+x^2)(1+y+y^2) = 2+2x+2y+1+2y+2x+xy+1+2y+x+2xy+x^2+2x^2y+1+y+y^2+2x+2xy+2xy^2+2+2y+2y^2+2x+2xy+2xy^2+2x^2+2x^2y+2x^2y^2 = 1+xy+x^2y+xy^2+2x^2y^2$.

L6) $\sim x \vee \nabla x = 1$.
$\sim x \vee \nabla x = (1+2x) \vee x^2 = 1+2x+x^2+x^2+2x+(1+x+x^2)x^2+(1+2x)x^2+2(1+x+x^2)x^2 = 1+x+2x^2+x^2+x+x^2+x^2+2x+2x^2+2x+2x^2 = 1$.

L7) $\sim x \wedge x = \sim x \wedge \nabla x$.
$\sim x \wedge x = 2(1+2x)x+2(1+x+x^2)x+2(1+2x)x^2+(1+x+x^2)x^2 = 2x+x^2+2x+2x^2+2x+2x^2+x+x^2+x+x^2 = 2x+x^2$.
$\sim x \wedge \nabla x = (1+2x) \wedge x^2 = 2(1+2x)x^2+2(1+x+x^2)x^2+2(1+2x)x^2+(1+x+x^2)x^2 = 2x^2+x+2x^2+2x+2x^2+2x^2+x+x^2+x+x^2 = 2x+x^2$.

L8) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$.
We already know that $(x \wedge y)^2 = x^2y^2$ so $\nabla(x \wedge y) = (x \wedge y)^2 = x^2y^2$.
$\nabla x \wedge \nabla y = x^2 \wedge y^2 = 2x^2y^2 + 2x^2y^2 + 2x^2y^2 + x^2y^2 = x^2y^2$

Furthermore, 2 is the center of A since $\sim 2 = 2.2 + 1 = 1 + 1 = 2$.

Let $(A, +, \cdot, 1)$ be a commutative ring with identity. With 0 we denote the identity for addition. We write xy instead of $x \cdot y$. A commutative ring with identity (1) A, verifying $3x = 0$ and $x^3 = x$, for all $x \in A$, is a 3-ring.

Theorem 1.9.1. (Gr. C. Moisil) *If A is a 3-ring, then defining the operations $\wedge$, $\vee$, $\sim$ and ∇ on A by the formulas (1.9.1)), (1.9.2)), (1.9.3)) and (1.9.4)) indicated above, the system $(A, 1, \sim, \nabla, \wedge, \vee)$, is a centered Lukasiewicz algebra.*

Proof. We prove first that Sholander's axioms [75] hold:

$$\text{L2) } x \wedge (x \vee y) = 2x(x \vee y) + 2x^2(x \vee y) + 2x(x \vee y)^2 + x^2(x \vee y)^2.$$

(1) $2x(x \vee y) = 2x(x+y+xy+xy(x+y)+2x^2y^2) = 2x^2+2xy+2x^2y+2x^3y+2x^2y^2+x^3y^2 = 2x^2+xy+xy^2+2x^2y+2x^2y^2$.

(2) $2x^2(x \vee y) = 2x^2(x+y+xy+xy(x+y)+2x^2y^2) = 2x^3+2x^2y+2x^3y+2x^2y+2x^3y^2+x^2y^2 = 2x+2xy+2xy^2+x^2y+x^2y^2$.

From (1) and (2) it follows that (3) $2x(x \vee y)^2 + 2x^2(x \vee y) = 2x + 2x^2$.

(4) $2x(x \vee y)^2 = 2x(x^2+y^2+2x^2y^2) = 2x^3+2xy^2+xy^2 = 2x$.

(5) $x^2(x \vee y)^2 = x^2(x^2+y^2+2x^2y^2) = x^2+x^2y^2+2x^2y^2 = x^2$.

From (3), (4) and (5) we get: $x \wedge (x \vee y) = 2x + 2x^2 + 2x + x^2 = x$.

$$\text{L3) } x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x).$$

$x \wedge (y \vee z) = 2x(y \vee z) + 2x^2(y \vee z) + 2x(y \vee z)^2 + x^2(y \vee z)^2 =$
$2x(y+z+yz+y^2z+yz^2+2y^2z^2)+2x^2(y+z+yz+y^2z+yz^2+2y^2z^2)+$
$2x(y^2+z^2+2y^2z^2)+x^2(y^2+z^2+2y^2z^2) =$
$2xy+2xz+2xyz+2xy^2z+2xyz^2+xy^2z^2+2x^2y+2x^2z+2x^2yz+2x^2y^2z+2x^2yz^2+x^2y^2z^2+2xy^2+2xz^2+xy^2z^2+x^2y^2+x^2z^2+2x^2y^2z^2 = 2xy+2xz+2xy^2+2xz^2+x^2y^2+x^2z^2+2x^2y+2x^2z+2xyz+2xyz^2+2xy^2z+2xy^2z^2+2x^2yz+2x^2y^2z+2x^2yz^2$.

On the other hand, $(z \wedge x) \vee (y \wedge x) = (z \wedge x) + (y \wedge x) + (z \wedge x)(y \wedge x) + (z \wedge x)^2(y \wedge x) + (z \wedge x)(y \wedge x)^2 + 2(z \wedge x)^2(y \wedge x)^2$. We now calculate:

(6) $z \wedge x = 2xz + 2xz^2 + 2x^2z + x^2z^2$,

(7) $y \wedge z = 2xy + 2xy^2 + 2x^2y + x^2y^2$,

(8) $(z \wedge x)(y \wedge x) = 2xyz + xy^2z^2 + 2x^2yz + 2x^2y^2z^2$,

(9) $(z \wedge x)^2(y \wedge x) = 2xyz^2 + 2xy^2z^2 + 2x^2yz^2 + x^2y^2z^2$,

(10) $(z\wedge x)(y\wedge x)^2 = 2xy^2z + 2xy^2z^2 + 2x^2y^2z + x^2y^2z^2$,

(11) $(z\wedge x)^2(y\wedge x)^2 = 2x^2y^2z^2$, so from (6) to (11) it follows that:

$(z\wedge x)\vee(y\wedge x) = 2xy + 2xz + 2xy^2 + 2xz^2 + x^2y^2 + x^2z^2 + 2x^2y + 2x^2z + 2xyz + 2xyz^2 + 2xy^2z + 2xy^2z^2 + 2x^2yz + 2x^2y^2z + 2x^2yz^2$.

We have already checked above the axioms L4)-L8) and that $c = 1+1$ is the center of A. $\square$

Theorem 1.9.2. *If* $(A, 1, \sim, \nabla, \wedge, \vee)$, *is a Łukasiewicz algebra with center* c *and we define:*

$$x+y = (\sim\nabla x\wedge\Delta y)\vee(\sim\nabla y\wedge\Delta x)\vee(\nabla x\wedge\nabla y\wedge\nabla\sim x\wedge\nabla\sim y)\vee$$
$$\Big(c\wedge\Big((\sim\nabla x\wedge\nabla y\wedge\nabla\sim y)\vee(\sim\nabla y\wedge\nabla x\wedge\nabla\sim x)\vee(\Delta x\wedge\Delta y)\Big)\Big),$$
$$x.y = (\Delta x\wedge\Delta y)\vee(\nabla x\wedge\nabla y\wedge\nabla\sim x\wedge\nabla\sim y)\vee$$
$$\Big(c\wedge\Big((\Delta x\wedge\nabla y\wedge\nabla\sim y)\vee(\Delta y\wedge\nabla x\wedge\nabla\sim x)\Big)\Big)$$

then the system $(A, 1, 0, +, \cdot)$ *is a* 3*-ring.* Moisil [27].

If $(A, 1, 0, +, \cdot)$ is a commutative ring with unity 1 such that $x^3 = x$ then $6x = 0$. Indeed:

$x+y = (x+y)(x+y)(x+y) = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$

so

$$x+y = x + xxy + xyx + xyy + yxx + yxy + yyx + y$$

and therefore

$$xxy + xyx + xyy + yxx + yxy + yyx = 0$$

then for $x = y$ we have that $x^3 + x^3 + x^3 + x^3 + x^3 + x^3 = 0$, this is $6x = x + x + x + x + x + x = 0$, so $-x = 5x$.

In a similar way to the one indicated above, we can prove that if in a commutative ring with unity verifying $x^3 = x$ we define:

$$x\wedge y := 2xy + 2x^2y + 2xy^2 + x^2y^2 = 2xy + 2xy(x+y) + x^2y^2$$
$$x\vee y := x + y + xy + x^2y + xy^2 + 2x^2y^2,$$

then $(A, 0, 1, \wedge, \vee)$ is a bounded distributive lattice.

Defining $\sim x = 1 + 5x$ and $\nabla x = x^2$ then $(A, 1, \sim, \nabla, \wedge, \vee)$, is a Łukasiewicz algebra. Furthermore, $e = 1+1$ is the axis of the algebra A.

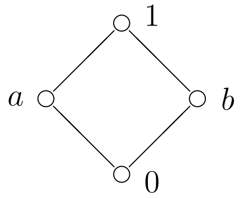
Theorem 1.9.3. *If* $(A, 1, \sim, \nabla, \wedge, \vee)$, *is a Łukasiewicz algebra with axis* e *and we define:*

$$x+y := (\sim\nabla x\wedge\Delta y)\vee(\sim\nabla y\wedge\Delta x)\vee(\nabla x\wedge\nabla y\wedge\nabla\sim x\wedge\nabla\sim y)\vee$$
$$\Big(e\wedge\Big((\sim\nabla x\wedge\nabla y\wedge\nabla\sim y)\vee(\sim\nabla y\wedge\nabla x\wedge\nabla\sim x)\vee(\Delta x\wedge\Delta y)\Big)\Big),$$
$$x\cdot y := (\Delta x\wedge\Delta y)\vee(\nabla x\wedge\nabla y\wedge\nabla\sim x\wedge\nabla\sim y)\vee$$
$$\Big(e\wedge\Big((\Delta x\wedge\nabla y\wedge\nabla\sim y)\vee(\Delta y\wedge\nabla x\wedge\nabla\sim x)\Big)\Big)$$

then the system $(A, 1, 0, +, \cdot)$ *is a commutative ring with unity such that* $x^3 = x$, Gr. C. Moisil [27].

1.10. Construction of Łukasiewicz algebras from monadic boolean algebras

In the preceding sections we have found analogies between monadic boolean algebras and Łukasiewicz algebras, but there are also fundamental differences. As an example, Moisil's determination principle does not hold in monadic boolean algebras. To see this, consider the monadic boolean algebra in the next figure:



x	$\exists x$	$\forall x$
0	0	0
a	1	0
b	1	0
1	1	1

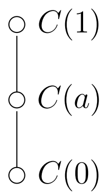
Here we have that $\exists a = \exists b = 1$, $\forall a = \forall b = 0$ and $a \neq b$.

This example suggests the idea of identifying two elements a and b of a monadic boolean algebra if :

$$\exists a = \exists b \text{ and } \forall a = \forall b,$$

and in this case denote $a \equiv b$.

It is clear that this is an equivalence relation. In the previous example there exist three equivalence classes, namely $C(0) = \{0\}, C(a) = \{a, b\}$ and $C(1) = \{1\}$. We consider the quotient set $A' = A/\equiv$. It is natural to think that A' is the chain indicated below.



We already know, as we have shown before, that on the preceding set a Łukasiewicz algebra structure can be defined.

Notice that the "≡" relation is not compatible with the "∨" operation defined on A. Indeed, $a \equiv a$ and $a \equiv b$ but $a \vee a = a \not\equiv 1 = a \vee b$. This means that the *join* operation in A' must be defined in a special way. A. Monteiro determined a general construction, which he called construction $\mathcal{L}$, and which allows to define a Łukasiewicz algebra $\mathcal{L}(A)$ from a monadic boolean algebra A. By analogy with the Łukasiewicz algebras, if A is a monadic boolean algebra, we can define the *weak implication* by:

$$x \to y = \exists - x \vee y,$$

and the *contraposed implication* by

$$x \rightarrowtail y = (x \to y) \wedge (-y \to -x).$$

Then it is natural to define in A the following operations:

$$x \cup y = (x \rightarrowtail y) \rightarrowtail y$$

$$x \cap y = -(-x \cup -y).$$

Notice that $x \rightarrowtail y = (\exists - x \vee y) \wedge (\exists y \vee -x)$, so

$$x \cup y = (x \rightarrowtail y) \rightarrowtail y = (\exists - (x \rightarrowtail y) \vee y) \wedge (\exists y \vee -(x \rightarrowtail y)) =$$

$$((\forall x \wedge \exists - y) \vee (\forall - y \wedge \exists x) \vee y) \wedge (\exists y \vee (\forall x \wedge -y) \vee (\forall - y \wedge x)) =$$

$$(\forall x \wedge \exists - y \wedge \exists y) \vee (\forall x \wedge \exists - y \wedge \forall x \wedge -y) \vee (\forall x \wedge \exists - y \wedge \forall - y \wedge x) \vee$$

$$(\forall - y \wedge \exists x \wedge \exists y) \vee (\forall - y \wedge \exists x \wedge \forall x \wedge -y) \vee (\forall - y \wedge \exists x \wedge \forall - y \wedge x) \vee$$

$$(y \wedge \exists y) \vee (y \wedge \forall x \wedge -y) \vee (y \wedge \forall - y \wedge x) =$$

$$(\forall x \wedge \exists - y \wedge \exists y) \vee (\forall x \wedge -y) \vee (\forall x \wedge \forall - y) \vee$$

$$0 \vee (\forall - y \wedge \forall x) \vee (\forall - y \wedge x) \vee y \vee 0 \vee 0 =$$

$$(\forall x \wedge \exists - y \wedge \exists y) \vee (\forall x \wedge -y) \vee (\forall x \wedge \forall - y) \vee (\forall - y \wedge x) \vee y$$

and since $\forall x \wedge \forall - y \leq \forall x \wedge -y$ we have that

$$x \cup y = (\forall x \wedge \exists - y \wedge \exists y) \vee (\forall x \wedge -y) \vee (\forall - y \wedge x) \vee y =$$

$$(\forall x \wedge \exists - y \wedge \exists y) \vee (\forall - y \wedge x) \vee ((\forall x \vee y) \wedge (-y \vee y)) =$$

$$(\forall x \wedge \exists - y \wedge \exists y) \vee (\forall - y \wedge x) \vee ((\forall x \vee y) \wedge 1) =$$

$$(\forall x \wedge \exists - y \wedge \exists y) \vee (\forall - y \wedge x) \vee \forall x \vee y$$

and since $\forall x \wedge \exists - y \wedge \exists y \leq \forall x$ then

$$x \cup y = \forall x \vee y \vee (x \wedge \forall - y) =$$

$$(\forall x \vee y \vee x) \wedge (\forall x \vee y \vee \forall - y) = (x \vee y) \wedge (\forall x \vee y \vee \forall - y).$$

Therefore we have that:

U1) $x \cup y = \forall x \vee y \vee (x \wedge \forall - y)$,
U2) $x \cup y = (x \vee y) \wedge (\forall x \vee y \vee \forall - y)$.

Then $x \cap y = -(-x \cup -y) = -(\forall -x \vee -y \vee (-x \wedge \forall y)) = -\forall - x \wedge y \wedge (x \vee -\forall y) = \exists x \wedge y \wedge (x \vee \exists - y) = (\exists x \wedge y \wedge x) \vee (\exists x \wedge y \wedge \exists - y) = (x \wedge y) \vee (\exists x \wedge y \wedge \exists - y) = y \wedge (x \vee (\exists x \wedge \exists - y)) = y \wedge (x \vee \exists x) \wedge (x \wedge \exists - y) = \exists x \wedge y \wedge (x \vee \exists - y)$.

Therefore we have that:

C1) $x \cap y = \exists x \wedge y \wedge (x \vee \exists - y)$,
C2) $x \cap y = (x \wedge y) \vee (\exists x \wedge y \wedge \exists - y)$,

Definition 1.10.1. *Given two elements a and b of a monadic boolean algebra A we shall say that they are equivalent and denote $a \equiv b$ if $a \rightarrowtail b = 1$ and $b \rightarrowtail a = 1$.*

Lemma 1.10.2. a) *$a \rightarrowtail b = 1$ and $b \rightarrowtail a = 1$ is equivalent to* b) *$\exists a = \exists b$ and $\forall a = \forall b$.*

PROOF. Since $a \rightarrowtail b = (a \to b) \wedge (-b \to -a) = (\exists - a \vee b) \wedge (\exists b \vee -a)$ and $b \rightarrowtail a = (b \to a) \wedge (-a \to -b) = (\exists - b \vee a) \wedge (\exists a \vee -b)$ then if a) holds we have that $\exists - a \vee b = 1$, $\exists b \vee -a = 1$, $\exists - b \vee a = 1$, $\exists a \vee -b = 1$ and this is equivalent to (1) $\forall a \leq b$, (2) $\forall b \leq a$, (3) $b \leq \exists a$, and (4) $a \leq \exists b$. (1) and (2) are equivalent to $\forall a \leq \forall b$ and $\forall b \leq \forall a$, then $\forall a = \forall b$. (3) and (4) are equivalent to $\exists b \leq \exists a$, and $\exists b \leq \exists a$, this is $\exists a = \exists b$.

Now we assume that b) holds. Then $a \to b = \exists - a \vee b = -\forall a \vee b = -\forall b \vee b$. Since $\forall b \leq b$, $1 = -b \vee b \leq -\forall b \vee b$, so $a \to b = 1$. In a similar way, $-b \to -a = \exists a \vee -a = 1$ so $a \rightarrowtail b = 1$. An analogous calculation shows that $b \rightarrowtail a = 1$ as well. □

To prove that the relation $\equiv$ is compatible with the operations $\cap$ and $\cup$, L. Monteiro [70] proved that:

Lemma 1.10.3. *(1) $\exists(x \cap y) = \exists x \wedge \exists y$, (2) $\exists(x \cup y) = \exists x \vee \exists y$, (3) $\forall(x \cap y) = \forall x \wedge \forall y$, (4) $\forall(x \cup y) = \forall x \vee \forall y$.*

The relation "$\equiv$" is an equivalence relation compatible with the operations $-$, $\exists$, $\cap$ and $\cup$. Consider the quotient set $\mathcal{L}(A) = A/_{\equiv}$, and represent by $C(x)$ the equivalence class containing the element $x \in A$, then if we define $I = C(1), \sim C(x) = C(-x), \nabla C(x) = C(\exists x), C(x) \cap C(y) = C(x \cap y)$, and $C(x) \cup C(y) = C(x \cup y)$, then we have the following theorem by A. Monteiro:

If $(A, \exists)$ is a monadic boolean algebra then the system $(\mathcal{L}(A), I, \sim, \nabla, \cap, \cup)$ is a Łukasiewicz algebra.

Notes

1) The proof of the preceding theorem was only published in 1967, [32], but the results were presented in the course given in 1963, [36] and in [40].
2) In the proof, Professor A. Monteiro used the theory of N-lattices, (see H. Rasiowa [72]) and in particular his results on semi-simple N-lattices.
3) The results on semi-simple N-lattices were only published in 1995 in the series *Informes Técnicos Internos*[2] No. 50 from the INMABB[3], and later in 1996, in the *Notas de Lógica Matemática*[4] No. 40, [48] also published by the INMABB.
4) Since in the statement of the preceding theorem only notions from the theory of monadic boolean algebras appear, Professor A. Monteiro posed to his students the problem of finding a proof without using the results on N-lattices.
5) This problem was solved by L. Monteiro and L. González Cóppola and published in 1964, [70].
6) Later on, Professor A. Monteiro [40] proved this result: Given a Łukasiewicz algebra L, there exists a monadic boolean algebra A such that

[2]Internal technical reports of the INMABB:
`http://inmabb-conicet.gob.ar/publicaciones/iti`
[3]Mathematics Institute of Bahía Blanca, Argentina.
[4]Notes on Mathematical Logic: `http://inmabb-conicet.gob.ar/publicaciones/nlm`

$\mathcal{L}(A)$ is isomorphic to L. This was presented in a 1966 seminar [44]. The proof, which will be presented in section 5.7, a representation theorem of Łukasiewicz algebras by Łukasiewicz algebras of sets, which will figure in section 5.6, and which uses the Axiom of Choice. This representation theorem was published in 1995 in No. 45 of the *Informes Técnicos Internos* from the INMABB, and later in 1996, in *Notas de Lógica Matemática* No. 40 [48].

7) L. Monteiro produced a purely algebraic proof of the preceding result and presented it in a Seminar conducted by Professor A. Monteiro [44], held in 1966. This proof was published in 1978, [65].

Remark 1.10.4. *In a monadic boolean algebra A, we denote*

$$K(A) = \{x \in A : \exists x = x\}.$$

If $k \in K(A)$, this is, if $\exists k = k$ then we have that $\nabla C(k) = C(\exists k) = C(k)$, so $C(k) \in B(\mathcal{L}(A))$. Conversely, if $C(k) \in B(\mathcal{L}(A))$, this is, $\nabla C(k) = C(k)$, and therefore $C(\exists k) = C(k)$, then $\exists k \equiv k$, from where in particular it results that $\forall\exists k = \forall k$, so $\exists k = \forall k$ and therefore $\exists k = k$ thus $k \in K(A)$.

Let $k_1, k_2 \in K(A)$ and assume that $C(k_1) = C(k_2)$, so $k_1 \equiv k_2$, therefore $k_1 = \exists k_1 = \exists k_2 = k_2$.

Then, if A is finite, we have that $|B(\mathcal{L}(A))| = |K(A)|$. Furthermore the boolean algebras $B(\mathcal{L}(A))$ and $K(A)$ are isomorphic. Indeed, if $k_1, k_2 \in K(A)$ are such that $k_1 \leq k_2$ then $k_1 \vee k_2 = k_2$ and therefore by Lemma 1.10.3*, $\exists(k_1 \cup k_2) = \exists k_1 \vee \exists k_2 = k_1 \vee k_2 = k_2$ and analogously $\forall(k_1 \cup k_2) = k_2$, so $k_1 \cup k_2 \equiv k_2$ and therefore $C(k_1) \cup C(k_2) = C(k_1 \cup k_2) = C(k_2)$, this is $C(k_1) \leq C(k_2)$. Conversely, if $C(k_1) \leq C(k_2)$, this is, $C(k_1 \cup k_2) = C(k_1) \cup C(k_2) = C(k_2)$, so $k_1 \cup k_2 \equiv k_2$ and therefore in particular $\exists(k_1 \cup k_2) = \exists k_2$, this is, $k_1 \cup k_2 = k_2$, so $k_1 \leq k_2$.*

Assume that A is a finite, non trivial monadic boolean algebra, and denote by $\mathcal{A}(A)$ the set of all the atoms of A.

If $x \in A$ we write $(C(x)] = \{C(y) \in \mathcal{L}(A) : C(y) \leq C(x)\}$.

Lemma 1.10.5. *If $a \in \mathcal{A}(A) \cap K(A)$ then $(C(a)] = \{C(0), C(a)\}$.*

PROOF. It is clear that $\{C(0), C(a)\} \subseteq (C(a)]$. From $a \in K(A)$ it follows that $C(a) \in B(\mathcal{L}(A))$. If $C(x) \in (C(a)]$, this is, $C(x) \leq C(a)$, then $C(x \cap a) = C(x) \cap C(a) = C(x)$ so $x \cap a \equiv x$ and in particular $\exists(x \cap a) = \exists x$, then by Lemma 1.10.3 $\exists x \wedge \exists a = \exists x$ and since $a \in K(A)$, we have that $\exists x \wedge a = \exists x$. Then $0 \leq \exists x \leq a$ whence since a is an atom of A it follows that (1) $\exists x = 0$ or (2) $\exists x = a$. If (1) holds then $x = 0$ and if (2) holds, since $0 \leq x \leq \exists x = a$ and a is an atom, we have that $x = 0$ or $x = a$. ☐

Lemma 1.10.6. *If $k \in K(A)$ y $(C(k)] = \{C(0), C(k)\}$ then k is an atom of A.*

PROOF. Assume that (1) $0 \leq y \leq k$, where $y \in A$, so $C(0) \leq C(y) \leq C(k)$ and therefore $C(y) = C(0)$ or $C(y) = C(k)$, this is (2) $y \equiv 0$ or (3) $y \equiv k$. If (2) holds, then in particular $0 = \exists 0 = \exists y \geq y$ so $y = 0$. If (3) holds then in particular (4) $k = \forall k = \forall y \leq y$. Then from (1) and (4) it follows that $y = k$. ☐

Lemma 1.10.7. *If $a \in \mathcal{A}(A)$ and $a \notin K(A)$ then $\exists a$ is an atom of $K(A)$.*

PROOF. Let $y \in \mathcal{A}(A)$ such that $0 \leq y \leq \exists a$. Then $y = \exists y \leq \exists a$ so (1) $\exists y = \exists y \wedge \exists a = \exists(a \wedge \exists y) = \exists(a \wedge y)$. Since $a \in \mathcal{A}(A)$, we have $a \wedge y = 0$ or $a \wedge y = a$. In the first case, from (1) it follows that $y = \exists y = \exists 0 = 0$ and in the second case, (1) implies that $y = \exists y = \exists a$. Therefore, $\exists a$ is an atom of $K(A)$. $\square$

Lemma 1.10.8. *If $a \in \mathcal{A}(A)$ and $a \notin K(A)$ then*

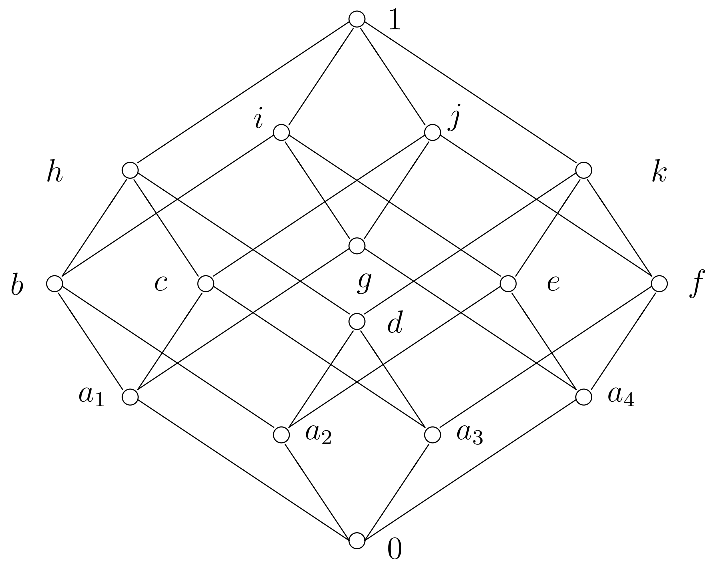
$$(C(\exists a)] = \{C(0), C(a), C(\exists a)\}.$$

PROOF. Since $0 \leq a \leq \exists a$, then $C(0) \leq C(a) \leq C(\exists a)$ so $\{C(0), C(a), C(\exists a)\} \subseteq (C(\exists a)]$.

Assume now that $C(x) \in (C(\exists a)]$, this is, $C(x) \leq C(\exists a)$ and therefore $C(x \cup \exists a) = C(x) \cup C(\exists a) = C(\exists a)$ then $x \cup \exists a \equiv \exists a$ and in particular $\exists x \vee \exists a = \exists(x \cup \exists a) = \exists a$. Therefore $0 \leq \exists x \leq \exists a$. By Lemma 1.10.7, $\exists a$ is an atom of $K(A)$ and since $\exists x \in K(A)$ we have that (1) $\exists x = 0$ or (2) $\exists x = \exists a$. If (1) holds then $x = 0$ and therefore $C(x) = C(0)$. If (2) holds then since $0 \leq \forall x \leq \exists x = \exists a$ and $\exists a$ is an atom of $K(A)$ and $\forall x \in K(A)$ we have that (3) $\forall x = 0$ or (4) $\forall x = \exists a$.

Since $a \in \mathcal{A}(A)$ and $a \notin K(A)$ then $\forall a = 0$, so if (3) holds we have that (5) $\forall x = 0 = \forall a$. From (5) and (2) it follows that $x \equiv a$ and therefore $C(x) = C(a)$. If (4) holds then $C(\forall x) = C(\exists a) = C(\exists x)$, this is $\Delta C(x) = \nabla C(x)$ and therefore $C(x) \in B(\mathcal{L}(A))$ so $\nabla C(x) = C(x)$, this is, $C(\exists x) = C(x)$ and therefore $x \equiv \exists x$. Then in particular $\forall x = \exists x$ and consequently $x \in K(A)$, this is, (6) $\forall x = x$. From (6) and (2) it follows that $x = \exists a$ and therefore $C(x) = C(\exists a)$. $\square$

Example 1.10.9. *Consider the following monadic boolean algebra A*

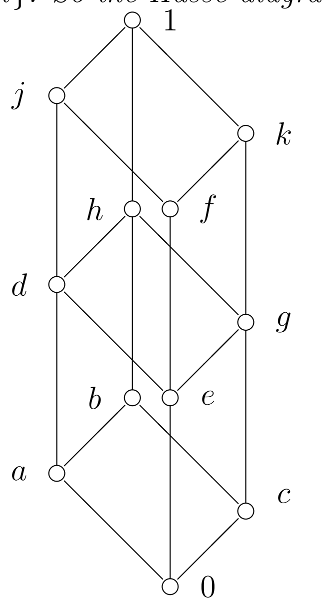


where the operators $\exists$ and $\forall$ are given in the table

x	0	a_1	a_2	a_3	a_4	b	c	d	e	f	g	h	i	j	k	1
$\exists x$	0	a_1	a_2	f	f	b	j	k	k	f	j	1	1	j	k	1
$\forall x$	0	a_1	a_2	0	0	b	a_1	a_2	a_2	f	a_1	b	b	j	k	1

then the elements of $\mathcal{L}(A)$ *are* $0 = C(0) = \{0\}, a = C(a_1) = \{a_1\}, b = C(b) = \{b\}, c = C(a_2) = \{a_2\}, j = C(j) = \{j\}, k = C(k) = \{k\}, f = C(f) = \{f\}$,

$1 = C(1) = \{1\}$ *y* $e = C(a_3) = \{a_3, a_4\}, d = C(c) = \{c, g\}, g = C(d) = \{d, e\}, h = C(h) = \{h, i\}$. *So the Hasse diagram of* $\mathcal{L}(A)$ *is the following:*



and the operations $\sim$ *and* ∇ *are given by:*

x	0	a	b	c	d	e	g	h	j	f	k	1
$\sim x$	1	k	f	j	g	h	d	e	c	b	a	0
∇x	0	a	b	c	j	f	k	1	j	f	k	1

Remark 1.10.10. *The monadic boolean algebra above is the monadic boolean algebra with a free generator* [22], [64] *and* $\mathcal{L}(A)$ *is, as we shall see later on, the Łukasiewicz algebra with a free generator. This example leads us to conjecture that:*

If $M(G)$ *is monadic boolean algebra with a set* G *of free generators then* $\mathcal{L}(M(G))$ *is the Łukasiewicz algebra with a set of free generators with the same cardinality as* G.

We shall see later that this is true only if G is a finite set with a single element.

1.11. Subalgebras

The concept of subalgebra allows to obtain new Łukasiewicz algebras from a given Łukasiewicz algebra, according to this definition: A non-empty part S of a Łukasiewicz algebra A is said to be a *L-subalgebra* of A if it is invariant with respect to the operations $\sim, \nabla$ and $\vee$.

Lemma 1.11.1. *Every L-subalgebra of a Łukasiewicz algebra is a Łukasiewicz algebra.*

It is easy to check that an L-subalgebra of A may be defined as a non-empty part S of A that is invariant with respect to any of the following groups of operations: (1) $\sim, \nabla, \wedge$, (2) $\sim, \Delta, \vee$, (3) $\sim, \Delta, \wedge$.

For that it is enough to consider that $\nabla x =\sim \Delta \sim x, \Delta x =\sim \nabla \sim x$ and the De Morgan laws. It is clear that if S is an L-subalgebra, then $0, 1 \in S$.

Therefore if S is an L-subalgebra de A we have:

$$\{0, 1\} \subseteq S \subseteq A.$$

It is clear that the intersection of L-subalgebras of A is an L-subalgebra of A. The notion of L-subalgebra generated by a part G of an algebra B, which we will denote by $LS(G)$, is defined as usual and one proves that $LS(G)$ is the least L-subalgebra of A containing G. It is clear that $LS(\emptyset) = \{0, 1\}$. If $LS(G) = A$, then G is said to be *a set of generators of* A.

If $G \subseteq A$ and we denote with $\mathcal{F}P(G)$ the family of all the finite parts of G, then:

Lemma 1.11.2. *$LS(G) = \bigcup\{LS(F) : F \in \mathcal{F}P(G)\}$.*

Proof. Let $X = \bigcup\{LS(F) : F \in \mathcal{F}P(G)\}$. If $F \subseteq G$, since $G \subseteq LS(G)$, then $F \subseteq LS(G)$ and therefore $LS(F) \subseteq LS(G)$ for all subsets of G, in particular for all $F \in \mathcal{F}P(G)$, so: (i) $X \subseteq LS(G)$. We prove now that (ii) $LS(G) \subseteq X$. In order to do that, we prove first (1) $G \subseteq X$ and (2) X is an L-subalgebra of A.

(1) Let $g \in G$ then $\{g\} \subseteq LS(\{g\}) \subseteq X$, and therefore $G = \bigcup\limits_{g \in G} \{g\} \subseteq X$.

(2) Since $0, 1 \in LS(F)$ for all $F \in \mathcal{F}P(G)$, then $0, 1 \in X$. Let $x, y \in X$ so $x \in LS(F_1)$, $y \in LS(F_2)$ where $F_1, F_2 \in \mathcal{F}P(G)$, therefore $F_1 \cup F_2 \in \mathcal{F}P(G)$. From $F_1 \subseteq F_1 \cup F_2$ and $F_2 \subseteq F_1 \cup F_2$ it follows that $LS(F_1) \subseteq LS(F_1 \cup F_2)$ and $LS(F_2) \subseteq LS(F_1 \cup F_2)$, then $x, y \in LS(F_1 \cup F_2)$ and therefore $x \wedge y, x \vee y \in LS(F_1 \cup F_2) \subseteq X$. It is clear that if $x \in X$ then $\sim x, \nabla x \in X$. $\square$

If G has a single element $G = \{g\}$, we write $LS(g)$ instead $LS(\{g\})$ and if $G = Y \cup \{x\}$ we write $LS(Y, x)$ instead of $LS(Y \cup \{x\})$.

Notice that if $G = \{g\}$ then

$$0, 1, g, \sim g, g\wedge \sim g, \nabla(g\wedge \sim g), \nabla g, \nabla \sim g, \Delta g, \Delta \sim g, g\vee \sim g, \Delta(g\vee \sim g) \in LS(g).$$

Further on we will see that if $G = \{g\}$ then $N[LS(g)] \leq 12$.

Professor A. Monteiro, posed the problem of finding a "simple" expression for the elements of $LS(S, g)$.

Lemma 1.11.3. (L. Monteiro) *If A is a Łukasiewicz algebra, S an L-subalgebra of A, and $g \in A$, then*

$$LS(S, g) = \{(s_1\wedge\Delta g)\vee(s_2\wedge\Delta \sim g)\vee(s_3\wedge\nabla g\wedge\nabla \sim g)\vee(s_4\wedge g\wedge \sim g) : s_1, s_2, s_3, s_4 \in S\}.$$

Proof. Let

$$S_0 = \{(s_1\wedge\Delta g)\vee(s_2\wedge\Delta \sim g)\vee(s_3\wedge\nabla g\wedge\nabla \sim g)\vee(s_4\wedge g\wedge \sim g) : s_1, s_2, s_3, s_4 \in S\}.$$

(i) $\underline{S_0 \subseteq LS(S, g).}$

Indeed, if $y \in S_0$, then

$$(1)\ y = (s_1 \wedge \Delta g) \vee (s_2 \wedge \Delta \sim g) \vee (s_3 \wedge \nabla g \wedge \nabla \sim g) \vee (s_4 \wedge g\wedge \sim g)$$

where (2) $s_1, s_2, s_3, s_4 \in S$. From (2) it follows that (3) $s_1, s_2, s_3, s_4 \in S \cup \{g\} \subseteq LS(S, g)$. Since $g \in S \cup \{g\} \subseteq LS(S, g)$ then (4) $\Delta g, \Delta \sim g, \nabla g \wedge \nabla \sim g, g\wedge \sim$

$g \in LS(S,g)$. From (3), (4) and (1) it follows that $y \in LS(S,g)$.
(ii) $LS(S,g) \subseteq S_0$.

We shall prove that:
(iii) $S \cup \{g\} \subseteq S_0$, and (iv) S_0 is an L-subalgebra of L.

We begin by noticing that since:
$\Delta g \vee \Delta \sim g \vee (\nabla g \wedge \nabla \sim g) \vee (g \wedge \sim g) = \Delta g \vee \Delta \sim g \vee (\nabla g \wedge \nabla \sim g) =$
$(\Delta g \vee \Delta \sim g \vee \nabla g) \wedge (\Delta g \vee \Delta \sim g \vee \nabla \sim g) = 1 \wedge 1 = 1$, then for all $x \in A$, we have that

$$x = x \wedge 1 = (x \wedge \Delta g) \vee (x \wedge \Delta \sim g) \vee (x \wedge \nabla g \wedge \nabla \sim g) \vee (x \wedge g \wedge \sim g),$$

thus if $s \in S$ we have that $s \in S_0$ and therefore (1) $S \subseteq S_0$. Since

$$(1 \wedge \Delta g) \vee (0 \wedge \Delta \sim g) \vee (0 \wedge \nabla g \wedge \nabla \sim g) \vee (1 \wedge g \wedge \sim g) =$$
$$\Delta g \vee (g \wedge \sim g) = (\Delta g \vee g) \wedge (\Delta g \vee \sim g) = g \wedge (g \vee \sim g) = g$$

then (2) $g \in S_0$. From (1) and (2), (iii) follows. Let us prove (iv) next.

Let $x, y \in S_0$, this is

$$x = (s_1 \wedge \Delta g) \vee (s_2 \wedge \Delta \sim g) \vee (s_3 \wedge \nabla g \wedge \nabla \sim g) \vee (s_4 \wedge g \wedge \sim g),$$

with $s_1, s_2, s_3, s_4 \in S$ and

$$y = (t_1 \wedge \Delta g) \vee (t_2 \wedge \Delta \sim g) \vee (t_3 \wedge \nabla g \wedge \nabla \sim g) \vee (t_4 \wedge g \wedge \sim g),$$

with $t_1, t_2, t_3, t_4 \in S$ so

$$x \vee y = (v_1 \wedge \Delta g) \vee (v_2 \wedge \Delta \sim g) \vee (v_3 \wedge \nabla g \wedge \nabla \sim g) \vee (v_4 \wedge g \wedge \sim g)$$

where $v_i = s_i \vee t_i \in S$ for $i = 1, 2, 3, 4$, so $x \vee y \in S_0$.

From $x \in S_0$ it follows that

$$\nabla x = (\nabla s_1 \wedge \Delta g) \vee (\nabla s_2 \wedge \Delta \sim g) \vee (\nabla s_3 \wedge \nabla g \wedge \nabla \sim g) \vee (\nabla s_4 \wedge \nabla g \wedge \nabla \sim g) =$$
$$(\nabla s_1 \wedge \Delta g) \vee (\nabla s_2 \wedge \Delta \sim g) \vee ((\nabla s_3 \vee \nabla s_4) \wedge \nabla g \wedge \nabla \sim g) \vee (0 \wedge g \wedge \sim g)$$

and since $\nabla s_i \in S$ for $i = 1, 2, 3, 4$ and $0 \in S$ it follows that $\nabla x \in S_0$.

From $x \in S_0$ it follows that

$$\sim x = (\sim s_1 \vee \nabla \sim g) \wedge (\sim s_2 \vee \nabla g) \wedge (\sim s_3 \vee \Delta \sim g \vee \Delta g) \wedge (\sim s_4 \vee g \vee \sim g).$$

To simplify the notation we write $\sim s_i = b_i$ for $i = 1, 2, 3, 4$, then:

$$\sim x = (b_1 \vee \nabla \sim g) \wedge (b_2 \vee \nabla g) \wedge (b_3 \vee \Delta \sim g \vee \Delta g) \wedge (b_4 \vee g \vee \sim g).$$

Let

$$y = (b_1 \vee \nabla \sim g) \wedge (b_3 \vee \Delta \sim g \vee \Delta g)$$

and

$$z = (b_2 \vee \nabla g) \wedge (b_4 \vee g \vee \sim g).$$

Then:

$$y = (b_1 \wedge b_3) \vee (b_1 \wedge \Delta \sim g) \vee (b_1 \wedge \Delta g) \vee (\nabla \sim g \wedge b_3) \vee (\nabla \sim g \wedge \Delta \sim g) \vee (\nabla \sim g \wedge \Delta g) =$$
$$(b_1 \wedge b_3) \vee (b_1 \wedge \Delta \sim g) \vee (b_1 \wedge \Delta g) \vee (\nabla \sim g \wedge b_3) \vee \Delta \sim g =$$
$$(b_1 \wedge b_3) \vee (b_1 \wedge \Delta g) \vee (\nabla \sim g \wedge b_3) \vee \Delta \sim g =$$
$$[(b_1 \wedge b_3) \wedge (\Delta g \vee \nabla \sim g)] \vee (b_1 \wedge \Delta g) \vee (\nabla \sim g \wedge b_3) \vee \Delta \sim g =$$
$$(b_1 \wedge b_3 \wedge \Delta g) \vee (b_1 \wedge b_3 \wedge \nabla \sim g) \vee (b_1 \wedge \Delta g) \vee (\nabla \sim g \wedge b_3) \vee \Delta \sim g =$$

$$(b_1 \wedge \Delta g) \vee (\nabla \sim g \wedge b_3) \vee \Delta \sim g$$

and

$$z = (b_2 \wedge b_4) \vee (b_2 \wedge g) \vee (b_2 \wedge \sim g) \vee (\nabla g \wedge b_4) \vee (\nabla g \wedge g) \vee (\nabla g \wedge \sim g) =$$
$$(b_2 \wedge b_4) \vee (b_2 \wedge g) \vee (b_2 \wedge \sim g) \vee (\nabla g \wedge b_4) \vee g \vee (g \wedge \sim g) =$$
$$(b_2 \wedge b_4) \vee (b_2 \wedge \sim g) \vee (\nabla g \wedge b_4) \vee g =$$
$$[(b_2 \wedge b_4) \wedge (\sim g \vee \nabla g)] \vee (b_2 \wedge \sim g) \vee (\nabla g \wedge b_4) \vee g =$$
$$(b_2 \wedge b_4 \wedge \sim g) \vee (b_2 \wedge b_4 \wedge \nabla g)] \vee (b_2 \wedge \sim g) \vee (\nabla g \wedge b_4) \vee g =$$
$$(b_2 \wedge \sim g) \vee (b_4 \wedge \nabla g) \vee g.$$

Then

$$\sim x = y \wedge z = [(b_1 \wedge \Delta g) \vee (\nabla \sim g \wedge b_3) \vee \Delta \sim g] \wedge [(b_2 \wedge \sim g) \vee (b_4 \wedge \nabla g) \vee g] =$$
$$(b_1 \wedge \Delta g \wedge b_2 \wedge \sim g) \vee (b_1 \wedge \Delta g \wedge b_4 \wedge \nabla g) \vee (b_1 \wedge \Delta g \wedge g) \vee$$
$$(\nabla \sim g \wedge b_3 \wedge b_2 \wedge \sim g) \vee (\nabla \sim g \wedge b_3 \wedge b_4 \wedge \nabla g) \vee (\nabla \sim g \wedge b_3 \wedge g) \vee$$
$$(\Delta \sim g \wedge b_2 \wedge \sim g) \vee (\Delta \sim g \wedge b_4 \wedge \nabla g) \vee (\Delta \sim g \wedge g) =$$
$$(b_1 \wedge \Delta g \wedge b_4) \vee (b_1 \wedge \Delta g) \vee (b_3 \wedge b_2 \wedge \sim g) \vee$$
$$(\nabla \sim g \wedge b_3 \wedge b_4 \wedge \nabla g) \vee (\sim g \wedge b_3 \wedge g) \vee (\Delta \sim g \wedge b_2) =$$
$$(b_1 \wedge \Delta g) \vee (b_3 \wedge b_2 \wedge \sim g) \vee (\nabla \sim g \wedge b_3 \wedge b_4 \wedge \nabla g) \vee (\sim g \wedge b_3 \wedge g) \vee (\Delta \sim g \wedge b_2) =$$
$$(b_1 \wedge \Delta g) \vee [(b_3 \wedge b_2 \wedge \sim g) \wedge (\Delta \sim g \vee \nabla g)] \vee$$
$$(b_3 \wedge b_4 \wedge \nabla \sim g \wedge \nabla g) \vee (b_3 \wedge \sim g \wedge g) \vee (b_2 \wedge \Delta \sim g) =$$
$$(b_1 \wedge \Delta g) \vee (b_3 \wedge b_2 \wedge \sim g \wedge \Delta \sim g) \vee (b_3 \wedge b_2 \wedge \sim g \wedge \nabla g) \vee$$
$$(b_3 \wedge b_4 \wedge \nabla \sim g \wedge \nabla g) \vee (b_3 \wedge \sim g \wedge g) \vee (b_2 \wedge \Delta \sim g) =$$
$$(b_1 \wedge \Delta g) \vee (b_3 \wedge b_2 \wedge \Delta \sim g) \vee (b_3 \wedge b_2 \wedge \sim g \wedge g) \vee$$
$$(b_3 \wedge b_4 \wedge \nabla \sim g \wedge \nabla g) \vee (b_3 \wedge \sim g \wedge g) \vee (b_2 \wedge \Delta \sim g) =$$
$$(b_1 \wedge \Delta g) \vee [((b_3 \wedge b_2) \vee b_2) \wedge \Delta \sim g)] \vee [((b_3 \wedge b_2) \vee b_3) \wedge \sim g \wedge g] \vee (b_3 \wedge b_4 \wedge \nabla \sim g \wedge \nabla g) =$$
$$(b_1 \wedge \Delta g) \vee (b_2 \wedge \Delta \sim g) \vee (b_3 \wedge \sim g \wedge g) \vee (b_3 \wedge b_4 \wedge \nabla \sim g \wedge \nabla g).$$

Thus since $b_1, b_2, b_3, b_4 \in S$ we have that $\sim x \in S_0$. □

Let A be a Łukasiewicz algebra, S an L-subalgebra of A and $g \in A$.

1) If $g \in B(A)$ then $LS(S, g) = \{(s_1 \wedge g) \vee (s_2 \wedge \sim g) : s_1, s_2 \in S\}$.
2) If $\Delta g = 0$, then $g \leq \sim g$ and:
$$LS(S, g) = \{(s_1 \wedge \Delta \sim g) \vee (s_2 \wedge \nabla g) \vee (s_3 \wedge g) : s_1, s_2, s_3 \in S\}.$$
3) If c is a center of A, then since $\Delta \sim c = \Delta c = 0$, $\nabla c \wedge \nabla \sim c = 1 \wedge \nabla c = 1$ and $c \wedge \sim c = c$ it follows that: $LS(S, c) = \{s_1 \vee (s_2 \wedge c) : s_1, s_2 \in S\}$.

We present now a method for finding $LS(G)$ when G is a finite, non-empty set of a Łukasiewicz algebra A.

Let $G = \{g_1, g_2, \ldots, g_n\}$, consider:

- $S_1 = LS(g_1)$
- $S_2 = LS(S_1, g_2)$
- ....
- $S_n = LS(S_{n-1}, g_n)$

We prove that $S_n = LS(G)$. Indeed, from the previous construction it follows that:

$$\{g_1\} \subseteq S_1 \subseteq S_1 \cup \{g_2\} \subseteq LS(S_1, g_2) = S_2 \subseteq \ldots \subseteq S_n,$$

therefore (1) $G \subseteq S_n$ and since by construction (2) S_n is an L-subalgebra, from (1) and (2) it follows that $LS(G) \subseteq S_n$. From $g_1 \in G$ it follows that $S_1 \subseteq LS(G)$, so $S_1 \cup \{g_2\} \subseteq LS(G)$, therefore $S_2 = LS(S_1, g_2) \subseteq LS(G), \ldots, S_n = LS(S_{n-1}, g_n) \subseteq LS(G)$.

We know that if $b \in B(L)$ then $\sim b$ is the boolean complement of b. If $b \in B(L)$, we denote with b^*, any of the elements b or $\sim b$.

Lemma 1.11.4. *If L is a Łukasiewicz algebra and X a finite subset of $B(L)$, then $SB(X) = LS(X)$.*

Proof. If $X = \emptyset$, then $SB(\emptyset) = \{0, 1\} = LS(\emptyset)$.

Assume that $X = \{x_1, x_2, \ldots, x_n\}$. Since $SB(X)$ is an L-subalgebra of L such that $X \subseteq SB(X)$ then $LS(X) \subseteq SB(X)$. Let $b \in SB(X)$, if $b = 0$ then evidently $b \in LS(X)$. If $b \neq 0$ then it is well known[5] that $b = \bigvee_{k=1}^{r} m_k$, where $m_k = \bigwedge_{i=1}^{n} x_i^*$. Since $x_i^* \in LS(X)$ for all $i, 1 \leq i \leq n$, then $m_k \in LS(X)$ for all $k, 1 \leq k \leq r$, so $b \in LS(X)$. □

Corollary 1.11.5. *If A is a Łukasiewicz algebra, and X a finite subset of A, then*
$SB(\triangle X \cup \nabla X) = LS(\triangle X \cup \nabla X)$.

Proof. This is a consequence of Lemma 1.11.4, since $\triangle X \cup \nabla X$ is a finite subset of $B(A)$. □

Lemma 1.11.6. *If A is a Łukasiewicz algebra, $g \in A$ and S an L-subalgebra of A such that $\nabla g, \sim \triangle g \in B(S)$, then $B(LS(S, g)) = B(S)$.*

Proof. From the hypothesis we deduce that:

$$(1)\ \triangle g, \triangle \sim g, \nabla(g \wedge \sim g) \in B(S).$$

If $z \in B(LS(S, g)) = LS(S, g) \cap B(L)$, then $\Delta z = z$, and

$$z = (s_1 \wedge \Delta g) \vee (s_2 \wedge \Delta \sim g) \vee (s_3 \wedge \nabla g \wedge \nabla \sim g) \vee (s_4 \wedge g \wedge \sim g), \text{where} s_1, s_2, s_3, s_4 \in S,$$

so

$$z = \Delta z = (\Delta s_1 \wedge \Delta g) \vee (\Delta s_2 \wedge \Delta \sim g) \vee (\Delta s_3 \wedge \nabla g \wedge \nabla \sim g) \vee (\Delta s_4 \wedge \Delta g \wedge \Delta \sim g).$$

From $\Delta s_i \in B(S)$, for $i = 1, 2, 3, 4$, we deduce, keeping in mind (1), that $z \in B(S)$.

Since $S \subseteq LS(S, g)$, we know that $B(S) = S \cap B(L) \subseteq LS(S, g) \cap B(L) = B(LS(S, g))$. □

[5]See, for example, [66].

Lemma 1.11.7. *If A is a Łukasiewicz algebra, $G = \{g_1, g_2, \ldots, g_n\} \subseteq A$ and*

$$L_0 = SB(\triangle G \cup \nabla G), L_1 = LS(L_0, g_1), L_2 = LS(L_1, g_2), \ldots, L_n = LS(L_{n-1}, g_n)$$

then: $L_n = LS(G)$ and $B(LS(G)) = SB(\triangle G \cup \nabla G)$.

Proof. By construction we have $L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n$, and $g_i \in L_i, 1 \leq i \leq n$, so $G \subseteq L_n$ and therefore $LS(G) \subseteq L_n$.

Since $\triangle G, \nabla G \subseteq LS(G)$, then $\triangle G \cup \nabla G \subseteq LS(G)$, so: (1) $LS(\triangle G \cup \nabla G) \subseteq LS(G)$.

By hypothesis G is a finite subset, then by the Corollary 1.11.5:

(2) $SB(\triangle G \cup \nabla G) = LS(\triangle G \cup \nabla G)$.

From (1) and (2) we have $L_0 = SB(\triangle G \cup \nabla G) \subseteq LS(G)$, then since $g_1 \in LS(G)$ we have $L_1 = LS(L_0, g_1) \subseteq LS(G)$. From $g_i \in LS(G)$, and $L_{i-1} \subseteq LS(G), 2 \leq i \leq n$, we have $L_i = LS(L_{i-1}, g_i) \subseteq LS(G), 2 \leq i \leq n$. This proves that $L_n = LS(G)$.

Since $\nabla g_i, \sim \triangle g_i \in B(L_0) = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n$, for $i = 1, 2, \ldots, n$, then by Lemma 1.11.6, $B(L_1) = B(LS(L_0, g_1)) = B(L_0) = L_0$, and therefore $B(L_j) = B(LS(L_{j-1}, g_j)) = B(L_{j-1}) = L_0$, $2 \leq j \leq n$. Thus $B(LS(G)) = L_0 = SB(\Delta G \cup \nabla G)$. □

Lemma 1.11.8. (A. Monteiro [36]) *If A is a Łukasiewicz algebra, $G \subseteq A$, and $L' = LS(G)$, then the subset $\triangle G \cup \nabla G$ of $B(A)$ verifies: $SB(\triangle G \cup \nabla G) = B(L')$.*

Proof. In 1994, L. Monteiro, obtained the following proof, indicated in [48], and which is simpler than the one by A. Monteiro.

Let G be a subset of the Łukasiewicz algebra A, then we saw that:

$$LS(G) = \bigcup \{LS(G') : G' \in \mathcal{F}P(G)\}.$$

We shall prove that: $B(LS(G)) = SB(\triangle G \cup \nabla G)$. Since $\triangle G \cup \nabla G \subseteq B(A)$ and $\triangle G \cup \nabla G \subseteq LS(G)$, then $\triangle G \cup \nabla G \subseteq LS(G) \cap B(A) = B(LS(G))$, so $SB(\triangle G \cup \nabla G) \subseteq B(LS(G))$. If $b \in B(LS(G))$ then $b = \triangle b$ and $b \in LS(G)$, so $b \in LS(G')$, where G' is a finite subset of G, then by Lemma 1.11.7,

$$B(LS(G')) = SB(\triangle G' \cup \nabla G') \subseteq SB(\triangle G \cup \nabla G),$$

and since $b \in B(LS(G'))$ we have that $b \in SB(\triangle G \cup \nabla G)$. □

Corollary 1.11.9. *If G is a set of generators of the Łukasiewicz algebra A, this is $LS(G) = A$, then $B(A) = SB(\triangle G \cup \nabla G)$.*

Lemma 1.11.10. (L. Monteiro, [61]) *If e is the axis of a Łukasiewicz algebra L, S an L-subalgebra of L such that $e \in S$, and $x \in L$ verifies: $\Delta x, \nabla x \in S$ then $x \in S$.*

Proof. It follows immediately from the hypothesis and from the fact that in an algebra with axis each element x can be written as $x = (\Delta x \vee e) \wedge \nabla x$. □

If L is a Łukasiewicz algebra with axis e and S an L-subalgebra of L such that (1) for every $x \in L$ $\Delta x, \nabla x \in S$ then not necessarily $e \in S$. Indeed, consider the L-subalgebra $B(L)$ of the Łukasiewicz algebra L from example 1.3.1, which has axis e and verifies condition (1), while $e \notin S$.

Lemma 1.11.11. (L. Monteiro, [61]) *If L is a Łukasiewicz algebra with center c and S is an L-subalgebra of L then the following conditions are equivalent:*

a) *S verifies if $\Delta x, \nabla x \in S$, then $x \in S$.*
b) *$c \in S$.*

PROOF. a) implies b): Since $\Delta c = 0$ and $\nabla c = 1$ then by a) $c \in S$.

b) implies a): Since c is an axis of L belonging to S, then by the preceding lemma, it clear that a) holds. □

Lemma 1.11.12. (L. Monteiro, [61]) *If L is a Łukasiewicz algebra with axis e and S is an L-subalgebra of L verifying* (1) *$B(L) \subseteq S$ and* (2) *$e \in S$, then $S = L$.*

PROOF. Given $x \in L$ then $\Delta x, \nabla x \in B(L)$ so, by (1) and (2) we have that $x = (\Delta x \vee e) \wedge \nabla x \in S$. □

Lemma 1.11.13. (L. Monteiro, [61]) *If L is a Łukasiewicz algebra with center c and S is an L-subalgebra of L then $LS(S, c) = L$ if and only if $B(L) \subseteq S$.*

PROOF. If $LS(S, c) = L$, since $LS(S, c) = \{x \in L : x = s_1 \wedge (s_2 \vee c), \text{where } s_1, s_2 \in S\}$ then if $b \in B(L) \subseteq L = LS(S, c)$ we have that $b = s_1 \wedge (s_2 \vee c)$, with $s_1, s_2 \in S$ so $b = \Delta b = \Delta(s_1 \wedge (s_2 \vee c)) = \Delta s_1 \wedge (\Delta s_2 \vee \Delta c) = \Delta s_1 \wedge (\Delta s_2 \vee 0) = \Delta s_1 \wedge \Delta s_2 \in S$.

If $B(L) \subseteq S \subseteq LS(S, c)$, since $c \in LS(S, c)$ then by Lemma 1.11.12, $LS(S, c) = L$. □

In the *IX Latin American Symposium on Mathematical Logic* held in the Universidad Nacional del Sur in 1992, M. Abad, L. Monteiro, S. Savini and J. Sewald [5] presented their determination of the number of subalgebras of a finite non trivial Łukasiewicz algebra L, the number of subisomorphic algebras to a given subalgebra of L, the number of non isomorphic subalgebras of L, and a method for constructing all the subalgebras of L.

1.12. Complete Łukasiewicz algebras

A Łukasiewicz algebra L is said to be complete if the underlying lattice L is complete.

Lemma 1.12.1. (L.Monteiro, [61])*If L is a Łukasiewicz algebra with axis e such that $B(L)$ is a complete boolean algebra, then L is complete.*

PROOF. Given a subset $F = \{a_i\}_{i \in I} \subseteq L$ consider these subsets of $B(L)$:

$$F_0 = \{\Delta a_i\}_{i \in I} \quad \text{and} \quad F_1 = \{\nabla a_i\}_{i \in I}.$$

Since $B(L)$ is complete, there exist the elements:

$$(1) \quad a_0 = \bigwedge_{i \in I} \Delta a_i \in B(L) \quad \text{and} \quad (2) \quad a_1 = \bigwedge_{i \in I} \nabla a_i \in B(L).$$

Consider the element (3) $a = (a_0 \vee e) \wedge a_1$. We shall prove that a is the infimum of the set F. From (3) it follows that (4) $a \leq a_0 \vee e$ and (5) $a \leq a_1$.

From (4) we obtain (6) $\Delta a \leq \Delta a_0 \vee \Delta e = \Delta a_0 \vee 0 = \Delta a_0 =$ (by (1)) $= a_0$.

From (1) we get that (7) $a_0 \leq \Delta a_i$ for all $i \in I$, so from (6) and (7) we have that (8) $\Delta a \leq \Delta a_i$ for all $i \in I$.

From (5) we get (9) $\nabla a \leq \nabla a_1 =$ (by (2))$= a_1$. By (2) we have that (10) $a_1 \leq \nabla a_i$ for all $i \in I$, so from (9) and (10) it follows that (11) $\nabla a \leq \nabla a_i$ for all $i \in I$.

From (8) and (11) we conclude, by the corollary to Moisil's determination principle, (Corollary 1.4.2) that $\underline{a \leq a_i \text{ for all } i \in I}$.

We prove next that $\underline{\text{if } x \in L \text{ verifies } x \leq a_i, \text{ for all } i \in I, \text{ then } x \leq a}$.

From $x \leq a_i$, for all $i \in I$, it follows that (12) $\Delta x \leq \Delta a_i$, for all $i \in I$, and (13) $\nabla x \leq \nabla a_i$, for all $i \in I$.

Since $\Delta x, \nabla x \in B(L)$ then from (12) and (1) it follows that (14) $\Delta x \leq a_0$ and therefore $\Delta x \vee e \leq a_0 \vee e$.

By (13) and (2) we have (15) $\nabla x \leq a_1$ so from (14) and (15) it follows that $x = (\Delta x \vee e) \wedge \nabla x \leq (a_0 \vee e) \wedge a_1 = a$. $\square$

Corollary 1.12.2. *If L is a Łukasiewicz algebra with center such that $B(L)$ is a complete boolean algebra then L is complete.*

Since every Łukasiewicz algebra is in particular a De Morgan algebra then

Lemma 1.12.3. *If L is a Łukasiewicz algebra and there exists the supremum (infimum) of a non-empty family $\{a_i\}_{i\in I}$ of elements of L then there also exists the infimum (the supremum) of the family $\{\sim a_i\}_{i\in I}$ and furthermore*

$$\bigwedge_{i\in I} \sim a_i = \sim \bigvee_{i\in I} a_i, \quad \text{and} \quad \bigvee_{i\in I} \sim a_i = \sim \bigwedge_{i\in I} a_i.$$

Lemma 1.12.4. *If L is a Łukasiewicz algebra, and there exists $a = \bigvee_{i\in I} a_i$ then there exists $\bigvee_{i\in I} \nabla a_i$ and $\nabla a = \bigvee_{i\in I} \nabla a_i$.*

Proof. From $a = \bigvee_{i\in I} a_i$ it follows that $a_i \leq a$ for all $i \in I$ so:

$\underline{\text{(i) } \nabla a_i \leq \nabla a \text{ for all } i \in I}$.

We prove now that $\underline{\text{(ii) If (1) } \nabla a_i \leq x \text{ for all } i \in I \text{ then } \nabla a \leq x}$.

From (1) it follows that $\nabla a_i \leq \Delta x$ for all $i \in I$ and since $a_i \leq \nabla a_i$ for all $i \in I$, we have that $a_i \leq \Delta x$ for all $i \in I$ and therefore $a = \bigvee_{i\in I} a_i \leq \Delta x$, so $\nabla a \leq \Delta x \leq x$. From (i) and (ii) it follows that $\nabla a = \bigvee_{i\in I} \nabla a_i$. $\square$

Corollary 1.12.5. *If L is a Łukasiewicz algebra, $\{b_i\}_{i\in I} \subseteq B(L)$ and there exists $b = \bigvee_{i\in I} b_i$ then $b \in B(L)$.*

Proof. $\nabla b = \nabla(\bigvee_{i\in I} b_i) =$ (by Lemma 1.12.4) $= \bigvee_{i\in I} \nabla b_i = \bigvee_{i\in I} b_i = b$. $\square$

Corollary 1.12.6. (L. Monteiro, [57]) *If L is a complete Łukasiewicz algebra, then $B(L)$ is a complete boolean algebra.*

Lemma 1.12.7. (A. Monteiro, [44]) *If L is a Łukasiewicz algebra, and there exists $a = \bigvee_{i\in I} a_i$ and $\Delta a_i = 0$ for all $i \in I$ then $\Delta a = 0$.*

Proof. By hypothesis $\Delta a_i = 0$ for all $i \in I$, by Lemma 1.4.10 this is equivalent to $a_i \leq\sim a_i$ for all $i \in I$, then since every Łukasiewicz algebra is a Kleene algebra, we have that

$$a_i = a_i \wedge \sim a_i \leq a_j \vee \sim a_j = \sim a_j, \text{ for all } \quad i,j \in I,$$

and therefore (1) $a = \bigvee_{i\in I} a_i \leq\sim a_j$ for all $j \in I$. By Lemma 1.12.3, we know that (2) $\sim a = \bigwedge_{i\in I} \sim a_i$. From (1) and (2) it follows that $a \leq\sim a$. Then, by Lemma 1.4.10, $\Delta a = 0$. □

Lemma 1.12.8. (A. Monteiro, [44]) *If L is a Łukasiewicz algebra, and there exists $a = \bigvee_{i\in I} a_i$ then there also exists $\bigvee_{i\in I} \Delta a_i$ and:*

$$\Delta\left(\bigvee_{i\in I} a_i\right) = \bigvee_{i\in I} \Delta a_i.$$

Proof. The proof below is due to L. Monteiro, [61]. See also A. Monteiro [45].

By hypothesis $a_i \leq a$ for all $i \in I$ so (i) $\Delta a_i \leq \Delta a$ for all $i \in I$. We prove next: (ii) If $t \in L$ verifies $\Delta a_i \leq t$ for all $i \in I$, then $\Delta a \leq t$.

From (1) $\Delta a_i \leq t$, for all $i \in I$ it follows that (2) $\Delta a_i \leq \Delta t$, for all $i \in I$.

For each $i \in I$ let (3) $a_i' = \sim \Delta t \wedge \Delta a \wedge a_i$, so by (i) and (2), we can deduce that:

$$\Delta a_i' = \sim \Delta t \wedge \Delta a \wedge \Delta a_i = \sim \Delta t \wedge \Delta a_i \leq \sim \Delta t \wedge \Delta t = 0,$$

this is (4) $\Delta a_i' = 0$ for all $i \in I$.

By theorem 1.6.8, from (3) it follows that (5) $\bigvee_{i\in I} a_i' = \bigvee_{i\in I}(\sim \Delta t \wedge \Delta a \wedge a_i) = \sim \Delta t \wedge \Delta a \wedge (\bigvee_{i\in I} a_i) = \sim \Delta t \wedge \Delta a \wedge a = \sim \Delta t \wedge \Delta a$.

By Lemma 1.12.7, from (4) and (5) it follows that $\Delta(\bigvee_{i\in I} a_i') = 0$, this is $\Delta(\sim \Delta t \wedge \Delta a) = 0$, so $\sim \Delta t \wedge \Delta a = 0$ and therefore $\Delta a \leq \Delta t \leq t$. □

As a consequence of the previous results we have

Lemma 1.12.9. *If in a Łukasiewicz algebra L there exists $\bigwedge_{i\in I} y_i$, then there also exist $\bigwedge_{i\in I} \nabla y_i$, $\bigwedge_{i\in I} \Delta y_i$ and furthermore*

$$\bigwedge_{i\in I} \nabla y_i = \nabla\left(\bigwedge_{i\in I} y_i\right) \quad \text{and} \quad \bigwedge_{i\in I} \Delta y_i = \Delta\left(\bigwedge_{i\in I} y_i\right).$$

Given a Łukasiewicz algebra L, consider the set $A(L) = \{x \in L : \Delta x = 0\}$, this is a non-empty set since $0 \in A(L)$.

Lemma 1.12.10. (A. Monteiro (1969)) *If L is a Łukasiewicz algebra with axis e, then e is an upper bound of $A(L)$.*

Proof. Since L has axis then $y = (\Delta y \vee e) \wedge \nabla y$, for all $y \in L$. Let $x \in A(L)$, this is $\Delta x = 0$, so $x = (\Delta x \vee e) \wedge \nabla x = (0 \vee e) \wedge \nabla x = e \wedge \nabla x \leq e$. □

Lemma 1.12.11. (A. Monteiro (1969)) *If L is a Łukasiewicz algebra and $e \in A(L)$ is an upper bound of the set $A(L)$ then e is an axis of L.*

Proof. By hypothesis we have that (1) $\Delta e = 0$ and (2) if $\Delta x = 0$ then $x \leq e$. We prove now that:

$$\underline{\text{(i) If } x \leq \sim \nabla e \text{ then } \Delta x = x, \text{ this is } x \in B(L).}$$

If $x \leq \sim \nabla e$ then (3) $x \wedge \sim x \leq x \leq \sim \nabla e$. Since $\Delta(x \wedge \sim x) = 0$ then by (2) it follows that (4) $x \wedge \sim x \leq e \leq \nabla e$. From (3) and (4) it follows that $x \wedge \sim x \leq \sim \nabla e \wedge \nabla e = 0$, this is $x \wedge \sim x = 0$ and therefore $x \vee \sim x = 1$, which proves that $x \in B(L)$, this is $\Delta x = x$.

From $x \wedge \sim \nabla e \leq \sim \nabla e$ it follows by (i) that:

$$\underline{\text{(ii) } \Delta(x \wedge \sim \nabla e) = x \wedge \sim \nabla e.}$$

$$\underline{\text{(iii) } x = \Delta x \vee (x \wedge \nabla e), \text{ for all } x \in L.}$$

Indeed $x = x \wedge 1 = x \wedge (\nabla e \vee \sim \nabla e) = (x \wedge \nabla e) \vee (x \wedge \sim \nabla e) = (\text{by (ii)}) = (x \wedge \nabla e) \vee (\Delta x \wedge \sim \nabla e) = (x \vee \Delta x) \wedge (x \vee \sim \nabla e) \wedge (\nabla e \vee \Delta x) \wedge 1 = x \wedge (x \vee \sim \nabla e) \wedge (\nabla e \vee \Delta x) = x \wedge (\nabla e \vee \Delta x) = (x \wedge \nabla e) \vee \Delta x$.

$$\underline{\text{(iv) } \Delta x \vee (e \wedge \nabla x) = \Delta x \vee (x \wedge \nabla e), \text{ for all } x \in L.}$$

This follows from considering (5) $\Delta(\Delta x \vee (e \wedge \nabla x)) = \Delta x \vee (\Delta e \wedge \nabla x) = \Delta x \vee (0 \wedge \nabla x) = \Delta x$.

(6) $\Delta(\Delta x \vee (x \wedge \nabla e)) = \Delta x \vee (\Delta x \wedge \nabla x) = \Delta x$.

(7) $\nabla(\Delta x \vee (e \wedge \nabla x)) = \Delta x \vee (\nabla e \wedge \nabla x) = \nabla(\Delta x \vee (x \wedge \nabla e))$.

From (5), (6) and (7), by Moisil's determination principle it follows that (iv) holds, and from (iii) and (iv) we have that

$$(8)\ x = \Delta x \vee (e \wedge \nabla x),$$

From (1) and (8) it follows by Lemma 1.4.6 that e is an axis of L. □

Theorem 1.12.12. (A. Monteiro (1969)) *Every complete Łukasiewicz algebra L has an axis.*

Proof. Let $A(L) = \{e_i\}_{i \in I}$, this is (1) $\Delta e_i = 0$ for all $i \in I$. Since L is a complete lattice then there exists (2) $e = \bigvee_{i \in I} e_i$. From (1) and (2) it follows by Lemma 1.12.7 that $\Delta e = \Delta\left(\bigvee_{i \in I} e_i\right) = \bigvee_{i \in I} \Delta e_i = 0$. So $e \in A(L)$ and by (2), it is an upper bound for $A(L)$, and by the preceding Lemma e is the axis of L. □

Corollary 1.12.13. *Every finite Łukasiewicz algebra has an axis.*

We indicate now an example of a Łukasiewicz algebra without axis. Consider the set $\mathbb{N}$ of the natural numbers and the Łukasiewicz algebra $\mathbf{T} = \{0, c, 1\}$ from Example 1.2.3. Let $L = \mathbf{T}^{\mathbb{N}}$ be the set of all the functions from $\mathbb{N}$ to $\mathbf{T}$, algebrized componentwise. We denote with $\mathbf{0}$ the bottom element of this Łukasiewicz algebra, this is, $\mathbf{0}(x) = 0$ for all $x \in \mathbb{N}$. Let S be the set of all the functions $f : \mathbb{N} \to \mathbf{T}$ such that $f(x) = c$ for $x \in F \subset \mathbb{N}$, where F is finite or empty. Clearly S is an L-subalgebra of L.

Consider $A(S) = \{f \in S : \Delta f = \mathbf{0}\}$. Then $f \in A(S) \iff f(x) \in \{0, c\}$ for all $x \in \mathbb{N}$. The set $A(S)$ does not have a top element. If $f \in A(S)$ then there exists a finite part $F \subset \mathbb{N}$ such that

$$f(x) = \begin{cases} 0 & \text{for} \quad x \notin F \\ c & \text{for} \quad x \in F. \end{cases}$$

Let $y \in \mathbb{N} \setminus F$ and $h : \mathbb{N} \to \mathbf{T}$ be defined by

$$h(x) = \begin{cases} 0 & \text{for} \quad x \notin F \cup \{y\} \\ c & \text{for} \quad x \in F \cup \{y\}. \end{cases}$$

Then $h \in A(S)$ and $f < h$, so $A(S)$ does not have a top element and therefore S has no axis.

CHAPTER 2

Homomorphisms, deductive systems and quotients

2.1. Homomorphisms

Definition 2.1.1. *A function h from a Łukasiewicz algebra A to a Łukasiewicz algebra A' is said to be a homomorphism from A to A', if the following conditions hold:*

H1) $h(x \vee y) = h(x) \vee h(y)$,
H2) $h(\sim x) = \sim h(x)$,
H3) $h(\nabla x) = \nabla h(x)$.

If h is surjective we say that h is an epimorphism *and if h is bijective, we say that h is an* isomorphism.

We say that a Łukasiewicz algebra A' is a homomorphic image of a Łukasiewicz algebra A if there exists an epimorphism from A to A'. If h is one to one as well, we say that A is isomorphic to A' and write $A \cong A'$.

The next lemma is proved without difficulty.

Lemma 2.1.2. *If $h : A \to A'$ is a homomorphism then:*

H4) $h(x \wedge y) = h(x) \wedge h(y)$,
H5) $h(1) = 1$,
H6) $h(0) = 0$,
H7) $h(x \to y) = h(x) \to h(y)$,
H8) $h(x \rightarrowtail y) = h(x) \rightarrowtail h(y)$,
H9) $h(x \Rightarrow y) = h(x) \Rightarrow h(y)$,
H10) $h(\Delta x) = \Delta h(x)$,
H11) $h(\partial x) = \partial h(x)$,
H12) $h(Ext\ x) = Ext\ h(x)$

It is also easy to prove:

Lemma 2.1.3. *If L and L' are Łukasiewicz algebras and h is a homomorphism from L to L' then $h(B(L)) \subseteq B(L')$ and the restriction h' of h to $B(L)$ is a boolean homomorphism from $B(L)$ to $B(L')$.*

The main goal of this section is the determination of the homomorphic images of a Łukasiewicz algebra L by means of an intrinsic construction on the algebra itself.

If L is a Łukasiewicz algebra then id_L is a surjective homomorphism, so L is a homomorphic image of L. If L' is a Łukasiewicz algebra with a single element, $L' = \{1'\}$ then the map $h : L \to L'$ defined by $h(x) = 1'$, for all $x \in L$, is a surjective homomorphism so L' is a homomorphic image of L. These two homomorphic images of L are called the trivial images of L.

If $h : A \to A'$ is a homomorphism, the *kernel* of h is the set:

$$Ker(h) = h^{-1}(1) = \{a \in A : h(a) = 1\}.$$

It is clear that this set has the following properties:

D1) $1 \in Ker(h)$,
D2) If $a, a \to b \in Ker(h)$, then $b \in Ker(h)$.

Lemma 2.1.4. *If A and A' are Lukasiewicz algebras and h is a homomorphism from A to A', then $h(a) = h(b)$ if and only if $a \rightarrowtail b \in Ker(h)$ and $b \rightarrowtail a \in Ker(h)$.*

Proof. $h(a \rightarrowtail b) = h(a) \rightarrowtail h(b) = h(a) \rightarrowtail h(a) = 1$ so $a \rightarrowtail b \in Ker(h)$. Analogously, from $h(b \rightarrowtail a) = 1$ it follows that $b \rightarrowtail a \in Ker(h)$.

Conversely, if $a \rightarrowtail b \in Ker(h)$ and $b \rightarrowtail a \in Ker(h)$ then $1 = h(a \rightarrowtail b) = h(a) \rightarrowtail h(b)$ and $1 = h(a \rightarrowtail b) = h(a) \rightarrowtail h(b)$, so by property IC13) from section 1.5 we have that $h(a) = h(b)$. □

Definition 2.1.5. *A part D of a Lukasiewicz algebra L is said to be a* deductive system *if*

D1) $1 \in D$,
D2) *If $a, a \to b \in D$, then $b \in D$* (modus ponens).

Definition 2.1.6. *A part F of a Lukasiewicz algebra L is said to be a* filter *if*

F1) $F \neq \emptyset$,
F2) *if $a, b \in F$, then $a \wedge b \in F$,*
F3) *if $a \in F$ and $a \leq b$ then $b \in F$.*

Lemma 2.1.7. *Every deductive system D of a Lukasiewicz algebra L is a filter of L.*

Proof. The properties of the operation $\to$ that we will cite were proved in section 1.5.

We prove first that if D is a deductive system then :

D3) If $b \in D$ then $a \to b \in D$ for all $a \in L$.

Indeed, from $b \in D$ and by property ID5) we know that $b \to (a \to b) = 1$, so from D1) and D2) it follows that $a \to b \in D$.

F1) $D \neq \emptyset$.
Obvious, given that $1 \in D$.

F2) If $a, b \in D$, then $a \wedge b \in D$.
By property ID8) we know that $a \to (a \wedge b) = a \to b$. Since $b \in D$ it follows by D3) that $a \to b \in D$, this is $a \to (a \wedge b) \in D$, so since $a \in D$ it follows by D2) that $a \wedge b \in D$.

F3) If $a \in D$ and $a \leq b$ then $b \in D$.
From $a \leq b$, it follows by ID6) that $1 = a \to a \leq a \to b$, so $a \to b = 1 \in D$ and since $a \in D$ it follows by D2 that $b \in D$.

□

Note that there exist filters that are not deductive systems. Indeed, in the Łukasiewicz algebra T from Example 1.2.3, $[c) = \{c, 1\}$ is a filter but not a deductive system since $c \in [c)$, $c \to 0 = 1 \in [c)$ and $0 \notin [c)$.

Definition 2.1.8. *A filter F of a Łukasiewicz algebra L is said to be a Δ-filter if it verifies: If $f \in F$ then $\Delta f \in F$.*

Lemma 2.1.9. *A subset D of a Łukasiewicz algebra L is a deductive system if and only if D is a Δ-filter.*

Proof. We already know that every deductive system is a filter. Let $d \in D$, so since by ID13), $d \to \Delta d = 1$ and $1 \in D$ it follows by D2) that $\Delta d \in D$.

Assume now that (1) D is a Δ-filter of L so $1 \in D$. If (2) $a \in D$ and (3) $a \to b \in D$ then from (2) and (1) it follows that (4) $\Delta a \in D$. Then since D is a filter, from (4) and (3): $\Delta a \wedge (a \to b) \in D$. But $\Delta a \wedge (a \to b) = \Delta a \wedge (\nabla \sim a \vee b) = (\Delta a \wedge \nabla \sim a) \vee (\Delta a \wedge b) = 0 \vee (\Delta a \wedge b) = \Delta a \wedge b$, this is $\Delta a \wedge b \in D$ and since $\Delta a \wedge b \leq b$, it follows that $b \in D$ because D is a filter. $\square$

We consider now a notion introduced by H. Rasiowa in her study of Nelson algebras [72] which we can also define in Łukasiewicz algebra.

Definition 2.1.10. *A subset D of a Łukasiewicz algebra L is called a special filter of the first kind if it verifies:*

R1) *$D \neq \emptyset$,*
R2) *If $a, b \in D$ then $a \wedge b \in D$,*
R3) *If $a \in D$ and $a \to b = 1$ then $b \in D$.*

Lemma 2.1.11. *A subset D of a Łukasiewicz algebra L is a deductive system if and only if D is a special filter of the first kind.*

Proof. We begin with the observation that a special *filter* of the first kind is a filter. It will be enough to prove that if (1) $a \in D$ and (2) $a \leq b$ then $b \in D$. From (2) it follows by ID6) that $1 = a \to a \leq a \to b$, so (3) $a \to b = 1 \in D$, and so from (1) and (3) it follows by R3) that $b \in D$.

If D is a deductive system then the conditions R1), R2) and R3) are clearly satisfied.

Assume now that D is a special filter of the first kind. From R1) it follows that there exists $d \in D$, so since $d \to 1 = 1$ it follows by R3) that $1 \in D$. We prove now that D2) holds, this is:

$$\text{If } a, a \to b \in D \text{ then } b \in D.$$

From the hypothesis it follows by R2) that (4) $(a \to b) \wedge a \in D$. But (5) $((a \to b) \wedge a) \to b =$ (by ID10)) $= (a \to b) \to (a \to b) = 1$, so from (4) and (5) it follows by R3) that $b \in D$.

We can also point out this proof: we already saw that D is a filter. If $a \in D$ then since $a \to \Delta a = 1$ it follows by R3) that D is a Δ-filter and therefore a deductive system. $\square$

Now we study another notion of deductive system.

We say that a part D of a Łukasiewicz algebra L is a contraposed deductive system if

Cd1) $1 \in D$,
Cd2) If $a, a \rightarrowtail b \in D$, then $b \in D$.

Lemma 2.1.12. *A subset D of a Łukasiewicz algebra L is a deductive system if and only if D is a contraposed deductive system.*

Proof. Assume that D verifies D1) and D2), so Cd1) holds. Assume that $a, a \rightarrowtail b \in D$. Since $a \rightarrowtail b = (a \to b) \wedge ({\sim} b \to {\sim} b) \leq a \to b$ and D is a filter, then $a \to b \in D$ and since $a \in D$, it follows by D1) that $b \in D$.

Assume now that D verifies Cd1) and Cd2), so D1) holds. Assume also that (1) $a \in D$ and (2) $a \to b \in D$. We saw in Lemma 1.5.6 that (3) $a \to b = a \rightarrowtail (a \rightarrowtail b)$ so from (1) and (3) it follows by Cd2) that (4) $a \rightarrowtail b \in D$, and from (1) and (4) it follows by Cd2) that $b \in D$. □

We see thus that both implications $\to$ and $\rightarrowtail$ give raise to the same deductive systems.

Lemma 2.1.13. *For a filter D to be a deductive system it is necessary and sufficient that the condition* C4) *If $a, a \twoheadrightarrow b = {\sim} a \vee b \in D$ then $b \in D$ holds.*

Proof. Assume that the filter D is a deductive system and that (1) $a \in D$ and (2) $a \twoheadrightarrow b = {\sim} a \vee b \in D$. Then since D is a filter, $a \wedge ({\sim} a \vee b) \in D$, but

$$a \wedge ({\sim} a \vee b) = (a \wedge {\sim} a) \vee (a \wedge b) = (a \wedge \nabla {\sim} a) \vee (a \wedge b) =$$

$$a \wedge (\nabla {\sim} a \vee b) = a \wedge (a \to b),$$

so $a \wedge ({\sim} a \vee b) \in D$ and since $a \wedge ({\sim} a \vee b) \leq a \to b$ it follows, given that D is a filter, that (3) $a \to b \in D$. From (1) and (3) it follows that since D is a deductive system, $b \in D$.

Assume now that D is a filter verifying condition C4). Since D is a filter then D1) $1 \in D$ holds. Let us check that D2) holds, this is that if (4) $a \in D$ and (5) $a \to b \in D$ then $b \in D$. Indeed from (4) and (5) it follows that since D is a filter $a \wedge (a \to b) \in D$, but $a \wedge (a \to b) = a \wedge (\nabla {\sim} a \vee b) = (a \wedge \nabla {\sim} a) \vee (a \wedge b) = (a \wedge {\sim} a) \vee (a \wedge b) = a \wedge ({\sim} a \vee b)$. Therefore (6) $a \wedge ({\sim} a \vee b) \in D$, and since (7) $a \wedge ({\sim} a \vee b) \leq {\sim} a \vee b$ then from (6) and (7) it follows that $a \twoheadrightarrow b = {\sim} a \vee b \in D$ because D is a filter, so by C4), $b \in D$. □

If X is a subset of a Łukasiewicz algebra L we write $\Delta X = \{\Delta x : x \in X\}$ and $\nabla X = \{\nabla x : x \in X\}$.

We are going to point out next the relations existing between the deductive systems of a Łukasiewicz algebra L and the filters of the boolean algebra $B(L)$.

Lemma 2.1.14. *If D is a deductive system of L, $D \cap B(L)$ is a filter of $B(L)$ and $D \cap B(L) = \Delta D = \nabla D$.*

Proof. F1) $1 \in D \cap B(L)$. Immediate since $1 \in D, B(L)$.

F2) Assume that $x, y \in D \cap B(L)$, this is (1) $x, y \in D$ and (2) $x, y \in B(L)$. From (1) it follows, since D is a filter that (3) $x \wedge y \in D$, and from (2) it follows that (4) $x \wedge y \in B(L)$ because $B(L)$ is a boolean algebra. From (3) and (4) we conclude that $x \wedge y \in D \cap B(L)$.

F3) Assume that (5) $x \in D \cap B(L)$ and (6) $y \in B(L)$ are such that (7) $x \leq y$. From (5) it follows that in particular (8) $x \in D$, so from (7) and (8), we get (9) $y \in D$ because D is a filter, and from (9) and (6) $y \in D \cap B(L)$.

Let $f \in D \cap B(L)$, this is (10) $f \in D$ and (11) $f \in B(L)$. From (11) it follows that $\Delta f = f$, so by (10) we have that $f \in \Delta D$. Assume now that $f \in \Delta D$, so $f = \Delta d$ with $d \in D$, and since D is a deductive system, hence a Δ-filter, we have that $\Delta d \in D$ and since $\Delta d \in B(L)$ we have that $f = \Delta d \in D \cap B(L)$. Analogously one can prove that $D \cap B(L) = \nabla D$. □

Recall the following results from lattice theory:

Lemma 2.1.15. *If R is a distributive lattice, with bottom element 0 and top element 1, $X \subseteq R$ then*

a) *the filter generated by X is the set*

$$F(X) = \left\{ y \in R : \text{there exists } x_1, x_2, \ldots, x_n \in X \text{ such that } \bigwedge_{i=1}^{n} x_i \leq y \right\}.$$

b) *If the set X verifies* F2) *"$x \wedge y \in X$ for all $x, y \in X$," then*

$$F(X) = \{y \in R : \text{there exists } x \in X \text{ such that } x \leq y\}.$$

If $0 \notin X$ then $F(X)$ is a proper filter of R.

Lemma 2.1.16. *If L is a Łukasiewicz algebra and X is a subset of $B(L)$ verifying condition* F2) *in the previous lemma then $F(X)$ is a Δ-filter.*

Proof. Since X verifies condition F2) then by Lemma 2.1.15 b), if $y \in F(X)$ then there exists $x \in X$ such that (3) $x \leq y$. Since $x \in X \subseteq B(L)$ we have that (4) $\Delta x = x$, then by (3) and (4) it follows that $x = \Delta x \leq \Delta y$, so by Lemma 2.1.15 b), we have that $\Delta y \in F(X)$. □

Corollary 2.1.17. *If Q is a filter of L then $F(\Delta Q)$ and $F(\nabla Q)$ are Δ-filters of L.*

Proof. Let $x, y \in \Delta Q$, so $x = \Delta q_1, y = \Delta q_2$, where $q_1, q_2 \in Q$, then $x \wedge y = \Delta(q_1 \wedge q_2)$ and since $q_1 \wedge q_2 \in Q$, we have that $x \wedge y \in \Delta Q$. By Lemma 2.1.16, $F(\Delta Q)$ is a Δ-filter of L. In a similar way, we can prove that $F(\nabla Q)$ is a Δ-filter de L. □

Lemma 2.1.18. *If Q is a filter of L then $B(L) \cap Q \subseteq F(\nabla Q)$.*

Proof. If $b \in B(L) \cap Q$ then $b = \nabla b \in \nabla Q \subseteq F(\nabla Q)$. □

We denote by $\mathbf{D}(L)$ and $\mathbf{F}(B(L))$ the sets of all the deductive systems of L and all the filters of the boolean algebra $B(L)$, respectively.

Lemma 2.1.19. *The transformation* $\alpha : \mathbf{D}(L) \to \mathbf{F}(B(L))$ *defined by* $\alpha(D) = D \cap B(L)$ *is an order isomorphism from the poset* $(\mathbf{D}(L), \subseteq)$ *to the poset* $(\mathbf{F}(B(L)), \subseteq)$ *and* $\alpha^{-1}(Q) = F(Q)$ *where* $Q \in \mathbf{F}(B(L))$.

Proof. We already saw in Lemma 2.1.14 that if $D \in \mathbf{D}(L)$ then $\alpha(D) = D \cap B(L) \in \mathbf{F}(B(L))$.

1) If $D_1, D_2 \in \mathbf{D}(L)$ are such that $D_1 \subseteq D_2$ then $\alpha(D_1) \subseteq \alpha(D_2)$.
   It is immediate to check this property.
2) $F(\Delta D) = F(D \cap B(L)) = D$.
   By Lemma 2.1.14 we know that $\Delta D = D \cap B(L)$.
   If $D \in \mathbf{D}(L)$ then $D \cap B(L) \subseteq D$ and therefore $F(D \cap B(L)) \subseteq F(D) = D$. If $d \in D$ then $\Delta d \in D \cap B(L) \subseteq F(D \cap B(L))$ so given that $\Delta d \leq d$, we have that $d \in F(D \cap B(L))$.
3) α is surjective.
   Indeed, for each $Q \in \mathbf{F}(B(L))$, since Q verifies condition F2), we have by Lemma 2.1.16 that $F(Q)$ is a Δ-filter and therefore $F(Q) \in \mathbf{D}(L)$. Then $\alpha(F(Q)) = F(Q) \cap B(L)$. Since $Q \subseteq B(L)$, then $Q = Q \cap B(L) \subseteq F(Q) \cap B(L)$. Conversely, if $y \in F(Q) \cap B(L)$ in particular $y \in F(Q)$, so by Lemma 2.1.15 there exists $x \in Q$ such that $x \leq y$, and since $y \in B(L)$ we have that (1) $\nabla x \leq \nabla y = y$. From $x \in Q$ and $x \leq \nabla x$ it follows that (2) $\nabla x \in Q$. From (1) and (2) we obtain $y \in Q$, so $F(Q) \cap B(L) \subseteq Q$, and therefore $\alpha(F(Q) \cap B(L)) = Q$.
4) If $D_1, D_2 \in \mathbf{D}(L)$ are such that $\alpha(D_1) \subseteq \alpha(D_2)$ then $D_1 \subseteq D_2$.
   By hypothesis $D_1 \cap B(L) \subseteq D_2 \cap B(L)$, so by 2) $D_1 = F(D_1 \cap B(L)) \subseteq F(D_2 \cap B(L)) = D_2$.

We have proved that α is an order isomorphism.

If $Q \in \mathbf{F}(B(L))$ then $\alpha(F(Q)) = F(Q) \cap B(L)$. We prove now that $F(Q) \cap B(L) = Q$. Indeed, since $Q \subseteq F(Q)$ then if $h \in Q \subseteq B(L)$, we have that $h \in F(Q) \cap B(L)$. Conversely if $x \in F(Q) \cap B(L)$ then there exists $t \in Q \subseteq B(L)$ such that $t \leq x$, and $t = \Delta t$. Then $t = \Delta t \leq \Delta x$. Since Q is a filter of $B(L)$, it follows that $\Delta x \in Q$, and since $x \in B(L)$, $\Delta x = x$ then $x \in Q$.

Thus $\alpha(F(Q)) = F(Q) \cap B(L) = Q$ and since α is a bijection, we have that $F(Q) = \alpha^{-1}\alpha(F(Q)) = \alpha^{-1}(Q)$. ☐

2.2. Quotient algebras

We shall indicate now a construction on a Łukasiewicz algebra L that determines all its homomorphic images.

Lemma 2.2.1. *If* D *is a deductive system of a Łukasiewicz algebra* L *and* $d \in D$ *then* $x \rightarrowtail d \in D$ *for all* $x \in L$.

Proof. Since $d \leq x \rightarrowtail d = \sim x \vee d \vee (\nabla \sim x \wedge \nabla d)$, $d \in D$ and D is a filter then $x \rightarrowtail d \in D$. ☐

Given a deductive system D of a Łukasiewicz algebra L and $a, b \in L$, we write $a \equiv b$ (mod D) to indicate that: Co1) $a \rightarrowtail b \in D$ and $b \rightarrowtail a \in D$.

Lemma 2.2.2. *For* $a \equiv b$ (mod D) *to hold it is necessary and sufficient that any of the following conditions are satisfied:*

Co1) $(a \rightarrowtail b) \wedge (b \rightarrowtail a) \in D$,

Co2) $a \to b, \sim b \to \sim a, b \to a, \sim a \to \sim b \in D$,

Co3) *There exists* $d \in D$ *such that* $a \wedge d = b \wedge d$.

PROOF. Co1) is equivalent to Co2):

Since by definition $a \rightarrowtail b = (a \to b) \wedge (\sim b \to \sim a)$ and D is a filter then $a \rightarrowtail b \in D$ is equivalent to $a \to b, \sim b \to \sim a \in D$, and in the same fashion, $b \rightarrowtail a \in D$ is equivalent to $b \to a, \sim a \to \sim b \in D$.

Co3) implies Co1):

Assume that there exists $d \in D$ such that $a \wedge d = b \wedge d$ then $b \rightarrowtail (a \wedge d) = b \rightarrowtail (b \wedge d)$ so by IC9) and IC3),

$$(b \rightarrowtail a) \wedge (b \rightarrowtail d) = (b \rightarrowtail b) \wedge (b \rightarrowtail d) = 1 \wedge (b \rightarrowtail d) = b \rightarrowtail d$$

this is (1) $b \rightarrowtail d \leq b \rightarrowtail a$. But as $d \in D$, it follows by Lemma 2.2.1 that (2) $b \rightarrowtail d \in D$, so from (1) and (2) it follows, since D is a filter, that (3) $b \rightarrowtail a \in D$. Analogously, from $a \rightarrowtail (a \wedge d) = a \rightarrowtail (b \wedge d)$ it follows that (4) $a \rightarrowtail b \in D$ and from (3) and (4) we conclude Co1).

Co1) implies Co3):

Assume now that Co1) holds, so since D is a Δ-filter we have that $d = \Delta((a \rightarrowtail b) \wedge (b \rightarrowtail a)) \in D$, this is

$$(5)\ d = (\nabla \sim a \vee \Delta b) \wedge (\nabla b \vee \Delta \sim a) \wedge (\nabla \sim b \vee \Delta a) \wedge (\nabla a \vee \Delta \sim b)$$

and calculating we get that

$$d = (\nabla \sim a \wedge \nabla \sim b \wedge \nabla a \wedge \nabla b) \vee (\Delta \sim a \wedge \Delta \sim b) \vee (\Delta a \wedge \Delta b).$$

Then, since $d \in B(L)$,

$$\Delta(d \wedge a) = \Delta d \wedge \Delta a = d \wedge \Delta a = 0 \vee 0 \vee (\Delta a \wedge \Delta b) = \Delta a \wedge \Delta b = \Delta(a \wedge b).$$

Analogously one proves that $\Delta(d \wedge b) = \Delta(a \wedge b)$ and therefore:

(6) $\Delta(d \wedge a) = \Delta(d \wedge b)$.

From (5) it follows that

$$\nabla(d \wedge a) = \nabla d \wedge \nabla a = d \wedge \nabla a = (\nabla \sim a \wedge \nabla \sim b \wedge \nabla a \wedge \nabla b) \vee 0 \vee (\Delta a \wedge \Delta b),$$

and that

$$\nabla(d \wedge b) = \nabla d \wedge \nabla b = d \wedge \nabla b = (\nabla \sim a \wedge \nabla \sim b \wedge \nabla a \wedge \nabla b) \vee 0 \vee (\Delta a \wedge \Delta b).$$

Thus (7) $\nabla(d \wedge a) = \nabla(d \wedge b)$.

From (6) and (7) it follows by Moisil's determination principle that $a \wedge d = b \wedge d$ and since $d \in D$, we have proved that Co3) holds. □

Technically condition Co3) is simpler than the others.

We prove now that $\equiv$ is an equivalence relation, and we shall use condition Co3) for doing so.

Eq1) $a \equiv a$ (mod D).

Since $1 \in D$ and $a \wedge 1 = a \wedge 1$ Eq1) holds.

It is clear that

Eq2) If $a \equiv b$ (mod D) then $b \equiv a$ (mod D).

Eq3) If $a \equiv b$ (mod D) and $b \equiv c$ (mod D) then $a \equiv c$ (mod D).

By hypothesis there exist $d_1, d_2 \in D$ such that $a \wedge d_1 = b \wedge d_1$ and $b \wedge d_2 = c \wedge d_2$ so $a \wedge d_1 \wedge d_2 = b \wedge d_1 \wedge d_2$ and $b \wedge d_2 \wedge d_1 = c \wedge d_2 \wedge d_1$ and therefore $a \wedge d_1 \wedge d_2 = c \wedge d_1 \wedge d_2$ and since $d_1 \wedge d_2 \in D$ then Eq3) holds.

If we use condition Co1) then the proofs are as follows:

Eq1) Since by IC3) $a \rightarrowtail a = 1$ then $a \equiv a$ (mod D).

It is clear that Eq2) holds.

Eq3) By hypothesis (1) $a \rightarrowtail b \in D$, (2) $b \rightarrowtail a \in D$, (3) $b \rightarrowtail c \in D$, and (4) $c \rightarrowtail b \in D$. By IC10) we know that

$$(5)\ (a \rightarrowtail b) \rightarrowtail ((b \rightarrowtail c) \rightarrowtail (a \rightarrowtail c)) = 1 \in D.$$

Then from (1) and (5) it follows by D2) (modus ponens) that

$$(6)\ (b \rightarrowtail c) \rightarrowtail (a \rightarrowtail c) \in D.$$

From (6) and (3) it follows again by D2) that

$$(7)\ a \rightarrowtail c \in D.$$

In a similar fashion, from

$$(c \rightarrowtail b) \rightarrowtail ((b \rightarrowtail a) \rightarrowtail (c \rightarrowtail a)) = 1 \in D.$$

Using (4) and (2) it follows that

$$(8)\ c \rightarrowtail a \in D.$$

We prove now that the relation "$\equiv$" is compatible with all the operations, using condition Co3).

Eq4) If $a \equiv b$ (mod D) then $\sim a \equiv \sim b$ (mod D).

By hypothesis there exists $d \in D$ such that $a \wedge d = b \wedge d$, so $\sim a \vee \sim d = \sim b \vee \sim d$ and therefore $(\sim a \vee \sim d) \wedge \Delta d = (\sim b \vee \sim d) \wedge \Delta d$, this is $(\sim a \wedge \Delta d) \vee (\sim d \wedge \Delta d) = (\sim b \wedge \Delta d) \vee (\sim d \wedge \Delta d)$ and since $\sim d \wedge \Delta d = 0$ we have that $\sim a \wedge \Delta d = \sim b \wedge \Delta d$, so since $\Delta d \in D$ it follows that $\sim a \equiv \sim b$ (mod D).

Eq5) If $a \equiv b$ (mod D) then $\nabla a \equiv \nabla b$ (mod D).

By hypothesis there exists $d \in D$ such that $a \wedge d = b \wedge d$, so $\nabla(a \wedge d) = \nabla(b \wedge d)$, this is $\nabla a \wedge \nabla d = \nabla b \wedge \nabla d$ then since $d \leq \nabla d$, $d \in D$ and D is a filter we have that $\nabla d \in D$, hence $\nabla a \equiv \nabla b$ (mod D).

Eq6) If $a \equiv a'$ (mod D) and $b \equiv b'$ (mod D) then $a \vee b \equiv a' \vee b'$ (mod D).

By hypothesis there exist $d_1, d_2 \in D$ such that $a \wedge d_1 = a' \wedge d_1$ and $b \wedge d_2 = b' \wedge d_2$, so $a \wedge d_1 \wedge d_2 = a' \wedge d_1 \wedge d_2$ and $b \wedge d_2 \wedge d_1 = b' \wedge d_2 \wedge d_1$ and therefore $(a \vee b) \wedge d_1 \wedge d_2 = (a' \vee b') \wedge d_1 \wedge d_2$ and since $d_1 \wedge d_2 \in D$ it follows that $a \vee b \equiv a' \vee b'$ (mod D).

The relation "$\equiv$" is compatible with $\wedge$, as an immediate consequence of the De Morgan laws and the fact that "$\equiv$" is compatible with $\sim$ and $\vee$.

Next we present a different proof based on condition Co1).

Eq4) By hypothesis $a \rightarrowtail b \in D$ and $b \rightarrowtail a \in D$, so since by IC16)

$$\sim a \rightarrowtail \sim b = b \rightarrowtail a \quad \text{and} \quad \sim b \rightarrowtail \sim a = a \rightarrowtail b$$

it follows immediately that $\sim a \equiv \sim b$ (mod D).

Eq5) By hypothesis $a \rightarrowtail b \in D$ and $b \rightarrowtail a \in D$, then (1) $\Delta(a \rightarrowtail b) \in D$ and (2) $\Delta(b \rightarrowtail a) \in D$.

By IC11) we know that (3) $\Delta(a \rightarrowtail b) \rightarrowtail (\nabla a \rightarrowtail \nabla b) = 1 \in D$ so from (3) and (1) it follows that $\nabla a \rightarrowtail \nabla b \in D$. In a similar way one proves $\nabla b \rightarrowtail \nabla a \in D$.

We prove now: (A) If $a \equiv a'$ (mod D) then $a \vee b \equiv a' \vee b$ (mod D).
Indeed:

$$(a \vee b) \rightarrowtail (a' \vee b) = (\sim a \wedge \sim b) \vee a' \vee b \vee (\nabla \sim a \wedge \nabla \sim b \wedge (\nabla a' \vee \nabla b)) =$$

$$(\sim a \wedge \sim b) \vee a' \vee b \vee (\nabla \sim a \wedge \nabla \sim b \wedge \nabla a') \vee (\nabla \sim a \wedge \nabla \sim b \wedge \nabla b)) =$$

$$(\sim a \wedge \sim b) \vee a' \vee [((\nabla \sim a \wedge \nabla a') \vee b) \wedge (\nabla \sim b \vee b)] \vee [((\nabla \sim a \wedge \nabla b) \vee b) \wedge (\nabla \sim b \vee b)] =$$

$$(\sim a \wedge \sim b) \vee a' \vee (\nabla \sim a \wedge \nabla a') \vee b \vee (\nabla \sim a \wedge \nabla b) \vee b =$$

$$(\sim a \wedge \sim b) \vee a' \vee (\nabla \sim a \wedge \nabla a') \vee b \vee (\nabla \sim a \wedge \nabla b) =$$

$$[(\sim a \vee \nabla \sim a) \wedge (\sim a \vee \nabla b) \wedge (\sim b \vee \nabla \sim a) \wedge (\sim b \vee \nabla b)] \vee a' \vee b \vee (\nabla \sim a \wedge \nabla a') =$$

$$[\nabla \sim a \wedge (\sim a \vee \nabla b) \wedge (\sim b \vee \nabla \sim a)] \vee a' \vee b \vee (\nabla \sim a \wedge \nabla a') =$$

$$[\nabla \sim a \wedge (\sim a \vee \nabla b)] \vee a' \vee b \vee (\nabla \sim a \wedge \nabla a') \geq$$

$$\sim a \vee a' \vee (\nabla \sim a \wedge \nabla a') = a \rightarrowtail a'$$

and since $a \rightarrowtail a' \in D$, from the preceding inequality it follows that $(a \vee b) \rightarrowtail (a' \vee b) \in D$. In a similar fashion one proves that $(a' \vee b) \rightarrowtail (a \vee b) \in D$.

Eq6) From $a \equiv a'$ (mod D) it follows by (A) that $a \vee b \equiv a' \vee b$ (mod D) and from $b \equiv b'$ (mod D) it follows that $b \vee a' \equiv b' \vee a'$ (mod D), so $a \vee b \equiv a' \vee b'$ (mod D).

Recall (see for instance [66]) that if E, E' are non-empty sets and f is a function from E to E' then the following relation, defined on E:

$$a, b \in E, aR_f b \textit{ iff } f(a) = f(b)$$

is an equivalence relation. We represent by $C(a)$ the equivalence of a with respect to R_f, this is:

$$C(a) = \{b \in E : bR_f a\} = \{b \in E : f(b) = f(a)\}.$$

Recall that if R is an equivalence relation over a set E, the *q*uotient set of E by R is the set of all the equivalence classes with respect to R. This set is denoted by E/R.

Consider the transformation $\varphi : E \to E/R$ defined by $\varphi(x) = C(x)$. It is well known that φ is well defined since if $x = y$ then $\varphi(x) = C(x) = C(y) = \varphi(y)$. Furthermore, φ is surjective, since given $x' \in E/R$ this is, $x' = C(x)$ with $x \in E$, then $\varphi(x) = C(x) = x'$. Therefore E/R is an image of E by means of φ.

φ is called the *c*anonical transformation ("according to the rules") or *n*atural transformation from E onto E/R.

We shall prove that every image of E may be obtained this way, this is, considering an equivalence relation R over E and constructing the quotient set E/R.

Lemma 2.2.3. *If $f_1 : E \to E_1, f_2 : E \to E_2$, $f_1(E) = E_1$, and $R_{f_1} \subseteq R_{f_2}$, (this is, $aR_{f_1}b \Rightarrow aR_{f_2}b$) then there exists a unique function h from E_1 to E_2 such that $h \circ f_1 = f_2$. If $f_2(E) = E_2$ then h is surjective.*

This is a standard algebraic construction and we just recall how the function h is defined. Given $a_1 \in E_1 = f_1(E)$, we have that $a_1 = f_1(a)$ with $a \in E$ and thus $f_2(a) = a_2 \in E_2$. Then $h(a_1)$ is defined to be $a_2 = f_2(a)$.

Lemma 2.2.4. *If $f_1(E) = E_1, f_2(E) = E_2$, and $R_{f_1} = R_{f_2}$, then E_1 and E_2 have the same cardinality.*

Lemma 2.2.5. *If $f(E) = E_1$ then $E' = E/R_f$ has the same cardinality as E_1.*

Therefore, all the images of a non-empty set E are obtained (up to a bijection) considering equivalence relations R on E and constructing E/R, since: 1) if R is an equivalence relation on E then $E' = E/R$ is a image of E and 2) if E' is an image of E there exists an equivalence relation R defined on E such that E' has the same cardinality as E/R.

Given a deductive system D of a Łukasiewicz algebra L, the quotient set of L by the congruence "$\equiv$" is denoted by $A/_\equiv$ or A/D. For each $x \in L$, we denote $C_D(x) = \{y \in L : y \equiv x\}$ or simply by $C(x)$ the equivalence class containing the element x.

If $x, y \in L$ and we algebrize the set A/D by

$$(1)\ C(x) \vee C(y) = C(x \vee y); \qquad (2)\ C(x) \wedge C(y) = C(x \wedge y)$$

$$(3)\ \sim C(x) = C(\sim x); \qquad (4)\ \nabla C(x) = C(\nabla x); \qquad (5)\ 1' = C(1)$$

then as we proved above, the system $(A' = A/D, 1', \nabla, \sim, \vee, \wedge)$ is a Łukasiewicz algebra which we denominate the quotient algebra of L by D.

Clearly the transformation $h : A \to A/D$ defined by $h(a) = C(a)$ is a homomorphism, called the natural homomorphism, from A onto A/D, and which has D as its kernel. Indeed, let $N = h^{-1}(1')$. We prove that $N = D$. If $x \in N$ then $C(x) = h(x) = 1' = C(1)$, so $x \equiv 1$ and therefore there exists $d \in D$ such that $x \wedge d = 1 \wedge d$ this is $x \wedge d = d$ hence $d \leq x$, so since $d \in D$ and D is a filter, we have that $x \in D$. Conversely let $d \in D$ then since $d \wedge d = 1 \wedge d$ it follows that $d \equiv 1$ (mod D), so $d \in C(1) = 1'$ this is $h(d) = 1'$, then $d \in N$.

Lemma 2.2.6. *Let L, L_1, L_2 be Łukasiewicz algebras, and let $h_1 : L \to L_1$, $h_2 : L \to L_2$ be homomorphisms such that $h_1(L) = L_1$, and $Ker(h_1) \subseteq Ker(h_2)$. Then there exists a homomorphism $h : L_1 \to L_2$. If furthermore $h_2(L) = L_2$, then L_2 is a homomorphic image of L_1.*

Proof. $Ker(h_1) \subseteq Ker(h_2)$, is equivalent to $R_{h_1} \subseteq R_{h_2}$, so by Lemma 2.2.3 the transformation $h : L_1 \to L_2$ defined by $h(a_1) = h_2(a)$, where $h_1(a) = a_1$ is a function from L_1 to L_2 that verifies $h \circ h_1 = h_2$ and h is the unique one in those conditions. Let us prove that in this case h is a homomorphism.

H1) $h(a_1 \vee b_1) = h(a_1) \vee h(b_1)$.

Let $a, b \in L$ be such that $h_1(a) = a_1, h_1(b) = b_1$ then $h(a_1) \vee h(b_1) = h(h_1(a)) \vee h(h_1(b)) = h_2(a) \vee h_2(b) = h_2(a \vee b) = (h \circ h_1)(a \vee b) = h(h_1(a \vee b)) = h(h_1(a) \vee h_1(b)) = h(a_1 \vee b_1)$.

H2) $h(\sim a_1) = h(\sim h_1(a)) = h(h_1(\sim a)) = h_2(\sim a) =\sim h_2(a) =\sim ((h \circ h_1)(a)) =\sim (h(h_1(a)) =\sim h(a_1)$.

H3) $h(\nabla a_1) = h(\nabla h_1(a)) = h(h_1(\nabla a)) = h_2(\nabla a) = \nabla h_2(a) = \nabla((h \circ h_1)(a)) = \nabla(h(h_1(a)) = \nabla h(a_1)$.

If $h_2(L) = L_2$, then by Lemma 2.2.3, h is surjective. □

Lemma 2.2.7. *Let L, L_1, L_2 be Łukasiewicz algebras, $h_1 : L \to L_1$, $h_2 : L \to L_2$ epimorphisms such that $Ker(h_1) = Ker(h_2)$. Then L_1 and L_2 are isomorphic Łukasiewicz algebras.*

Proof. $Ker(h_1) = Ker(h_2)$, is equivalent to $R_{h_1} = R_{h_2}$, so by Lemma 2.2.4 the transformation $h : L_1 \to L_2$ defined by $h(a_1) = h_2(a)$, where $h_1(a) = a_1$ is a bijection from L_1 to L_2. By Lemma 2.2.6, h is a homomorphism, so h is an isomorphism. □

Corollary 2.2.8. *If L and L' are Łukasiewicz algebras and h is a homomorphism from L onto L' then L' is isomorphic to $L/Ker(h)$.*

Proof. Let $F = Ker(h)$, so F is a deductive system of L and in consequence $L'' = L/Ker(h)$ is a Łukasiewicz algebra. Furthermore, $\varphi(x) = C_{Ker(h)}(x)$ is an epimorphism from L to L'' such that $Ker(\varphi) = F = Ker(h)$ and therefore $L'' = L/Ker(h) \cong L'$. □

Thus we have proved the following result by A. Monteiro, [36]:
Every homomorphic image of a Łukasiewicz algebra is obtained (up to isomorphism) considering deductive systems D of L and constructing L/D.

Lemma 2.2.9. *If L and L' are Łukasiewicz algebra, $h : L \to L'$ is a homomorphism, and $G \subseteq L$ is such that $LS(G) = L$ then $LS(h(G)) = h(L)$. This is if G generates L then $h(G)$ generates $h(L)$.*

Proof. Let $S' = LS(h(G))$, so S' is an L-subalgebra of L', and in consequence $S = h^{-1}(S')$ is an L-subalgebra of L. Furthermore (1) $G \subseteq S$ given that if $g \in G$ then $h(g) \in h(G) \subseteq S'$ and therefore $g \in h^{-1}(S') = S$. From (1) it follows that $L = LS(G) \subseteq S$ and therefore $S = L = LS(G)$. Since h is a function from L onto $h(L)$ we have that $h(L) = h(S) = h(h^{-1}(S')) = S' = LS(h(G))$. □

Corollary 2.2.10. *If L and L' are Łukasiewicz algebras, $h : L \to L'$ is an epimorphism, and $G \subseteq L$ is such that $LS(G) = L$ then $LS(h(G)) = L'$.*

Lemma 2.2.11. *Let L and L' be Łukasiewicz algebras. For an epimorphism $h : L \to L'$ to be an isomorphism, it is necessary and sufficient that $Ker(h) = \{1\}$.*

Proof. Assume that h is a isomorphism, then $a \in Ker(h)$ this is $h(a) = 1 = h(1)$, so since h is one to one, $a = 1$. Assume now that (1) $Ker(h) = \{1\}$ and that $h(a) = h(b)$ so $1 = h(a \rightarrowtail b) = h(b \rightarrowtail a)$ and therefore $a \rightarrowtail b, b \rightarrowtail a \in Ker(h)$ then $a \rightarrowtail b = 1 = b \rightarrowtail a$ and in consequence $a = b$. □

If X is a subset of a Łukasiewicz algebra L we denote with $\sim X$ the set $\{\sim x : x \in X\}$.

Lemma 2.2.12. *If R is a congruence relation defined over a Łukasiewicz algebra L then:*

a) $C(1) = \{x \in L : xR1\}$ *is a deductive system.*
b) *The following conditions are equivalent:*
  b1) aRb.
  b2) $a \to b, \sim a \to \sim b, b \to a, \sim b \to \sim a \in C(1)$.
  b3) *There exists* $n \in C(1)$ *such that* $a \wedge n = b \wedge n$.
c) $C(1) = \sim C(0) = \{\sim x : x \in C(0)\}$.

Proof. a) Since $1R1$ then F1) $1 \in C(1)$.

F2) Let $x, y \in C(1)$, so $xR1, yR1$, and since R is compatible with $\wedge$ we have that $(x \wedge y)R(1 \wedge 1) = 1$, therefore $x \wedge y \in C(1)$.

F3) If $x \in C(1)$, then $xR1$, and since yRy, and R is compatible with $\wedge$, we have that $(x \wedge y)R(1 \wedge y) = y$. Therefore if $x \leq y$ where $y \in L$, we have that $x \wedge y = x$, and therefore xRy. Since $xR1$ we get, given that R is an equivalence relation, that $yR1$, this is $y \in C(1)$.

Note that it is enough that R is compatible with $\wedge$ to prove that $C(1)$ is a filter.

F4) If $x \in C(1)$, this is, $xR1$ then as R is compatible with Δ we have that $\Delta x R \Delta 1 = 1$, and therefore $\Delta x \in C(1)$. We have thus proved that $C(1)$ is a deductive system.

b) b1) implies b2):

Recall that by L13), $\nabla \sim a \vee a = 1$, so from aRb, since R is compatible with $\vee$ it follows that:

$$1 = (\nabla \sim a \vee a)R(\nabla \sim a \vee b) = a \to b,$$

therefore $a \to b \in C(1)$. From aRb, since R is compatible with $\sim$ it follows that $\sim aR \sim b$ then, as R is compatible with $\vee$ we have that $1 = (\nabla a \vee \sim a)R(\nabla a \vee \sim b) = \sim a \to \sim b$ therefore $\sim a \to \sim b \in C(1)$. Analogously, we can prove $b \to a \in C(1)$ and $\sim a \to \sim b \in C(1)$.

b2) implies b3):

From $a \to b, \sim a \to \sim b \in C(1)$ and $b \to a, \sim b \to \sim a \in C(1)$, since $C(1)$ a filter, it follows that $a \rightarrowtail b, b \rightarrowtail a \in C(1)$, so

$$n = (a \rightarrowtail b) \wedge (b \rightarrowtail a) \in C(1),$$

and as in the proof of Lemma 2.2.2, we can show that $a \wedge n = b \wedge n$.

b3) implies b1):

Assume now that there exists $n \in C(1)$ such that $a \wedge n = b \wedge n$. From $n \in C(1)$ it follows that $nR1$, so, since R is compatible with $\wedge$, we have that $a \wedge nRa \wedge 1 = a$ and $(b \wedge n)R(b \wedge 1) = b$. Since by hypothesis $a \wedge n = b \wedge n$, it follows that aRb.

c) $x \in C(1) \Leftrightarrow xR1 \Leftrightarrow$ (R is compatible with $\sim$) $\sim xR \sim 1 = 0 \Leftrightarrow \sim x \in C(0) \Leftrightarrow x \in \sim C(0)$.

□

We can now conclude that there exists a bijective correspondence between the deductive systems of a Łukasiewicz algebra L and the set of congruences defined on L.

2.3. Construction of deductive systems

We will indicate methods to obtain the deductive systems of a Łukasiewicz algebra. In the development of a deductive theory, as in mathematics, a set H of statements is considered, and each of those statements is called an axiom or hypothesis. The statements in H are not, in general, tautologies of the propositional calculus, and thus they are said to be true by hypothesis.

The goal is then to obtain the *logical consequences* of the hypothesis in H, which is done through *proofs* that must follow the rules of logic.

We consider first the case in which H is a finite sequence of elements of a Łukasiewicz algebra L.

Definition 2.3.1. *Given a sequence $h_1, h_2, \ldots, h_n$ of elements of L, $x \in L$ is said to be a consequence of the sequence, and denoted by $h_1, h_2, \ldots, h_n \vdash x$ if*

$$(h_1 \wedge h_2 \wedge \ldots \wedge h_n) \to x = 1.$$

The intuitive concept of logical consequence is such that if x is a consequence of the sequence $h_1, h_2, \ldots, h_n$ then, if we alter the order of the hypothesis, then x is also a consequence of the new sequence thus obtained. This is what we point out in the following results:

Lemma 2.3.2. *If $h_1, h_2, \ldots, h_{k-1}, h_k, h_{k+1}, \ldots, h_n \vdash x$ then*

$$h_1, h_2, \ldots, h_{k-1}, h_{k+1}, h_k, \ldots, h_n \vdash x.$$

Proof. The lemma is immediate from the commutative property of the meet operation. □

From the lemma above, by trivial procedures, we prove:

Lemma 2.3.3. *If $h_1, h_2, \ldots, h_n \vdash x$ and $k_1, k_2, \ldots, k_n$ is a permutation of the elements $h_1, h_2, \ldots, h_n$ then $k_1, k_2, \ldots, k_n \vdash x$.*

We see thus that the fact that x is a consequence of the hypothesis appearing in a sequence does not depend on their order, but just on the hypothesis in the set.

We can then replace the definition above by the following one:

Definition 2.3.4. *Given a finite, non-empty set of elements of L, $H = \{h_1, h_2, \ldots, h_n\}$ we say that $x \in L$ is consequence of H and denote by $(h_1, h_2, \ldots, h_n) \vdash x$ if*

$$(h_1 \wedge h_2 \wedge \ldots \wedge h_n) \to x = 1.$$

Lemma 2.3.5. *If $(h_1, h_2, \ldots, h_n) \vdash x$ then $(h_0, h_1, h_2, \ldots, h_n) \vdash x$.*

PROOF. Since

$$h_0 \wedge h_1 \wedge h_2 \wedge \ldots \wedge h_n \leq h_1 \wedge h_2 \wedge \ldots \wedge h_n$$

then by the hypothesis $(h_1 \wedge h_2 \wedge \ldots \wedge h_n) \to x = 1$ and property ID7) of $\to$ we have that

$$1 = (h_1 \wedge h_2 \wedge \ldots \wedge h_n) \to x \leq (h_0 \wedge h_1 \wedge h_2 \wedge \ldots \wedge h_n) \to x,$$

which proves the lemma. □

Definition 2.3.6. *We say that $x \in L$ is consequence of the empty subset of L if $x = 1$ and denote this by $\emptyset \vdash x$, this is $\emptyset \vdash x$ if and only if $x = 1$.*

Definition 2.3.7. *Given a part H of a Łukasiewicz algebra L, we say that x is a consequence of H, if x is a consequence of a finite part of H and denote this by $H \vdash x$.*

Given a subset H of a Łukasiewicz algebra L we denote by $\mathbf{C}(H)$ the set of all the consequences of H. This operator will be called the *consequence operator.*

Given a subset H of a Łukasiewicz algebra L, we call the *deductive system generated* by H the intersection of all the deductive systems containing H, and we denote it by $D(H)$. Since L is a deductive system containing H and that the intersection of deductive systems is a deductive system, the operator D is well defined. Clearly $D(H)$ is the least deductive system containing H.

The definition of $D(H)$ isn't constructive since it involves the family of all the deductive systems containing H.

Theorem 2.3.8. *If H is a subset of a Łukasiewicz algebra then $D(H) = \mathbf{C}(H)$.*

PROOF. First case: $H = \emptyset$. By definition $\mathbf{C}(\emptyset) = \{1\}$. On the other hand, since $\{1\}$ is a deductive system, and it is the least deductive system of L, it is contained in any other deductive system so $D(\emptyset) = \{1\}$.

Second case: $H \neq \emptyset$.

(i) $\mathbf{C}(H)$ is a deductive system.

D1) $1 \in \mathbf{C}(H)$. Indeed, since $H \neq \emptyset$ there exists $h \in H$ and since $h \to 1 = 1$ it follows that $1 \in \mathbf{C}(H)$.

D2) If $x \in \mathbf{C}(H)$ and $x \to y \in \mathbf{C}(H)$ then $y \in \mathbf{C}(H)$.

By hypothesis there exist finite subsets of $H, \{h_1, h_2, \ldots, h_n\}$ and $\{k_1, k_2, \ldots, k_m\}$ such that:

$$(1) \quad \left(\bigwedge_{i=1}^{n} h_i\right) \to x = 1,$$

$$(2) \quad \left(\bigwedge_{j=1}^{m} k_j\right) \to (x \to y) = 1.$$

Put $h = \bigwedge_{i=1}^{n} h_i$ and $k = \bigwedge_{j=1}^{m} k_j$, then

$$(3) \quad h \to x = 1$$

$$(4) \quad k \to (x \to y) = 1.$$

Since for all $a \in L$, $a \to 1 = 1$ from (4) it follows that

$$(5) \quad h \to (k \to (x \to y)) = 1,$$

so by property ID10) of $\to$, (5) can be written as

$$(6) \quad (h \wedge k) \to (x \to y) = 1.$$

By ID15) we know that

$$(7) \quad ((h \wedge k) \to (x \to y)) \to ((h \wedge k) \to x) \to ((h \wedge k) \to y) = 1.$$

Then since by ID4), $1 \to a = a$, from (6) and (7) it follows that

$$(8) \quad ((h \wedge k) \to x) \to ((h \wedge k) \to y) = 1.$$

By ID10) we know that

$$(9) \quad (h \wedge k) \to x = (k \wedge h) \to x = k \to (h \to x)$$

and by (3) we have

$$(10) \quad (h \wedge k) \to x = 1.$$

So, since $1 \to a = a$, from (8) and (10) it follows that

$$(11) \quad (h \wedge k) \to y = 1$$

this is

$$(12) \quad \left(\bigwedge_{i=1}^{n} h_i \wedge \bigwedge_{j=1}^{m} k_j \right) \to y = 1$$

where $h_i \in H, 1 \leq i \leq n$ and $k_j \in H, 1 \leq j \leq m$, and therefore $y \in \mathbf{C}(H)$.

From D1) and D2) it follows that $\mathbf{C}(H)$ is a deductive system.

(ii) $\underline{H \subseteq \mathbf{C}(H)}$.

Let $h \in H$, so since $h \to h = 1$ it follows that $H \vdash h$, this is $h \in \mathbf{C}(H)$.

From (i) and (ii) it follows that

(iii) $D(H) \subseteq \mathbf{C}(H)$.

(iv) $\underline{\mathbf{C}(H) \subseteq D(H)}$.

If $x \in \mathbf{C}(H)$ then

$$(13) \quad \left(\bigwedge_{i=1}^{n} h_i \right) \to x = 1, \quad h_i \in H, 1 \leq i \leq n$$

so since $h_i \in H \subseteq D(H)$ for $1 \leq i \leq n$ and $D(H)$ is a filter it follows that

$$(14) \quad \bigwedge_{i=1}^{n} h_i \in D(H).$$

From (13) and (14), it follows by modus ponens that $x \in D(H)$.

From (iii) and (iv) it follows that $D(H) = \mathbf{C}(H)$. □

We have thus a constructive way of obtaining deductive systems generated by a given set. We will now prove a result known as the *Deduction Theorem* which was proved first, in classical logic, by Alfred Tarski and Jacques Herbrand.

Theorem 2.3.9. Deduction Theorem

$$H \cup \{x\} \vdash y \iff H \vdash x \to y$$

Proof. If $H = \emptyset$ then $H \cup \{x\} = \{x\}$ so $H \cup \{x\} = \{x\} \vdash y \iff x \to y = 1 \iff \emptyset \vdash x \to y$. Assume now that $H \neq \emptyset$.

Necessity: If $H \cup \{x\} \vdash y$ then there exist $g_1, g_2, \ldots, g_n \in H \cup \{x\}$ such that

$$(1) \qquad \left(\bigwedge_{i=1}^{n} g_i\right) \to y = 1$$

Let $G = \{g_1, g_2, \ldots, g_n\}$. There are three cases to be considered:

- First case: $g_i \in H$ for all $i, 1 \leq i \leq n$.
  By ID10),
  $$(2) \qquad \left(\bigwedge_{i=1}^{n} g_i\right) \to (x \to y) = x \to \left(\bigwedge_{i=1}^{n} g_i \to y\right)$$
  Since $a \to 1 = 1$, from (2) and (1) we have that
  $$\left(\bigwedge_{i=1}^{n} g_i\right) \to (x \to y) = 1$$
  with $g_i \in H$ for all $i, 1 \leq i \leq n$, so $H \cup \vdash x \to y$.
- Second case: $g_i = x$ for all $i, 1 \leq i \leq n$.
  From (1) we have that
  $$x \to y = \left(\bigwedge_{i=1}^{n} g_i\right) \to y = 1$$
  and in consequence $h \to (x \to y) = 1$, for all $h \in H$, so $H \cup \vdash x \to y$.
- Third case: $G \cap H \neq \emptyset$ and $G \cap \{x\} \neq \emptyset$.
  Assume that $G \cap H = \{h_1, h_2, \ldots, h_r\}$ then by (1) we have that
  $$1 = \left(\bigwedge_{i=1}^{n} g_i\right) \to y = \left(\left(\bigwedge_{i=1}^{r} h_i\right) \wedge x\right) \to y$$
  Then by ID10),
  $$\left(\bigwedge_{i=1}^{r} h_i\right) \to (x \to y) = 1$$
  where $h_i \in H$ for $1 \leq i \leq r$, and therefore $H \vdash x \to y$.

Sufficiency: Assume that $H \vdash x \to y$. Then there exist $h_1, h_2, \ldots, h_n \in H$ such that

$$\left(\bigwedge_{i=1}^{n} h_i\right) \to (x \to y) = 1$$

then by property ID10)

$$\left(\left(\bigwedge_{i=1}^{n} h_i\right) \wedge x\right) \to y = 1 \tag{2.3.1}$$

this is $H \cup \{x\} \vdash y$. □

Given a deductive system H of a Łukasiewicz algebra L and $a \in L$ we denote with $D(H,a)$ the deductive system generated by the set $H \cup \{a\}$.

Theorem 2.3.10. *If H is a deductive system of a Łukasiewicz algebra L and $a \in L$ then*

$$D(H,a) = \{x \in L : a \to x \in H\}.$$

Proof. Since H is a deductive system we have that $\mathbf{C}(H) = D(H) = H$, and by Theorem 2.3.8 we know that $D(H,a) = \mathbf{C}(H \cup \{a\})$ so

$$x \in D(H,a) \iff x \in \mathbf{C}(H \cup \{a\}) \iff H \cup \{a\} \vdash x \iff$$

$$H \vdash (a \to x) \iff a \to x \in \mathbf{C}(H) = H$$

□

2.4. Arithmetics of the deductive systems

We saw that the determination of all the homomorphic images of a Łukasiewicz algebra L can be reduced to the determination of all its deductive systems. We will study now the properties of the family $\mathbf{D}(L)$ of all the deductive systems of L. It is clear that $(\mathbf{D}(L), \subseteq)$ is a poset.

Among the deductive systems of a Łukasiewicz algebra L are L and the set $\{1\}$, which are different if L has more than one element. Clearly the intersection of deductive systems is a deductive system. A deductive system D of a Łukasiewicz algebra L is said to be proper if $D \neq L$.

Lemma 2.4.1. *If D is a proper deductive system of a Łukasiewicz algebra L then $a \wedge \sim a \notin D$, for all $a \in L$.*

Proof. Indeed, if $a \wedge \sim a \in D$ then since D is a Δ-filter, $0 = \Delta(a \wedge \sim a) \in D$ and $D = L$. □

Calling *contradictions* the elements of the form $a \wedge \sim a$, then in the Łukasiewicz algebras no proper deductive system contains contradictions.

Lemma 2.4.2. *If $\mathcal{K} = \{D_i\}_{i \in I}$ is a chain of deductive systems of a Łukasiewicz algebra L then $D = \bigcup_{i \in I} D_i$ is a deductive system of L.*

Proof. Since $1 \in D_i$ for all $i \in I$ then: $\underline{\text{D1) } 1 \in D}$.

We prove now that: $\underline{\text{D2) If } x \in D \text{ and } x \to y \in D \text{ then } y \in D}$.

From $x \in D$ it follows that there exists $j \in I$ such that $x \in D_j$ and from $x \to y \in D$ it follows that there exists $h \in I$ such that $x \to y \in D_h$. Since $D_j, D_h \in \mathcal{K}$ and $\mathcal{K}$ is a chain then (i) $D_j \subseteq D_h$ or (ii) $D_h \subseteq D_j$. Assuming that (i) is the case, then $x, x \to y \in D_h$ and since D_h is a deductive system we have that $y \in D_h$ and in consequence $y \in D$. If (ii) is the case, the proof is similar. □

Let D be a proper deductive system of L and $a \notin D$. We consider the family

$$\mathbf{D}(D, a) = \{D' \in \mathbf{D}(L) : D \subseteq D', a \notin D'\}$$

It is clear that $D \in \mathbf{D}(D, a)$ and that $(\mathbf{D}(D, a), \subseteq)$ is a poset.

Lemma 2.4.3. *The poset $(\mathbf{D}(D, a), \subseteq)$ is inductive. This is, it is a poset in which every chain has an upper bound.*

Proof. We have to prove that every chain $\mathcal{K}$ of elements of the set $\mathbf{D}(D, a)$ has an upper bound in $\mathbf{D}(D, a)$.

Let $\mathcal{K} = \{D_i\}_{i \in I}$ be a chain of $\mathbf{D}(D, a)$ and consider the set $D' = \bigcup_{i \in I} D_i$, then by Lemma 2.4.2 we have: (1) $D' \in \mathbf{D}(L)$, and (2) $a \notin D'$, since $a \notin D_i$ for all $i \in I$. Since $D \subseteq D_i$ for all $i \in I$ then (3) $D \subseteq \bigcup_{i \in I} D_i = D'$, and therefore from (1), (2) and (3) it follows that (4) $D' \in \mathbf{D}(D, a)$, and from (3) and (4), that D' is an upper bound of $\mathcal{K}$ belonging to the set $\mathbf{D}(D, a)$. Therefore the poset $(\mathbf{D}(D, a), \subseteq)$ is inductive.

Notice that D' is a proper deductive system of L given that $a \notin D'$. □

Since the set $\mathbf{D}(D, a)$ is inductive, by Zorn's Lemma, this set has at least a maximal element.

If L is a non trivial Łukasiewicz algebra and $a \in L$, is such that $a \neq 1$, then $D = \{1\}$ is a deductive system of L not containing the element a. Let

$$\mathbf{D}(a) = \mathbf{D}(\{1\}, a) = \{D' \in \mathbf{D}(L) : \{1\} \subseteq D', a \notin D'\} = \{D' \in \mathbf{D}(L) : a \notin D'\}.$$

This is, if $a \neq 1$, $\mathbf{D}(a)$ represents the family of all the deductive systems of L not containing the element a.

Each maximal element of this poset will be called a *d*eductive system bounded to the element a and denoted by D_a.

Each maximal element of the poset $\mathbf{D}(0)$ will be called a *m*aximum deductive system, this is the maximum deductive systems are the maximal elements of the poset of all the proper deductive systems. If L is a Łukasiewicz algebra with more than one element we denote with $\mathbf{M}(L)$ the set of all the maximum deductive systems of L.

Lemma 2.4.4. *For a principal filter $F(x) = [x)$ of a Łukasiewicz algebra L to be a deductive system it is necessary and sufficient that $x \in B(L)$.*

Proof. If $[x)$ is a deductive system, this is a Δ-filter, then from $x \in [x)$ it follows that $\Delta x \in [x)$, this is $x \leq \Delta x$, so $\Delta x = x$, this is $x \in B(L)$. Conversely if $x \in B(L)$, let $y \in [x)$ so $x \leq y$ and in consequence $x = \Delta x \leq \Delta y$, then $\Delta y \in [x)$ which proves that $[x)$ is a Δ-filter, thus a deductive system. □

Lemma 2.4.5. *Let $D \in \mathbf{D}(L)$ and $a \notin D$, then if M is a maximal element of $\mathbf{D}(D, a)$, M is a maximal element of $\mathbf{D}(a)$.*

Proof. (i) $\mathbf{D}(D, a) \subseteq \mathbf{D}(a)$.

Let $D' \in \mathbf{D}(D, a)$, so $a \notin D'$. It follows that $D' \in \mathbf{D}(a)$.

(ii) Assume that M is a maximal element of $\mathbf{D}(D,a)$, in particular $M \in \mathbf{D}(D,a)$ then by (i) we have that $M \in \mathbf{D}(a)$. If M were not a maximal element of $\mathbf{D}(a)$, there would exist (1) $C \in \mathbf{D}(a)$ such that (2) $M \subset C$.

From (1) it follows that (3) $a \notin C$. By hypothesis (4) $D \subseteq M$, so from (2) and (4) we have (5) $D \subset C$. From (5) and (3), it follows that since C is a deductive system, $C \in \mathbf{D}(D,a)$. Thus we have that $M \in \mathbf{D}(D,a)$, M is maximal, $C \in \mathbf{D}(D,a)$ and $M \subset C$. Contradiction!

□

Notice that the converse of the preceding lemma is not true. For that consider the Łukasiewicz algebra L indicated in Example 1.2.4, where $B(L) = \{0,1,d,e\}$, with diagram indicated in figure A. The poset $(\mathbf{D}(L), \subseteq)$ is depicted in figure B. The elements of the set $\mathbf{D}(a)$ are marked with $\bullet$, therefore $[d)$ and $[e)$ are the maximal elements of $\mathbf{D}(a)$.

The set $\mathbf{D}([d),a)$ has a single element $[d)$, so $[e)$ is a maximal element of $\mathbf{D}(a)$ and not a maximal element of $\mathbf{D}([d),a)$ since $[e) \notin \mathbf{D}([d),a)$.

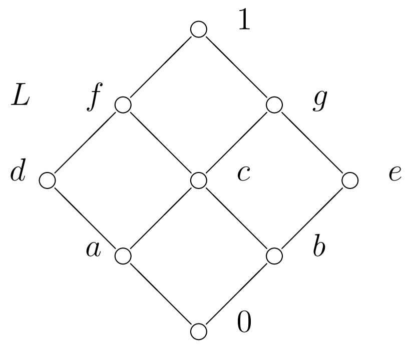


Figure A

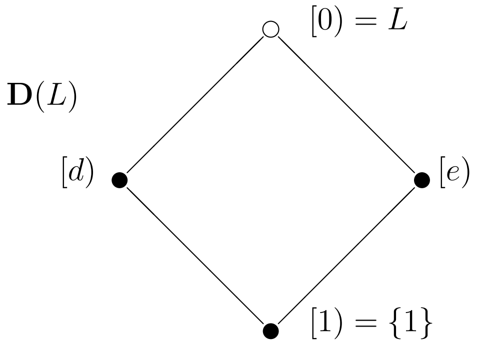


Figure B

Definition 2.4.6. *A deductive system D of a Łukasiewicz algebra L is* completely irreducible *if*

CI1) *D is proper,*

CI2) *If $\{D_i\}_{i\in I}$ is a family of deductive systems of L such that $D = \bigcap_{i\in I} D_i$ then there exists an index $i \in I$ such that $D = D_i$.*

D is irreducible *if:*

Ir1) *D is proper,*

Ir2) *If D_1 and D_2 are deductive systems of L such that $D = D_1 \cap D_2$ then $D = D_1$ or $D = D_2$.*

From the definition above, it is clear that every completely irreducible deductive system is irreducible.

Lemma 2.4.7. *For a deductive system of a Łukasiewicz algebra L to be completely irreducible it is necessary and sufficient for it to be bounded to some element of L.*

PROOF. Necessity: Let C be a completely irreducible deductive system, so C is a proper deductive system and therefore there exists at least an element $a \notin C$. Let D be a maximal element of $\mathbf{D}(C,a)$ so $C \subseteq D$. By Lemma 2.4.5, D is a deductive system bounded to a. Then for each $x \notin C$ there exists a deductive system D_x, bounded to x such that (1) $C \subseteq D_x$. Let us prove that $C = \bigcap\limits_{x\notin C} D_x$. By (1) it follows that (2) $C \subseteq \bigcap\limits_{x\notin C} D_x$. Now we prove that (3) $\bigcap\limits_{x\notin C} D_x \subseteq C$, which is equivalent to prove that $\complement C \subseteq \complement\left(\bigcap\limits_{x\notin C} D_x\right) = \bigcup\limits_{x\notin C} \complement D_x$. Let $y \in \complement C$ this is $y \notin C$ so since $y \notin D_y$ we have that $y \in \complement D_y \subseteq \bigcup\limits_{x\notin C} \complement D_x$. From (2) and (3) it follows that $C = \bigcap\limits_{x\notin C} D_x$ and therefore given that C is completely irreducible there exists $x \notin C$ such that $C = D_x$.

Sufficiency: Given a deductive system D_y bounded to y, since $y \notin D_y$ then D_y is proper. Assume that $D_y = \bigcap\limits_{i\in I} D_i$ where $\{D_i\}_{i\in I}$ is a family of deductive systems of L. Since $y \notin D_y$, there exists $i_0 \in I$ such that $y \notin D_{i_0}$, so $D_{i_0} \in \mathbf{D}(y)$. Furthermore, $D_y = \bigcap\limits_{i\in I} D_i \subseteq D_{i_0}$ and since D_y is a maximal element of $\mathbf{D}(y)$ it follows that $D_y = D_{i_0}$. □

Lemma 2.4.8. *Every proper deductive system of a Łukasiewicz algebra L is intersection of completely irreducible deductive systems.*

PROOF. Let H be a proper deductive system of L, so there exists $x \in L$ such that $x \notin H$. Therefore, there exists a maximal element M of $\mathbf{D}(H,x)$ and in consequence M is maximal in $\mathbf{D}(x)$ this is $M = D_x$. Furthermore, $H \subseteq D_x$.

Then for each $x \notin H$ there exists a D_x such that $H \subseteq D_x$ and therefore $H \subseteq \bigcap\limits_{x\notin H} D_x$. Let us prove next that (1) $\bigcap\limits_{x\notin H} D_x \subseteq H$.

To prove (1) is equivalent to prove that $\complement H \subseteq \complement\left(\bigcap\limits_{x\notin H} D_x\right) = \bigcup\limits_{x\notin H} \complement D_x$. Let $y \in \complement H$ then $y \notin H$ so $y \notin D_y$ and in consequence $y \in \complement D_y \subseteq \bigcup\limits_{x\notin H} \complement D_x$.

Thus $H = \bigcap\limits_{x\notin H} D_x$, where the deductive systems D_x are completely irreducible for all $x \notin H$. □

Corollary 2.4.9. *Every proper deductive system of a Łukasiewicz algebra L is intersection of irreducible deductive systems.*

PROOF. It is enough to note that every completely irreducible deductive system is irreducible. □

We will prove that in the Łukasiewicz algebras the notions of completely irreducible deductive system, irreducible deductive system and maximal deductive system are equivalent. We begin by proving:

Lemma 2.4.10. *Every irreducible deductive system P of a Łukasiewicz algebra L is a prime filter.*

Proof. By hypothesis P is proper. Assume that $x \vee y \in P$ and consider the deductive systems $D_1 = D(P, x), D_2 = D(P, y)$, so $D = D_1 \cap D_2 \neq \emptyset$, $P \subseteq D_1$ and $P \subseteq D_2$. In consequence $P \subseteq D_1 \cap D_2$.

We prove now that $D_1 \cap D_2 \subseteq P$. Let $t \in D_1 \cap D_2$, this is $t \in D(P, x)$ and $t \in D(P, y)$, so by Theorem 2.3.10 we have that: (1) $x \to t \in P$ and (2) $y \to t \in P$, so since P is a filter, from (1) and (2) it follows that (3) $(x \to t) \wedge (y \to t) \in P$, but by the property ID12), $(x \to t) \wedge (y \to t) = (x \vee y) \to t$ and therefore (4) $(x \vee y) \to t \in P$, so since by hypothesis (5) $x \vee y \in P$, from (4) and (5) it follows that $t \in P$.

We have thus proved that $P = D_1 \cap D_2$, so given that P is irreducible it follows that (6) $P = D_1$ or (7) $P = D_2$. Since $x \in D_1$ and $y \in D_2$ we have that $x \in P$ or $y \in P$ therefore P is a prime filter. □

The converse to this result is not valid in general. If we consider the Łukasiewicz algebra $\mathbf{T}$ from Example 1.2.3, then $[c)$ is a prime filter of $\mathbf{T}$ but it is not a deductive system since $c \in [c)$ and $0 = \Delta c \notin [c)$.

Lemma 2.4.11. *In a Łukasiewicz algebra L every proper deductive system is intersection of prime filters.*

Proof. By Corollary 2.4.9 we know that every proper deductive system is intersection of irreducible deductive systems and by Lemma 2.4.10 every irreducible deductive system is a prime filter. □

Lemma 2.4.12. *Every irreducible deductive system D of a Łukasiewicz algebra L is a maximal deductive system.*

Proof. Let $M \in \mathbf{D}(L)$ be such that (1) $D \subseteq M$ and (2) $m \in M$. By Lemma 2.4.10, D is a prime filter. From $m \vee \nabla \sim m = 1 \in D$ it follows that (3) $m \in D$ or (4) $\nabla \sim m \in D$. If (3) occurs then $M \subseteq D$, so by (1) we have $M = D$. If (4) occurs, then from (4) and (1) it follows that (5) $\nabla \sim m \in M$. From (2) it follows that since M is a Δ-filter, (6) $\Delta m \in M$, so by (5) and (6) we have that $0 = \Delta m \wedge \nabla \sim m \in M$ and therefore $M = L$. □

Corollary 2.4.13. *In Łukasiewicz algebras, the notions of completely irreducible deductive system, irreducible deductive system and maximal deductive system coincide.*

Proof. It is enough to notice that:

- By Definition 2.4.6 every completely irreducible deductive system is an irreducible deductive system.
- By Lemma 2.4.12 every irreducible deductive system is a maximal deductive system.
- Every maximal deductive system is a deductive system bounded to the bottom element of the algebra and therefore by Lemma 2.4.7 is also a completely irreducible deductive system.

□

Corollary 2.4.14. *In the Łukasiewicz algebras, every proper deductive system is intersection of maximal deductive systems.*

Proof. Immediate consequence of Lemma 2.4.8 and Corollary 2.4.13. □

Corollary 2.4.15. *If L is a non trivial Łukasiewicz algebra then $\{1\}$ is the intersection of all the maximal deductive systems of L.*

We characterize now the maximal deductive systems of a Łukasiewicz algebra, this is the completely irreducible deductive systems.

Theorem 2.4.16. *For a deductive system C of a Łukasiewicz algebra L to be completely irreducible it is necessary and sufficient that there exists $a \notin C$ such that for all $x \notin C$, $x \to a \in C$ holds.*

Proof. If C is a completely irreducible deductive system, then it is bounded to some element $a \notin C$, this is C is a maximal deductive system among the deductive systems not containing the element a. Let $x \notin C$ and consider the deductive system $D = D(C, x)$. Then $C \subseteq C \cup \{x\} \subseteq D(C, x)$ and (1) $a \in D(C, x)$, because otherwise C would not be maximal among the deductive systems that do not contain the element a. By Theorem 2.3.10 we know that (2) $D(C, x) = \{y \in L : x \to y \in C\}$. So from (1) and (2) we have that $x \to a \in C$.

Conversely assume that C is a deductive system verifying: there exists $a \in L$ such that (1) $a \notin C$ and (2) for all $x \notin C$, $x \to a \in C$ holds. Then C is a deductive system bounded to the element a. Otherwise, there would exist a deductive system D such that (3) $C \subset D$ and (4) $a \notin D$, so if x is an element verifying (5) $x \in D$ and (6) $x \notin C$, from the hypothesis it follows that (7) $x \to a \in C$ so by (3) we have (8) $x \to a \in D$ and from (5) and (8) it follows that $a \in D$, a contradiction. □

Corollary 2.4.17. *For a proper deductive system M of a Łukasiewicz algebra L to be maximal, it is necessary and sufficient that the following condition holds: (*) if $x \notin M$ then $x \to y \in M$ for all $y \in L$.*

Proof. If M is a maximal deductive system then it is bounded to any $x \notin M$, so $D(M, x) = L$. By lemma 2.3.10 we have that $x \to y \in M$ for all $y \in L$.

Conversely, let M be a proper deductive system verifying (*). Assume that there exists a deductive system D such that (1) $M \subset D \subseteq L$. Let $x \in D \setminus M$ so (2) $x \in D$ and (3) $x \notin M$. By (3), condition (*) implies that $x \to y \in M$ for all $y \in L$. Then by (1), $x \to y \in D$. From the fact that D is a deductive system and (2), it follows that $y \in D$ for all $y \in L$, therefore $D = L$. □

Theorem 2.4.18. *For a proper deductive system M of a Łukasiewicz algebra L to be maximal, it is necessary and sufficient that for all $x \in L$ either $x \in M$ or $\nabla \sim x \in M$ hold.*

Proof. If M is maximal then by Corollary 2.4.13, it is an irreducible deductive system so by Lemma 2.4.10, it is a prime filter and so from $x \vee \nabla \sim x = 1 \in M$ it follows that $x \in M$ or $\nabla \sim x \in M$.

Let $x \notin M$ then from the hypothesis it follows that $\nabla \sim x \in M$ and since $\nabla \sim x \leq \nabla \sim x \vee y = x \to y$ for all $y \in L$ and M is a filter we have that $x \to y \in M$ for all $y \in L$ and in particular $x \to y \in M$ for all $y \notin M$, so by Corollary 2.4.17 M is a maximal deductive system. □

Recall that every boolean algebra B can be regarded as a Łukasiewicz algebra, where $\nabla x = x$, for all $x \in B$, and therefore $\sim x$ is the boolean complement of x, this is $\sim x = -x$ and therefore $\nabla \sim x = -x$.

In this case, the notions of deductive system and filter coincide. Indeed, we know that every deductive system is a filter, and if D is a filter of B, since $\Delta x = x$ for all $x \in B$ then clearly D is a Δ-filter, and thus a deductive system. Therefore in this case, the notion of maximal deductive system coincides with the notion of ultrafilter. As a corollary of the previous theorem we have:

Theorem 2.4.19. (Stone's theorem) *For a proper filter U of a boolean algebra B to be an ultrafilter of B, it is necessary and sufficient that given $x \in B$ then $x \in U$ or $-x \in U$.*

Lemma 2.4.20. *In a Łukasiewicz algebra L, if M is a deductive system, and $a \in L$ we have $D(M,a) = F(M,\Delta a)$*

Proof. ($\subseteq$) From Lemma 2.1.15 (a), we have (1) $F(M,\Delta a) = \{x \in L : m \wedge \Delta a \leq x, \text{where } m \in M\}$. Let (2) $x \in D(M,a)$. By Theorem 2.3.10 and (2) it follows that $m = a \to x \in M$. Since $m \wedge \Delta a = (a \to x) \wedge \Delta a = (\nabla \sim a \vee x) \wedge \Delta a = x \wedge \Delta a \leq x$ we conclude by (1) that $x \in F(M,\Delta a)$.

($\supseteq$) By definition, $M \cup \{a\} \subseteq D(M,a)$. Since $D(M,a)$ is a Δ-filter, it also contains the set $M \cup \{\Delta a\}$. Therefore, $F(M,\Delta a) \subseteq D(M,a)$. □

Next theorem is a particular instance of the theorem given by L. Monteiro in 1971, for monadic Łukasiewicz algebras [62].

Theorem 2.4.21. *In a Łukasiewicz algebra L the following statements are equivalent:*

a) *M is a maximal deductive system,*
b) *if $a \notin M$, there exists $m \in M$ such that $\Delta a \wedge m = 0$,*
c) *if $\Delta a \vee b \in M$ then $a \in M$ or $b \in M$,*
d) *if $a \notin M$, then $\nabla \sim a \in M$,*
e) *if $a, b \notin M$, then $a \to b \in M$ and $b \to a \in M$.*

Proof. a) implies b): Consider the deductive system

$$D = D(M,a) = (\text{by Lemma 2.4.20}) = F(M,\Delta a).$$

Since $F(M,\Delta a) = \{x \in L : m \wedge \Delta a \leq x, \text{where } m \in M\}$, if $m \wedge \Delta a \neq 0$ for all $m \in M$ then D would be a proper deductive system such that $M \subset D$, a contradiction.

b) implies c): If $a \notin M$, by b) there exists $m \in M$ such that $\Delta a \wedge m = 0$. Since $b \wedge m = (\Delta a \vee b) \wedge m \in M$, then $b \in M$.

c) implies d): Since $\Delta a \vee \nabla \sim a = 1 \in M$, then by c) $\nabla \sim a \in M$.

d) implies e): If $a \notin M$, then $\nabla \sim a \in M$ and therefore $a \to b = \nabla \sim a \vee b \in M$. Analogously one can prove $b \to a \in M$.

e) implies a): If M were not maximal, there would exist a deductive system M' such that $M \subset M' \subset L$. Let (1) $a \in M' \setminus M$ and (2) $b \in L \setminus M'$, so $a, b \notin M$ and therefore from e) it follows that in particular $a \to b \in M$ and therefore (3) $a \to b \in M'$. From (1) and (3) it follows that $b \in M'$, which contradicts (2). □

Lemma 2.4.22. *If M is a maximal deductive system of a Łukasiewicz algebra L then*

$$(M \cap B(L)) \cup (\sim M \cap B(L)) = B(L) \text{ and } (M \cap B(L)) \cap (\sim M \cap B(L)) = \emptyset.$$

Proof. Let $X = (M \cap B(L)) \cup (\sim M \cap B(L))$ so $X \subseteq B(L)$. Let $b \in B(L)$. If $b \in M$ then clearly $b \in X$. If $b \notin M$ then by the previous theorem $\nabla \sim b \in M$, but since $b \in B(L)$ then $\sim b \in B(L)$ so $\sim b \in M$ and therefore $b \in \sim M$. It follows that $b \in \sim M \cap B(L)$, so $b \in X$.

If $a \in (M \cap B(L)) \cap (\sim M \cap B(L))$ then (1) $a \in B(L)$, (2) $a \in M$ and (3) $a \in \sim M$ so from (3) it follows that (4) $a = \sim m$ with (5) $m \in M$, then (6) $\sim a = m \in M$ and therefore from (2) and (6), $a \wedge \sim a \in M$, which is impossible, as we saw that no proper deductive system can contain contradictions. □

Corollary 2.4.23. *If M is a maximal deductive system of a Łukasiewicz algebra L and $b \in B(L)$ then $b \in M$ or $\sim b \in M$.*

Lemma 2.4.24. $D(X) = F(\Delta X)$.

Proof. $\underline{\text{(i) } \Delta X \subseteq D(X)}$. Let $y \in \Delta X$, this is $y = \Delta x$ where (1) $x \in X$. Since (2) $X \subseteq D(X)$ from (1) and (2) it follows that $x \in D(X)$ and since $D(X)$ is a deductive system we have that $y = \Delta x \in D(X)$.

Since $D(X)$ is a filter, from (i) it follows that $\underline{(3)\ F(\Delta X) \subseteq D(X)}$.

Now we prove that $\underline{\text{(ii) } X \subseteq F(\Delta X)}$. Given $x \in X$, $\Delta x \in \Delta X \subseteq F(\Delta X)$, so $\Delta x \in F(\Delta X)$ and since $\Delta x \leq x$ and $F(\Delta X)$ is a filter we have $x \in F(\Delta X)$.

We prove next that $\underline{\text{(iii) } F(\Delta X) \text{ is a } \Delta\text{-filter}}$. Let $y \in F(\Delta X)$, so by Lemma 2.1.15, (4) there exist $z_1, z_2, \ldots, z_n \in \Delta X$ such that $\bigwedge_{i=1}^{n} z_i \leq y$ so $\bigwedge_{i=1}^{n} \Delta z_i = \Delta\left(\bigwedge_{i=1}^{n} z_i\right) \leq \Delta y$ and since $z_i \in B(L)$, we have that (5) $\bigwedge_{i=1}^{n} z_i \leq \Delta y$. From (4) and (5) it follows that $\Delta y \in F(\Delta X)$. From (ii) and (iii) it follows that $\underline{D(X) \subseteq F(\Delta X)}$. □

We denote with $D(a)$ the deductive system generated by the set $\{a\}$. Every deductive system generated by a singleton set is called a *principal deductive system*. We saw in Lemma 2.4.24 that if $X \subseteq L$ then $D(X) = F(\Delta X)$, so if $X = \{a\}$ we have that $D(a) = F(\Delta a) = \{x \in L : \Delta a \leq x\}$. Since $\Delta a \leq x \iff 1 = \sim \Delta a \vee \Delta a \leq \sim \Delta a \vee x = a \to x$ then

$$D(a) = \{x \in L : a \to x = 1\}.$$

It is clear that $D(1) = \{1\} = D(\emptyset)$. We shall prove that if X is a finite non-empty subset of a Łukasiewicz algebra L then $D(X)$ is a principal deductive system. More precisely:

Lemma 2.4.25. $D(\{a_1, a_2, \ldots, a_n\}) = D\left(\bigwedge_{i=1}^{n} a_i\right)$.

PROOF. Let $a = \bigwedge_{i=1}^{n} a_i$, we will prove that $D(\{a_1, a_2, \ldots, a_n\}) = D(a)$. By Lemma 2.4.24 we have that

$$D(\{a_1, a_2, \ldots, a_n\}) = F(\Delta(\{a_1, a_2, \ldots, a_n\})) = F(\{\Delta a_1, \Delta a_2, \ldots, \Delta a_n\}),$$

From lattice theory we know that $F(\{x, y\}) = F(x \wedge y)$ so

$$F(\{\Delta a_1, \Delta a_2, \ldots, \Delta a_n\}) = F\left(\bigwedge_{i=1}^{n} \Delta a_i\right) = F\left(\Delta\left(\bigwedge_{i=1}^{n} a_i\right)\right) = F(\Delta a) = D(a).$$

□

Adapting Tarski's terminology, indicated in an analogous construction, we denominate *axiomatizable* deductive systems those that have a finite number of generators. Generally, in a deductive theory, a finite number of propositions $h_1, h_2, \ldots, h_n$ are taken to be true by hypothesis and their logical consequences are deduced. This situation has its algebraic counterpart when considering a finite number of elements $h_1, h_2, \ldots, h_n$ and studying the deductive system generated by the set $\{h_1, h_2, \ldots, h_n\}$, this is $D(\{h_1, h_2, \ldots, h_n\})$.

Since it is natural to admit as true the propositions $h_1, h_2, \ldots, h_n$, which is the same as admitting as true $h = \bigwedge_{i=1}^{n} h_i$, then the set of consequences of $h_1, h_2, \ldots, h_n$ coincides with the set of consequences of $h = h_1 \wedge h_2 \wedge \ldots \wedge h_n$, this is:

$$D(\{h_1, h_2, \ldots, h_n\}) = D(h_1 \wedge h_2 \wedge \ldots \wedge h_n).$$

It is natural to name *not axiomatizable* deductive system those that *do not* have a finite number of generators.

Observe finally that if L is a Łukasiewicz algebra then the ordered set $\mathbf{D}(L)$ is more precisely a bounded complete distributive lattice.

Indeed, if $D_1, D_2 \in \mathbf{D}(L)$ and we put by definition:

- $D_1 \sqcap D_2 = D_1 \cap D_2$
- $D_1 \sqcup D_2 = D(D_1 \cup D_2)$

then $D_1 \sqcap D_2$ and $D_1 \sqcup D_2$ are the meet and join of the deductive systems D_1 and D_2 respectively. The first and last elements of $\mathbf{D}(L)$ are $\{1\}$ and L respectively.

To prove that the lattice is distributive it remains to be proven that:

$$D_1 \sqcap (D_2 \sqcup D_3) \subseteq (D_1 \sqcap D_2) \sqcup (D_1 \sqcap D_3),$$

for every $D_1, D_2, D_3 \in \mathbf{D}(L)$ (the other inclusion holds for every lattice).

If $x \in D_1 \sqcap (D_2 \sqcup D_3) = D_1 \cap (D_2 \sqcup D_3)$ then (1) $x \in D_1$ and (2) $x \in (D_2 \sqcup D_3)$. From $x \in D_2 \sqcup D_3 = D(D_2 \cup D_3) =$ (Lemma 2.4.24) $= F(\Delta(D_2 \cup D_3))$, then using Lemma 2.1.15 it follows that (3) $\Delta y \leq x$ for some $y \in D_2 \cup D_3$.

If $y \in D_2$, since D_2 is a Δ-filter, it follows that $\Delta y \in D_2$, and by (3) we have that (4) $x \in D_2$ because D_2 is a filter.

Therefore, (1) and (4) implies $x \in D_1 \cap D_2 \subseteq (D_1 \sqcap D_2) \cup (D_1 \sqcap D_3) \subseteq (D_1 \sqcap D_2) \sqcup (D_1 \sqcap D_3)$. If $y \in D_3$ the proof is analogous.

Furthermore, since there always exist $\bigcap_{i \in I} D_i$ for every family $\mathcal{F} = \{D_i\}_{i\in I}$ of deductive systems of L, then (see, for example [73], p. 42) there also exists the least upper bound of $\mathcal{F}$ and therefore the lattice is complete.

2.5. Prime filters and deductive systems

This section includes results indicated by A. Monteiro in a Seminar in 1963, [37] and published in 1996, [48], [49].

Represent by $\mathbf{P}(L)$ the set all the prime filters of a Łukasiewicz algebra L and by $\mathbf{p}(L)$ the set of all the minimal prime filters of L, this is the set of the minimal elements of the ordered set $(\mathbf{P}(L), \subseteq)$. Therefore P is a minimal prime filter, if P is prime and there exists no prime filter properly included in P.

If $P \in \mathbf{P}(L)$, we can define $\varphi(P) = \complement \sim P \in \mathbf{P}(L)$. The function φ is denominated the Birula-Rasiowa transformation, [6], [7], and it verifies $\varphi(\varphi(P)) = P$ for all $P \in \mathbf{P}(L)$ and if $P, Q \in \mathbf{P}(L)$ then $P \subseteq Q \iff \varphi(Q) \subseteq \varphi(P)$.

We will study the connections between the prime filters of a Łukasiewicz algebra L and the maximal deductive systems of L.

Lemma 2.5.1. *If L is a Łukasiewicz algebra then:*

a) *If $b \notin P \in \mathbf{P}(L)$ and $b \in B(L)$ then $\sim b \in P$.*
b) *If F is a proper filter of L and $b \in F \cap B(L)$ then $\sim b \notin F$.*

Proof. a) We know that $\sim b \vee \nabla b = 1 \in P$ and since $b \in B(L)$, $\nabla b = b$ and therefore $\sim b \vee b = 1 \in P$, then since P is a prime filter and $b \notin P$ it follows that $\sim b \in P$.

b) If $\sim b \in F$ then $b \wedge \sim b \in F$, but since $b \in B(L)$ then $b \wedge \sim b = 0$ and therefore $F = L$, a contradiction.

□

Lemma 2.5.2. *If $P \in \mathbf{P}(L)$ then $F(\nabla P) \in \mathbf{M}(L)$ and $F(\nabla P) \subseteq P$.*

Proof. (i) $F(\nabla P) \subseteq P$.

We prove first (1) :$\nabla P \subseteq P$. Let $t \in \nabla P$, then $t = \nabla p$, where $p \in P$. Since $p \leq \nabla p = t$ and P is a filter, then $t \in P$. Since (2) P is a filter, then from (1) and (2) we deduce (i).

(ii) By the Corollary 2.1.17, $F(\nabla P)$ is a deductive system. We prove next that it is a maximal deductive system.

Assume that there exists $M \in \mathbf{M}(L)$ such that: (3) $F(\nabla P) \subset M$. We show that (4) $M \not\subseteq P$. From (3) it follows that there exists an element $m \in L$ such that (5) $m \in M \setminus F(\nabla P)$, so since M is a deductive system, from (5) we deduce that (6) $\Delta m \in M$. Since $\Delta m \leq m$, having (5) in mind it follows that (7) $\Delta m \notin F(\nabla P)$.

If (8) $M \subseteq P$, then from (6) it follows that, $\Delta m \in P$ and therefore $\Delta m = \nabla \Delta m \in \nabla P$, so $\Delta m \in F(\nabla P)$, which contradicts (7).

From (4) it follows that there exists $z \in L$, such that (9) $z \in M \setminus P$, then (10) $\Delta z \in M$, (11) $\Delta z \notin P$.

From (10) it follows, applying Lemma 2.5.1 b), that (12) $\sim \Delta z \notin M$. From (11) and Lemma 2.5.1 a), $\sim \Delta z \in P$, so $\sim \Delta z = \nabla \sim \Delta z \in \nabla P$, and therefore $\sim \Delta z \in F(\nabla P) \subseteq M$, which contradicts (12).

□

Lemma 2.5.3. *Every prime filter P of a Łukasiewicz algebra L contains one and only one maximal deductive system.*

Proof. By Lemma 2.5.2 we know that there exists a maximal deductive system contained in P, namely $F(\nabla P)$. Let $M, M' \in \mathbf{M}(L)$ be such that $M \subseteq P$, $M' \subseteq P$ and $M \neq M'$. Then there exists $x \in M \setminus M'$ or there exists $x \in M' \setminus M$. If $x \in M \setminus M'$ then $\Delta x \in M$ and $\Delta x \notin M'$. Since M' is a maximal deductive system then $L = D(M', \Delta x) = \{y \in L : \sim \Delta x \vee y \in M'\}$, so $\sim \Delta x \in L = D(M', \Delta x)$, this is $\sim \Delta x = \sim \Delta x \vee \sim \Delta x \in M'$, and since $M' \subseteq P$ we have $\sim \Delta x \in P$. Since $\Delta x \in M \subseteq P$, then $\Delta x \in P$ and by Lemma 2.5.1 (b), $\sim \Delta x \notin P$, a contradiction. If $x \in M' \setminus M$, we also arrive to a contradiction. □

The next lemma is a well known fact.

Lemma 2.5.4. *If A is a Kleene algebra and P is a prime filter of A then $P \subseteq \varphi(P)$ or $\varphi(P) \subseteq P$.*

Lemma 2.5.5. *If $P \in \mathbf{P}(L)$ then $F(\nabla P) \subseteq \varphi(P)$.*

Proof. Assume that (1) $F(\nabla P) \not\subseteq \varphi(P) = \complement \sim P$, then there exists (2) $m \in F(\nabla P)$, (3) $m \notin \varphi(P)$.

From (2) we deduce that (4) $\Delta m \in F(\nabla P)$ and since $\Delta m \leq m$, from (3) and (4) it follows that (5) $\Delta m \notin \varphi(P)$.

Since L is a Kleene algebra we know that every prime filter P of L, is comparable with $\varphi(P)$.

If $P \subseteq \varphi(P)$, then since $F(\nabla P) \subseteq P$, we have $F(\nabla P) \subseteq \varphi(P)$, which contradicts (1). Then (6) $\varphi(P) \subset P$. Since $\Delta m \vee \sim \Delta m = 1 \in \varphi(P)$ and $\varphi(P)$ is a prime filter, then from (5), we have $\sim \Delta m \in \varphi(P)$, and by (6) we have $\sim \Delta m \in P$, so by (4) $0 = \Delta m \wedge \sim \Delta m \in P$. This contradicts the fact that P is a proper filter. Therefore, $F(\nabla P) \subseteq \varphi(P)$.

Proof by L. Monteiro. Let $m \in \nabla P$, then $m = \nabla p$, where $p \in P$. Since $\sim p \vee \nabla p = 1 \in \varphi(P)$ and $\varphi(P)$ is a prime filter then $\sim p \in \varphi(P)$ or $\nabla p \in \varphi(P)$. If $\sim p \in \varphi(P) = \complement \sim P$, then $\sim p \notin \sim P$. This contradiction shows that $m = \nabla p \in \varphi(P)$. Thus we have shown that $\nabla P \subseteq \varphi(P)$, so $F(\nabla P) \subseteq \varphi(P)$. □

Lemma 2.5.6. *If $P \in \mathbf{P}(L)$, is such that $P \subseteq \varphi(P)$ and $\nabla a \in P$, then $a \in \varphi(P)$.*

Proof. By hypothesis (1) $P \subseteq \varphi(P)$, and (2) $\nabla a \in P$. Assume that (3) $a \notin \varphi(P)$, so $a \in \sim P$, this is, (4) $\sim a \in P$. From (2) and (4), we have: $a \wedge \sim a = \nabla a \wedge \sim a \in P$, therefore $a \in P$, and thus by (1), $a \in \varphi(P)$, a contradiction. □

Lemma 2.5.7. *If $P \in \mathbf{P}(L)$ verifies $\varphi(P) \subseteq P$ then $F(\nabla P) = \varphi(P)$.*

Proof. By Lemma 2.5.5, we know that (1) $F(\nabla P) \subseteq \varphi(P)$. Assume that $F(\nabla P) \subset \varphi(P)$, then there exists (2) $a \in \varphi(P)$ such that (3) $a \notin F(\nabla P)$. If $\sim \Delta a = \nabla \sim a \in P$ then since by Lemma 2.1.18, $P \cap B(L) \subseteq F(\nabla P)$ we have that (4) $\nabla \sim a \in F(\nabla P)$. From (1) and (4) we deduce (5) $\nabla \sim a \in \varphi(P) = Q \in \mathbf{P}(L)$. Since by hypothesis $\varphi(P) \subseteq P$ then (6) $Q = \varphi(P) \subseteq \varphi(\varphi(P)) = \varphi(Q)$. From (5) and (6) it follows, by Lemma 2.5.6, that $\sim a \in \varphi(Q) = P$ and therefore $a \notin \varphi(P) = Q$, which contradicts (2). Then (7) $\sim \Delta a \notin P$ and since $\Delta a \vee \sim \Delta a = 1 \in P \in \mathbf{P}(L)$ we have that $\Delta a \in P$, and therefore $\Delta a \in P \cap B(L)$. Since by Lemma 2.1.18 $P \cap B(L) \subseteq F(\nabla P)$ we have that $\Delta a \in F(\nabla P)$ and since $\Delta a \leq a$, it follows that $a \in F(\nabla P)$, which contradicts (3). □

Lemma 2.5.8. *If $P \in \mathbf{P}(L)$ verifies $P \subseteq \varphi(P)$ then $F(\nabla\varphi(P)) = P$.*

Proof. Let $Q = \varphi(P)$ so from the hypothesis we have that $\varphi(Q) = \varphi(\varphi(P)) \subseteq \varphi(P) = Q$, then by Lemma 2.5.7 $F(\nabla Q) = \varphi(Q)$, this is, $F(\nabla\varphi(P)) = P$. □

Corollary 2.5.9. *If $P \in \mathbf{P}(L)$ then $P \in \mathbf{M}(L)$ or $\varphi(P) \in \mathbf{M}(L)$, this is if P is a prime filter of L then either P or $\varphi(P)$ is a maximal deductive system.*

Proof. Since L is in particular a Kleene algebra we have that (1) $\varphi(P) \subseteq P$ or (2) $P \subseteq \varphi(P)$. If (1) occurs then by Lemma 2.5.7 we have that $F(\nabla P) = \varphi(P)$. But by Lemma 2.5.2 we know that $\varphi(P) = F(\nabla P) \in \mathbf{M}(L)$.

If (2) occurs then by Lemma 2.5.8 we have that $F(\nabla\varphi(P)) = P$. But since $\varphi(P)$ is a prime filter, by Lemma 2.5.2 we know that $P = F(\nabla\varphi(P)) \in \mathbf{M}(L)$. □

Corollary 2.5.10. $\mathbf{M}(L) \subseteq \mathbf{P}(L)$*, this is every maximal deductive system of L is a prime filter of L.*

Proof. Let $\mathbf{U}(L)$ be the set of all the maximal filters of L and M a maximal deductive system. Then there exists $U \in \mathbf{U}(L)$ such that (1) $M \subseteq U$. Since $\mathbf{U}(L) \subseteq \mathbf{P}(L)$, then U is a prime filter and since $\varphi(U)$ is comparable to U, we have necessarily (2) $\varphi(U) \subseteq U$. So by the Corollary 2.5.9 $\varphi(U)$ is a maximal deductive system, and since by Lemma 2.5.3 there exists a unique maximal deductive system contained in U, we have that $M = \varphi(U) \in \mathbf{P}(L)$. □

The following results are also well known:

Lemma 2.5.11. *If R is a bounded distributive lattice (with 0 and 1 as lower and upper bound, respectively) and U is a proper filter of R, then the following conditions are equivalent:*

a) *U is a maximal filter of R.*
b) *Given $x \notin U$ there exists $u \in U$ such that $x \wedge u = 0$.*

Lemma 2.5.12. *If R is a bounded distributive lattice (with 0 and 1 as lower and upper bound, respectively), then the following conditions are equivalent:*

a) *P is a minimal prime filter of R.*
b) *$P = R \setminus I$, where I is a maximal ideal of R.*

and the following conditions are equivalent for a proper ideal I of R:

c) *I is a maximal ideal of R.*
d) *Given $p \notin I$ there exists $q \in I$ such that $p \vee q = 1$.*

Lemma 2.5.13. a) *$\mathbf{p}(L) \subseteq \mathbf{D}(L)$;*

b) $\mathbf{p}(L) = \mathbf{M}(L)$.

This is, every minimal prime filter of L is a deductive system of L and the set of the minimal prime filters of L coincides with the set of the maximal deductive systems of L.

PROOF. a) Let $P \in \mathbf{p}(L)$. If $0 \in \Delta P$, this is $0 = \Delta p$ where $p \in P$, then $p \notin L \setminus P = I$. Then by Lemma 2.5.12 d), there exists $q \in I$, so $q \notin P$ and $1 = p \vee q$, therefore $1 = \Delta(p \vee q) = \Delta p \vee \Delta q = 0 \vee \Delta q = \Delta q \leq q$, hence $q = 1 \in P$, a contradiction. Thus we have $0 \notin \Delta P$ from where we deduce, using Corollary 2.1.17, that: (1) $F(\Delta P)$ is a proper deductive system of L. Let us prove now that $P = F(\Delta P)$. Let $p \in P$. Since $\Delta p \leq p$ and $\Delta p \in F(\Delta P)$ then $p \in F(\Delta P)$, so $P \subseteq F(\Delta P)$. Assuming that $P \subset F(\Delta P)$, then there exists a filter P verifying (2) $x \in F(\Delta P)$, and (3) $x \notin P$. From (2) it follows that there exists $p \in P$ such that (4) $\Delta p \leq x$. From (3) and (4) we deduce that (5) $\Delta p \notin P$, and since $\Delta p \vee \sim \Delta p = 1 \in P$, we have that (6) $\sim \Delta p \in P$. Since $\Delta p \in F(\Delta P)$ and $P \subset F(\Delta P)$, from (6) we deduce $\sim \Delta p \in F(\Delta P)$, and therefore $0 = \Delta p \wedge \sim \Delta p \in F(\Delta P)$, which contradicts (1).

b) (i) Let $P \in \mathbf{p}(L)$ then by part a) P is a deductive system. Assume that there exists $M \in \mathbf{M}(L)$ such that $P \subset M$, then there exists (7) $x \in M$ such that (8) $x \notin P$. Then (9) $\Delta x \in M$ and (10) $\Delta x \notin P$. Since $\Delta x \vee \sim \Delta x = 1 \in P$, from (10) we have that $\sim \Delta x \in P$, and then (11) $\sim \Delta x \in M$. From (9) and (11): $0 = \Delta x \wedge \sim \Delta x \in M$, a contradiction.

(ii) Let $M \in \mathbf{M}(L)$ then by Corollary 2.5.10, $M \in \mathbf{P}(L)$. If $M \notin \mathbf{p}(L)$, then there exists (12) $P \in \mathbf{p}(L)$ such that (13) $P \subset M$. From (12) it follows using part (i) that $P \in \mathbf{M}(L)$, which contradicts (13).

□

Corollary 2.5.14. *Every prime filter of a Łukasiewicz algebra L contains a unique minimal prime filter of L.*

PROOF. By Lemma 2.5.3, each prime filter contains a single maximal deductive system, and by Lemma 2.5.13 b), the set of maximal deductive systems of L coincides with the set of the minimal prime filters of L. □

Lemma 2.5.15. *If $P \in \mathbf{p}(L)$ and $P \notin \mathbf{U}(L)$ then there exists a single $P' \in \mathbf{P}(L)$ such that $P \subset P'$, and more precisely, $P' = \varphi(P)$.*

PROOF. Let $P \in \mathbf{p}(L)$, we know that (1) $\varphi(P) \subset P$ or (2) $P \subseteq \varphi(P)$. Since $\varphi(P) \in \mathbf{P}(L)$ and P is a minimal prime filter minimal, condition (1) cannot hold. Let $U \in \mathbf{U}(L)$, be such that $\varphi(P) \subseteq U$, then $\varphi(U) \subseteq \varphi(\varphi(P)) = P$, so since P is a minimal prime filter, we must have $\varphi(U) = P$, so $U = \varphi(P)$.

Then $\varphi(P)$ is an ultrafilter that contains P. We can't have $\varphi(P) = P$ since by hypothesis $P \notin \mathbf{U}(L)$, so :

$$P \subset \varphi(P) \text{ and } \varphi(P) \text{ is an ultrafilter.}$$

Let $P' \in \mathbf{P}(L)$ be such that (3) $P \subset P'$. We shall prove that (4) $P' \subseteq U = \varphi(P)$. Indeed, if $P' \nsubseteq U$ then there exists (5) $p' \in P'$ such that (6) $p' \notin U$. From (6), (see Lemma 2.5.11 (b)), we infer that there exists (7) $u \in U$ such that (8) $u \wedge p' = 0$. If $u \in P'$ then $0 = u \wedge p' \in P'$, and therefore $P' = L$, a contradiction.

Since $P' \in \mathbf{P}(L)$ then by Lemma 2.5.2:

$$(9)\ F(\nabla P') \in \mathbf{M}(L) \quad \text{and} \quad (10)\ F(\nabla P') \subseteq P'.$$

By Lemma 2.5.13 (II), $\mathbf{p}(L) = \mathbf{M}(L)$, then (11) $P \in \mathbf{M}(L)$. By Lemma 2.5.3 we know that each prime filter of a Łukasiewicz algebra contains a single maximal deductive system, then from (9), (10), (11) and (3), we have that: (12) $F(\nabla P') = P$.

From (5) it follows that (13) $\nabla p' \in \nabla P' \subseteq F(\nabla P') = P$. Since $P \subseteq \varphi(P) = U$ and $P \in \mathbf{M}(L)$, by Lemma 2.5.3 we have that $P = F(\nabla U) \subseteq U$.

From (7) it follows that (14) $\nabla u \in F(\nabla U) = P$, and from (8), (13) and (14): $0 = \nabla 0 = \nabla(u \wedge p') = \nabla u \wedge \nabla p' \in P$, a contradiction. Thus $P' \subseteq U = \varphi(P)$.

Assume now that $P' \subset U = \varphi(P)$, then we have $P \subset P' \subset U$, so

$$P = \varphi(U) \subset \varphi(P') \subset \varphi(P) = U.$$

From $\varphi(P') \subset U$, we deduce that there exists $u \in U$ such that $u \notin \varphi(P')$. We know that P' and $\varphi(P')$ are comparable. Assume that $P' \subseteq \varphi(P')$. Since $u \wedge \sim u \leq u$ then: (i) $\sim u \wedge \nabla u = u \wedge \sim u \notin \varphi(P')$. From $u \notin \varphi(P')$ we get $\sim u \in P'$ and since $\nabla u \in F(\nabla U) = P \subset P'$, then $\sim u \wedge \nabla u \in P' \subseteq \varphi(P')$ which contradicts (i).

If $\varphi(P') \subseteq P'$, we also arrive to a contradiction.

Thus $U = \varphi(P)$ is the unique prime filter containing P as a proper subset. □

Corollary 2.5.16. *If $P \in \mathbf{p}(L)$ and $P \notin \mathbf{U}(L)$ then the unique proper filter containing P as a proper subset is $F = \varphi(P)$.*

Proof. Let F be a proper filter such that: (1) $P \subset F$, and assume that $F \notin \mathbf{U}(L)$, then there exists (2) $U \in \mathbf{U}(L)$ such that (3) $F \subset U$. From (1) and (3) we have: $P \subset U$, whence by Lemma 2.5.15, $U = \varphi(P)$. Let (4) $x \in U \setminus F$.

Since $F = \bigcap\{P' : P' \in \mathbf{P}(L), F \subseteq P'\}$ and $x \notin F$, then there exists $P' \in \mathbf{P}(L)$ such that (5) $F \subseteq P'$ and (6) $x \notin P'$. From (4) and (6) we have $P' \neq U = \varphi(P)$. From (1) and (5): (7) $P \subset P'$. Thus there exists a prime filter P', different from $\varphi(P)$ containing P as a proper subset, which is impossible by Lemma 2.5.15. □

2.6. Principal deductive systems and their quotient algebras

If L is a Łukasiewicz algebra and $u \in L$, we will present a construction to determine the quotient algebra $L/D(u)$.

If $p, u \in B(L)$ are such that $p \leq u$, $L' = [p, u] = \{x \in L : p \leq x \leq u\}$, and we define $\approx x = p \vee (\sim x \wedge u)$, for $x \in L'$, then as we proved in Theorem 1.4.1 that the system $(L', u, \approx, \nabla, \wedge, \vee)$ is a Łukasiewicz algebra.

Therefore, if $u \in L$, and $L' = [0, \Delta u] = (\Delta u]$ then L' is a Łukasiewicz algebra where if $x \in L'$, $\approx x = 0 \vee (\sim x \wedge \Delta u) = \sim x \wedge \Delta u$.

Lemma 2.6.1. *The Łukasiewicz algebra $L/D(u)$ is isomorphic to $L' = [0, \Delta u]$.*

Proof. Given $x \in L$, let $h(x) = x \wedge \Delta u$, then since $0 \leq x \wedge \Delta u \leq \Delta u$ then h is a function from L to $[0, \Delta u]$. Furthermore, given $y \in [0, \Delta u]$, this is, $0 \leq y \leq \Delta u$, with $y \in L$ then $h(y) = y \wedge \Delta u = y$. Therefore h is a surjective function with the elements of $[0, \Delta u]$ as invariants. We also have

H1) $h(x \vee y) = (x \vee y) \wedge \Delta u = (x \wedge \Delta u) \vee (x \wedge \Delta u) = h(x) \vee h(y)$.

H2) $\approx h(x) = \approx (x \wedge \Delta u) = \sim (x \wedge \Delta u) \wedge \Delta u = (\sim x \vee \sim \Delta u) \wedge \Delta u = \sim x \wedge \Delta u = h(\sim x)$.

H3) $\nabla h(x) = \nabla(x \wedge \Delta u) = \nabla x \wedge \Delta u = h(\nabla x)$.

Since

$$Ker(h) = \{x \in L : h(x) = \Delta u\} = \{x \in L : x \wedge \Delta u = \Delta u\} =$$
$$\{x \in L : \Delta u \leq x\} = F(\Delta u) = D(u),$$

and the kernel of the natural epimorphism $L \to L/D(u)$ is $D(u)$ we have by Lemma 2.2.7 that $L/D(u)$ and $[0, \Delta u]$ are isomorphic. $\square$

Lemma 2.6.2. *If C is an equivalence class* $(\mathrm{mod}\ D(u))$ *then $C \cap [0, \Delta u]$ contains a unique element.*

PROOF. (i) $C \cap [0, \Delta u] \neq \emptyset$. Let $x \in C$, so: $h(x) \wedge \Delta u = (x \wedge \Delta u) \wedge \Delta u = x \wedge \Delta u$, and since $\Delta u \in D(u)$ it follows by condition C3) in Lemma 2.2.2 that $h(x) \equiv x \pmod{D(u)}$ so since $x \in C$ we have that $h(x) \in C$ and therefore $h(x) \in C \cap [0, \Delta u]$.

(ii) The element is unique. Assume that $x, y \in C \cap [0, \Delta u]$ then (1) $x, y \in C$ and (2) $x, y \in [0, \Delta u]$. From (2) it follows by the previous lemma that $h(x) = x$ and $h(y) = y$. By (1) we have that $x \equiv y \pmod{D(u)}$, so by condition C3) in Lemma 2.2.2, $x \wedge d = \wedge d$ with (3) $d \in D(u)$, then (4) $x \wedge h(d) = h(x) \wedge h(d) = h(x \wedge d) = h(y \wedge d) = h(y) \wedge h(d) = y \wedge h(d)$.

From (3), since $D(u) = Ker(h)$ we have that (5) $h(d) = \Delta u$, so from (5) and (4) it follows that $x \wedge \Delta u = y \wedge \Delta u$ and since by (2) $x, y \leq \Delta u$ we have that $x = y$.

$\square$

Lemma 2.6.3. (L. Monteiro (2002)) *If $x \in [0, \Delta u]$ then*

$$C_{D(u)}(x) = [x, x \vee \nabla \sim u].$$

PROOF. Let $y \in [x, x \vee \nabla \sim u]$, so (1) $x \leq y$ and (2) $y \leq x \vee \nabla \sim u$.

From (1) it follows that $1 = \nabla \sim x \vee x \leq \nabla \sim x \vee y = x \to y$ then $x \to y = 1$ and therefore (3) $x \to y \in D(u)$.

From (2) it follows that $1 = \nabla \sim y \vee y \leq \nabla \sim y \vee x \vee \nabla \sim u$, so $\nabla \sim y \vee x \vee \nabla \sim u = 1$ and therefore $\Delta u \wedge (\nabla \sim y \vee x \vee \nabla \sim u) = \Delta u$, this is $(\Delta u \wedge (\nabla \sim y \vee x)) \vee (\Delta u \wedge \nabla \sim u) = \Delta u \wedge (\nabla \sim y \vee x) = \Delta u$, and therefore $\Delta u \leq \nabla \sim y \vee x = y \to x$ then since $\Delta u \in D(u)$ we have that (4) $y \to x \in D(u)$.

From (1) it follows that $\sim y \leq \sim x$ and therefore $1 = \nabla y \vee \sim y \leq \nabla y \vee \sim x = \sim y \to \sim x$ so $\sim y \to \sim x = 1$ and therefore (5) $\sim y \to \sim x \in D(u)$.

From (2) it follows that $\sim x \wedge \Delta u \leq \sim y$, so $\nabla x \vee (\sim x \wedge \Delta u) \leq \nabla x \vee \sim y = \sim x \to \sim y$ then $\Delta u \leq \nabla x \vee \Delta u = 1 \wedge (\nabla x \vee \Delta u) = (\nabla x \vee \sim x) \wedge (\nabla x \vee \Delta u) = \nabla x \vee (\sim x \wedge \Delta u) \leq \sim x \to \sim y$, and since $\Delta u \in D(u)$, we have (6) $\sim x \to \sim y \in D(u)$.

From (3), (4), (5) and (6) it follows by Lemma 2.2.2 that $y \equiv x \pmod{D(u)}$ this is $y \in C_{D(u)}(x)$.

Conversely if $y \in C_{D(u)}(x)$, where (7) $0 \leq x \leq \Delta u$, this is $y \equiv x \pmod{D(u)}$, then by Lemma 2.2.2, there exists $d \in D(u) = F(\Delta u)$ such that $x \wedge d = y \wedge d$ and therefore $x \wedge d \wedge \Delta u = y \wedge d \wedge \Delta u$ so since $d \in F(\Delta u)$, this is $\Delta u \leq d$ and by (7) $x \leq \Delta u$ it follows that (8) $x = y \wedge \Delta u$, then since $y \wedge \Delta u \leq y$, we have that (9) $x \leq y$. From (8) it follows that

$$\nabla \sim u \vee x = \nabla \sim u \vee (y \wedge \Delta u) = \nabla \sim u \vee y$$

and since $y \leq \nabla \sim u \vee y$, we have that (10) $y \leq x \vee \nabla \sim u$. From (9) and (10) we have that $x \leq y \leq x \vee \nabla \sim u$, this is $y \in [x, x \vee \nabla \sim u]$. $\square$

The previous lemma generalizes a result by L. Monteiro in [66].

CHAPTER 3

Products and factors

3.1. Simple algebras

Definition 3.1.1. *A Łukasiewicz algebra L is said to be* simple *if:*

Si1) *L has more than one element,*
Si2) *every homomorphic image of L has a single element or is isomorphic to L, this is, the only homomorphic images of L are the trivial ones.*

Lemma 3.1.2. *A Łukasiewicz algebra L is simple if and only if:*

Si1) *L has more than one element,*
Si2′) *the only deductive systems of L are $D(1) = \{1\}$ and $D(0) = L$.*

Proof. Let L be a simple Łukasiewicz algebra so Si1) holds. Let D be a deductive system of L. By hypothesis L/D is isomorphic to L or to the singleton algebra. If $L \cong L/D$, then $C_D(x) = \{x\}$ and since $C_D(1) = D$ we have that $D = \{1\} = D(1)$. If L/D has a single element then $C_D(x) = L$ for all $x \in L$, so $D = C_D(1) = L = D(0)$.

Conversely if L is a Łukasiewicz algebra such that Si1) and Si2′) hold, then the only homomorphic images of L are $L/D(1)$ and $L/D(0)$, which are isomorphic to L and the singleton algebra respectively.

□

Corollary 3.1.3. *If L is a simple Łukasiewicz algebra then $\{1\}$ is a maximal deductive system of L.*

Lemma 3.1.4. *A Łukasiewicz algebra L is simple if and only if:*

Si1) *L has more than one element,*
Si2″) *$B(L) = \{0, 1\}$.*

This is a Łukasiewicz algebra L is simple if and only if the boolean algebra $B(L)$ is simple.

Proof. Assume that L is a simple Łukasiewicz algebra and that there exists (1) $b \in B(L)$ such that (2) $b \neq 0$ and (3) $b \neq 1$. We know that $D(x) = F(\Delta x)$ for all $x \in L$. Then by (1) $D(b) = F(\Delta b) = F(b)$. By (2), $D(b) = F(b) \neq L$ and by (3), $D(b) = F(b) \neq \{1\} = D(1)$. So Si2′) does not hold, a contradiction.

Conversely assume that the Łukasiewicz algebra L verifies Si1) and Si2″). We know that $D(1)$ is a deductive system. Let D be a deductive system such that $D \neq D(1)$ so there exists (1) $d \in D$ such that (2) $d \neq 1$. From (1) it follows that (3) $\Delta d \in D$ and by Si2″) we have (4) $\Delta d = 0$ or (5) $\Delta d = 1$. Since $\Delta d \leq d$, if (5) occurs then $d = 1$, which contradicts (2) then (4) must hold. From (3) and (4) it follows that $0 \in D$ and therefore $D = L = D(0)$. □

Corollary 3.1.5. *If D is a deductive system of a Łukasiewicz algebra L, with more than one element, it is necessary and sufficient that D is a maximal deductive system of L for $L/D = L'$ to be simple.*

Proof. Assume that $L' = L/D$ is simple, then L' is an algebra with more than one element and therefore D is a proper deductive system and by Lemma 3.1.4, $B(L') = \{0', 1'\}$. Let h be the natural homomorphism from L to L/D. Since $h(\nabla x) = \nabla h(x)$ then $h : B(L) \to B(L/D)$. Furthermore $D = h^{-1}(1')$ and $I = h^{-1}(0')$ is an ideal of the lattice L, $B(L) \subseteq D \cup I$, $\Delta D = D \cap B(L)$ is a proper filter of $B(L)$ and it is easy to prove that $\Delta I = I \cap B(L)$ is a proper ideal of $B(L)$. Also, $\Delta D \cap \Delta I = \emptyset$ and $\Delta D \cup \Delta I = B(L)$, so ΔD is a prime filter of $B(L)$, this is ΔD is an ultrafilter of the boolean algebra $B(L)$, then by Lemma 2.1.19, $F(\Delta D)$ is a maximal deductive system of L and by Lemma 2.1.19, $F(\Delta D) = D$.

Assume now that D is a maximal deductive system of L, so D is proper and therefore $L' = L/D$ has more than one element. Let $b' \in B(L')$ be such that $b' \neq 1'$ and let h be the natural epimorphism from L to L/D, so there exists $b \in L$ such that $h(b) = b'$ and since $b' \in B(L')$ we have that $b \in B(L)$ and $b \neq 1$. Since D is a maximal deductive system, we know by Corollary 2.4.23 that (1) $b \in D$ or (2) $\sim b \in D$. If (1) occurs then $1' = h(b) = b'$, a contradiction, so (2) holds and therefore $\sim b' = h(\sim b) = 1'$ and therefore $b' = 0'$. Then $B(L') = \{0', 1'\}$, this is, L' is simple. □

From the previous corollary it follows that it is important to study the maximal deductive systems to determine simple algebras.

We shall denote with $\mathbf{B}$ the boolean algebra with two elements and with $\mathbf{T}$ the centered Łukasiewicz algebra with three elements, as shown in Example 1.2.3. The next lemma is a special case of the lemma proved by L. Monteiro in 1971 for monadic Łukasiewicz algebras [62].

Lemma 3.1.6. *If L is a Łukasiewicz algebra, with more than one element, then the following conditions are equivalent:*

a) *L is simple,*
b) *For every $a \in L$, if $a \neq 1$ then $\Delta a = 0$,*
c) *$L \cong \mathbf{B}$ or $L \cong \mathbf{T}$.*

Proof. *a*) implies *b*): By Corollary 3.1.3, $M = \{1\}$ is a maximal deductive system so if $a \neq 1$, $a \notin M$ then by the Theorem 2.4.21, there exists $m \in M = \{1\}$ such that $0 = \Delta a \wedge m = \Delta a \wedge 1 = \Delta a$.

b) implies *c*):

First case: $L \setminus B(L) = \emptyset$, this is $L = B(L)$. Then if (1) $b \in B(L)$ and (2) $b \neq 1$, by (1) $\Delta b = b$ and from (2) it follows by the hypothesis that $\Delta b = 0$, which proves that $B(L) = \{0, 1\}$ and therefore $L \cong \mathbf{B}$.

Second case: $L \setminus B(L) \neq \emptyset$. If $c \in L \setminus B(L)$ we have that (3) $c \neq 0$ and (4) $c \neq 1$. From (4) it follows by the hypothesis that $\Delta c = 0$. If $\sim c = 1$ then $c = 0$, which contradicts (3), so $\sim c \neq 1$ and it follows by the hypothesis that $\Delta \sim c = 0$ this is $\nabla c = 1$. Therefore we have that for all $c \in L \setminus B(L)$, c is a center of the algebra L and since the center is unique $L = \{0, c, 1\} \cong \mathbf{T}$.

c) implies a): If $L \cong \mathbf{B}$ or $L \cong \mathbf{T}$ then L has more than one element and the only deductive systems of L are $D(0)$ and $D(1)$. □

Note that $\mathbf{B}$ is isomorphic to a subalgebra of $\mathbf{T}$.

Lemma 3.1.7. *If M is a maximal deductive system of a Łukasiewicz algebra L then $L/M \cong \mathbf{B}$ or $L/M \cong \mathbf{T}$.*

PROOF. Since $M \in \mathbf{M}(L)$ then by Corollary 3.1.5, L/M is a simple Łukasiewicz algebra, so by Lemma 3.1.6, 3) $L/M \cong \mathbf{B}$ or $L/M \cong \mathbf{T}$. □

Lemma 3.1.8. *If L is a finite Łukasiewicz algebra, with more than one element, then $F(b)$ is a maximal deductive system of L if and only if b is an atom of the boolean algebra $B(L)$.* (L. Monteiro, 2002)

PROOF. Let $F(b)$ be a maximal deductive system, so in particular it is a deductive system, so by Lemma 2.4.4 we have that $b \in B(L)$. Assume there exists $x \in B(L)$ such that $0 \leq x \leq b$ so $F(b) \subseteq F(x) \subseteq F(0) = A$, and since by Lemma 2.4.4, $F(x)$ is a deductive system and $F(b)$ is maximal, it follows that $F(x) = F(b)$ or $F(x) = F(0)$, this is $x = b$ or $x = 0$, which proves b is an atom of $B(L)$.

Conversely assume that b is an atom of $B(L)$. We know that $D(x) = F(\Delta x)$ for all $x \in L$, so from $b \in B(L)$ it follows that $D(b) = F(\Delta b) = F(b)$. Since $b \neq 0$ the deductive system $F(b)$ is proper. Assume that (1) D is a deductive system such that (2) $F(b) \subseteq D$. Since D is a filter and L is finite (3) $D = F(x)$, for some $x \in L$. From (2) and (3) we have $F(b) \subseteq F(x)$ and therefore (4) $x \leq b$ so (5) $0 \leq \Delta x \leq \Delta b = b$. Since b is an atom of $B(L)$ and $0, \Delta x \in B(L)$ then (6) $\Delta x = 0$ or (7) $\Delta x = b$. Since $x \in D$ and D is a deductive system we have that (8) $\Delta x \in D$. If (6) holds, from (6) and (8) it follows that $0 \in D$ and therefore $D = L$. If (7) holds, since $b = \Delta x \leq x$ then $b = x$ and therefore $D = F(b)$, which proves that $F(b)$ is a maximal deductive system. □

Therefore if L is a finite Łukasiewicz algebra with more than one element, the number of maximal deductive systems is the same as the number of atoms in the boolean algebra $B(L)$.

3.2. Cartesian product

Given a family of Łukasiewicz algebras $\{L_i\}_{i\in I}$, let $L = \prod_{i\in I} L_i$ be the cartesian product of the family of sets $\{L_i\}_{i\in I}$, this is, the set of all the functions $x : I \to \bigcup_{i\in I} L_i$ such that for each element $i \in I$ they take a value $x(i) = x_i \in L_i$. Then x_i is the coordinate with index i of the element $x \in L$ and it is denoted by $x = [x_i]_{i\in I}$, $x = [x_i]$, $x = (x_i)_{i\in I}$ or $x = (x_i)$.

We represent with 0_i and 1_i the bottom and top elements respectively of the algebras $L_i, i \in I$. Let $0 = (0_i)_{i\in I}, 1 = (1_i)_{i\in I}$ and given $x = (x_i)_{i\in I} \in L, y = (y_i)_{i\in I} \in L$, put by definition:

$$\nabla x = (\nabla x_i)_{i\in I},\ \sim x = (\sim x_i)_{i\in I},\ x \wedge y = (x_i \wedge y_i)_{i\in I},\ x \vee y = (x_i \vee y_i)_{i\in I}.$$

It is easy to prove that $(L, 1, \sim, \nabla, \vee, \wedge)$ is a Łukasiewicz algebra which we call the *cartesian product* or *direct product* of the family of Łukasiewicz algebras

$\{L_i\}_{i\in I}$. Each one of the sets L_i is called the *i-th coordinate axis* or *i-th axis*. If $L_i = L, \forall i \in I$, then $\prod_{i\in I} L_i$ is the set of all the functions from I to L. In this case we write L^I instead of $\prod_{i\in I} L_i$. If I is finite, for example $I = \{1, 2, \ldots, n\}$ then any of the following notations may be used:

$$\prod_{i=1}^{n} L_i \qquad \text{or} \qquad L_1 \times L_2 \times \ldots \times L_n.$$

In this case if $x \in \prod_{i=1}^{n} L_i$ then: $x = (x(1), x(2), \ldots, x(n)) = (x_1, x_2, \ldots, x_n)$.

It is clear that $L_1 \times L_2 \neq L_2 \times L_1$, but $L_1 \times L_2$ and $L_2 \times L_1$ are isomorphic Łukasiewicz algebras. It is also easy to prove that $L_1 \times (L_2 \times L_3) \cong (L_1 \times L_2) \times L_3$.

Notice that if L, L' are Łukasiewicz algebras and f is a fixed element of L, then the subset C_f of $L \times L'$ defined by $C_f = \{(f, y) : y \in L'\}$ is a Łukasiewicz algebra isomorphic to L', if we define $\sim (f, y) = (f, \sim y)$, $\nabla(f, y) = (f, \nabla y)$ and in the same manner, if f' is a fixed element of L' then $C_{f'} = \{(x, f') : x \in L\} \subseteq L \times L'$ is a Łukasiewicz algebra isomorphic to L as long as we define $\sim (x, f') = (\sim x, f')$, $\nabla(x, f') = (\nabla x, f')$.

3.3. Factorization of an axled Łukasiewicz algebra

Notice the following facts:

- Every boolean algebra A is a Łukasiewicz algebra where $\nabla x = \Delta x = x$ for all $x \in A$ and conversely, every Łukasiewicz algebra in which $\nabla x = \Delta x = x$ for all x is a boolean algebra.
- If A is a boolean algebra then $e = 0$ is the axis of A regarded as a Łukasiewicz algebra, since $\Delta e = e = 0$ and $(\Delta x \vee e) \wedge \nabla x = (x \vee 0) \wedge x = x$ for all $x \in A$.
- If c is a center of a Łukasiewicz algebra L then c is an axis of the Łukasiewicz algebra L.
- In the remaining part of this section we will only consider Łukasiewicz algebras which are not boolean algebras nor centered algebras.

Gr. Moisil [27], p. 66–90 proved the following theorem.

Theorem 3.3.1. *Every axled Łukasiewicz algebra is the cartesian product of a boolean algebra by a centered Łukasiewicz algebra.*

To prove this theorem, Moisil used some results from ring theory (see section 1.9).

L. Monteiro, [62] presented a simpler proof of this theorem, using only results from the theory of Łukasiewicz algebras.

Recall this result about distributive lattices: Let R be a distributive lattice with bottom element 0 and top element 1. If $x \in R$ we write $[x) = F(x) = \{y \in R : x \leq y\}$ and $(x] = I(x) = \{y \in R : y \leq x\}$. These sets are respectively a filter and an ideal of R and are called principal filter and principal ideal. Furthermore, $F(x)$ is a distributive lattice with bottom element x and top element 1, while $I(x)$ is a distributive lattice with bottom element 0 and top element x. We denote with $B(R)$ the set of all the boolean elements of R, so $\{0,1\} \subseteq B(R)$. If $x \in B(R)$ we denote with $-x$ its boolean complement.

Lemma 3.3.2. *If R is a distributive lattice with bottom element 0 and top element 1 and $b \in B(R) \setminus \{0,1\}$ then R is isomorphic to the cartesian product of the distributive lattices $I(b)$ and $I(-b)$ this is $R \cong I(b) \times I(-b)$.*

The isomorphism from R to $I(b) \times I(-b)$ is defined by $h(x) = (x \wedge b, x \wedge -b)$.

Lemma 3.3.3. *If R is a finite, non trivial, reducible distributive lattice, i.e. $R \cong R_1 \times R_2$, where R_1 and R_2 are nontrivial distributive lattices then $R \cong \prod_{i=1}^{t}(a_i]$, where $\mathcal{A}(B(R)) = \{a_1, a_2, \ldots, a_t\}$.*

Lemma 3.3.4. (L. Monteiro, [62]) *If L is a Łukasiewicz algebra and $b \in B(L) \setminus \{0,1\}$ then $L \cong L/F(b) \times L/F(\sim b)$.*

Proof. Recall that if $b \in B(L)$ then its boolean complement is $\sim b$. Since $b, \sim b \in B(L)$ then we know that $F(b)$ and $F(\sim b)$ are deductive systems so by Lemma 2.6.1, $L/F(b) \cong I(b)$ and $L/F(\sim b) \cong I(\sim b)$. Then by Lemma 3.3.2, the distributive lattice L is isomorphic to the distributive lattice

$$I(b) \times I(\sim b) \cong L \cong L/F(b) \times L/F(\sim b).$$

Since the function h defined above is a lattice isomorphism we have that

(i) $h(0) = 0$,
(ii) $h(1) = 1$,
(iii) $h(x \wedge y) = h(x) \wedge h(y)$,
(iv) $h(x \vee y) = h(x) \vee h(y)$,

We prove now that

(v) $h(\nabla x) = \nabla h(x)$. Indeed, $\nabla h(x) = \nabla(x \wedge b, x \wedge \sim b) =$
$(\nabla x \wedge \nabla b, \nabla x \wedge \nabla \sim b) = (\nabla x \wedge b, \nabla x \wedge \sim b) = h(\nabla x)$.

In a similar manner we can prove

(vi) $h(\Delta x) = \Delta h(x)$.

Then since h verifies (i) to (vi), by the results by L. Monteiro [52], it turns out that h respects the operator $\sim$ and in consequence h is a Łukasiewicz algebra homomorphism. Since h is bijective then h is a isomorphism. □

Lemma 3.3.5. *The cartesian product of a boolean algebra and a centered Łukasiewicz algebra is an axled Łukasiewicz algebra.*

Proof. If B is a boolean algebra and C is a centered Łukasiewicz algebra with center c then it easy to check that the element $e = (0, c) \in B \times C$ is an axis of the Łukasiewicz algebra $B \times C$. □

Lemma 3.3.6. *If L is a Łukasiewicz algebra and $a \in L$ verifies $\Delta a = 0$ then the quotient algebra $E = L/F(\nabla a)$ is a centered Łukasiewicz algebra* (L. Monteiro [62]).

Proof. We shall prove that $c = C(a)$, where $C(a)$ is the equivalence class, mod $F(\nabla a)$, containing the element a, is the center of the quotient algebra $E = L/F(\nabla a)$. Indeed, we prove that $\sim C(a) = C(a)$ this is, we prove that $\sim a \equiv a \bmod F(\nabla a)$. It will be enough to notice that the following conditions are equivalent:

$\Delta a = 0 \iff$ by Lemma 1.4.10, $a \leq \sim a \iff a = a \wedge \sim a \iff$ by L9), $a = a \wedge \nabla a = a \wedge \sim a \iff$ by L7), $a = \sim a \wedge \nabla a \iff \sim a \equiv a \bmod F(\nabla a)$. $\square$

Lemma 3.3.7. (L. Monteiro [62]) *If L is an axled Łukasiewicz algebra with axis e then the quotient algebra $B = L/F(\sim \nabla e) = L/F(\Delta \sim e)$ is a boolean algebra.*

Proof. We prove that $\Delta C(x) = C(x)$ for all $x \in L$, this is, $\Delta x \equiv x \bmod F(\Delta \sim e)$.

Since L has an axis then $x = (\Delta x \vee e) \wedge \nabla x = \Delta x \vee (e \wedge \nabla x)$ for all $x \in L$, so $x \wedge \Delta \sim e = (\Delta x \wedge \Delta \sim e) \vee (e \wedge \Delta \sim e \wedge \nabla x) = (\Delta x \wedge \Delta \sim e) \vee (0 \wedge \nabla x) = \Delta x \wedge \Delta \sim e$ and therefore $\Delta x \equiv x \bmod F(\Delta \sim e)$. $\square$

We prove now Theorem 3.3.1. Let L be an axled Łukasiewicz algebra with axis e that is not a boolean algebra nor a centered algebra. Then $\nabla e \neq 1$ and $\nabla e \neq 0$, because if $\nabla e = 0$ then $e = 0$ and L would be a boolean algebra, contradicting the hypothesis. If $\nabla e = 1$ then e would be a center of L, contradicting the other hypothesis.

Thus we have that $\nabla e \in B(L) \setminus \{0, 1\}$, so by Lemma 3.3.4, we know that

$$L \cong L/F(\sim \nabla e) \times L/F(\nabla e)$$

where by Lemma 3.3.7, $L/F(\sim \nabla e)$ is a boolean algebra and by Lemma 3.3.6, $L/F(\nabla e)$ is a centered Łukasiewicz algebra.

Remark 3.3.8.
- *If L is a centered Łukasiewicz algebra, so an axled algebra, then we can write $L \cong P \times L$ where P is a boolean algebra with a single element.*
- *If L is a boolean algebra, then $L \cong L \times P$ where P is the centered Łukasiewicz algebra with a single element.*

Lemma 3.3.9. *If L is a finite non trivial Łukasiewicz algebra, different from $\mathbf{B}$ and $\mathbf{T}$, and $\mathcal{A}(B(L)) = \{a_1, a_2, \ldots, a_t\}$, then $L \cong \prod_{i=1}^{t}(a_i]$.*

Proof. By hypothesis $B(L)$ is finite, non trivial, and $B(L) \neq \{0, 1\}$, so by Lemma 3.3.3, $L \cong \prod_{i=1}^{t}(a_i]$. $\square$

Remark 3.3.10. *Let $(A, \exists)$ be a finite monadic boolean algebra with n atoms, $n \in \mathbb{N}$, and assume that the boolean algebra $K(A) = \{x \in A : \exists x = x\}$ has h atoms $1 \leq h \leq n$. If the partition $\{X_1, X_2, \ldots, X_h\}$ of $\mathcal{A}(A)$ associated to $K(A)$ is such that: $N[X_i] = 1$ for $1 \leq i \leq k$ and $N[X_i] > 1$ for $k+1 \leq i \leq h$, then the atoms of $K(A)$ are $k_i = \bigvee_{x \in X_i} x$, $1 \leq i \leq h$. By the results in section 1.10 we know that $\mathcal{L}(A)$ is a Łukasiewicz algebra such that the boolean algebras $B(\mathcal{L}(A))$*

and $K(A)$ are isomorphic, so $B(\mathcal{L}(A))$ has h atoms, and the atoms of $B(\mathcal{L}(A))$ are $C(k_i), 1 \leq i \leq h$. Moreover, if $1 \leq i \leq k$, then $k_i \in \mathcal{A}(A)$. By Lemma 1.10.5, $(C(k_i)] \cong \mathbf{B}$. If $k+1 \leq i \leq h$, then $k_i = \bigvee_{x \in X_i} x$. By Lemma 1.10.7 it follows that $\exists x$ is an atom of $K(A)$ for any $x \in X_i$, and since $x \leq k_i$, $\exists x \leq \exists k_i = k_i$, so since k_i is an atom of $K(A)$, we must have $k_i = \exists x$ for some $x \in X_i$. Applying Lemma 1.10.8 it follows that $(C(\exists x)] = (C(k_i)] \cong \mathbf{T}$ so from the assumptions taken and the previous lemmas we have that:

$$\mathcal{L}(A) \cong \prod_{i=1}^{h} (C(k_i)] \cong \mathbf{B}^k \times \mathbf{T}^{h-k}.$$

3.4. Subdirect product of Łukasiewicz algebras

Given a non-empty family of Łukasiewicz algebras $\{L_i\}_{i \in I}$, consider the Łukasiewicz algebra $P = \prod_{i \in I} L_i$. Given $i \in I$, consider the i-th projection π_i of P over L_i defined by $\pi_i(a) = a_i \in L_i$. We know that π_i is a lattice homomorphism from P onto L_i such that $\pi_i(1) = 1_i$ and $\pi_i(0) = 0_i$. Furthermore, if $a \in P$ then $\sim \pi_i(a) = \sim a_i = \pi_i((\sim a_j)) = \pi_i(\sim a)$, $\nabla \pi_i(a) = \nabla a_i = \pi_i((\nabla a_j)) = \pi_i(\nabla a)$ therefore each one of the i-th projections is an epimorphism from P onto L_i.

Definition 3.4.1. *If S is a subalgebra of the Łukasiewicz algebra $P = \prod_{i \in I} L_i$ such that $\Pi_i(S) = L_i$, for all $i \in I$, then we say that S is a subdirect product of the Łukasiewicz algebras L_i.*

Lemma 3.4.2. *Every subalgebra S of the cartesian product $P = \prod_{i \in I} L_i$ is a subdirect product of Łukasiewicz algebras.*

Proof. For each $i \in I$ let $L'_i = \pi_i(S)$. Since the projections are homomorphisms from P to L_i then L'_i is a subalgebra of L_i. Let $P' = \prod_{i \in I} L'_i$ and π'_i be the i-th projection of P' on L'_i. We prove that S is subdirect product of P', this is, that S is a subalgebra of P' and $\pi'_i(S) = L'_i$ for all $i \in I$. By the definition of P' it is immediate that the second condition holds. Given $s = (s_i)_{i \in I} \in S$, since $s_i = \pi_i(s) \in L'_i$, then $s = (a_i)_{i \in I} \in P' = \prod_{i \in I} L'_i$. Therefore S is a subset of P', and therefore S is a subalgebra of P'. □

Definition 3.4.3. *A Łukasiewicz algebra L is said to be* subdirectly reducible *if L is isomorphic to a subalgebra L' of a direct product $P = \prod_{i \in I} L_i$, and such that:*

1. *$\pi_i(L') = L_i$, for all $i \in I$.*
2. *None of the projections is an isomorphism.*

A Łukasiewicz algebra is said to be subdirectly irreducible *if it is not is subdirectly reducible.*

3.5. Moisil's representation theorem

Once we know the simple algebras, we can build new ones using elemental methods as indicated before. The homomorphic images of the simple algebras, do not yield new ones.

Therefore, it remains to build cartesian products and to determine subalgebras of those products. This leads naturally to the following definition: A Łukasiewicz algebra is said to be *semisimple* if it is isomorphic to a subdirect product of simple Łukasiewicz algebras.

Theorem 3.5.1. (Moisil's representation Theorem) *Every Łukasiewicz algebra with more than one element is subdirect product of simple Łukasiewicz algebras.*

Proof. If the Łukasiewicz algebra L is simple, then the theorem holds. Assume then L is not simple, so $B(L) \neq \{0,1\}$ and there exists $b \in B(L)$ such that $b \neq 0, b \neq 1$. Therefore $\sim b \in B(L)$ and $\sim b \neq 0, \sim b \neq 1$, so there exist maximal deductive systems M_1 and M_2 such that $b \in M_1$ and $\sim b \in M_2$. Furthermore, $M_1 \neq M_2$ because if $M_1 = M_2$ then $b, \sim b \in M_1$, which contradicts Lemma 2.4.22. Therefore if L is not simple there exist at least two different maximal deductive systems. Let $\mathbf{M}(L)$ be the set of the maximal deductive systems of L. For each $M \in \mathbf{M}(L)$ let h_M be the natural epimorphism from L to L/M. We know that if $M \in \mathbf{M}(L)$ then $L/M \cong \mathbf{B}$ or $L/M \cong \mathbf{T}$, so since $\mathbf{B}$ is isomorphic to a subalgebra of $\mathbf{T}$ we can assume that for each $M \in \mathbf{M}(L)$, h_M is a homomorphism from L to $\mathbf{T}$. Let $\mathcal{F} = \mathbf{T}^{\mathbf{M}(L)}$. We already know that $\mathcal{F}$ is a Łukasiewicz algebra. We shall prove now that L is isomorphic to a subalgebra $\mathcal{A}$ of $\mathcal{F}$. For this, consider the following transformation: given $f \in L$ put $\varphi(f) = F$ where F is defined by:

$$F(M) = h_M(f), \text{ for every } M \in \mathbf{M}(L),$$

so $F \in \mathcal{F}$. Then

H1) $\varphi(f \vee g) = \varphi(f) \vee \varphi(g)$, for all $f, g \in L$.
Let $k = f \vee g$, $\varphi(f) = F, \varphi(g) = G$ and $\varphi(k) = K$, so $K(M) = h_M(k) = h_M(f \vee g) = h_M(f) \vee h_M(g) = F(M) \vee G(M) = (F \vee G)(M)$ which proves H1).

H2) $\varphi(\sim f) = \sim \varphi(f)$, for all $f \in L$.
Let $g = \sim f$, $\varphi(f) = F$, and $\varphi(g) = G$, so $G(M) = h_M(g) = h_M(\sim f) = \sim h_M(f) = \sim F(M) = (\sim F)(M)$, which proves H2).

H3) $\varphi(\nabla f) = \nabla\varphi(f)$, for all $f \in L$.
Let $g = \nabla f$, $\varphi(f) = F$, and $\varphi(g) = G$, so $G(M) = h_M(g) = h_M(\nabla f) = \nabla h_M(f) = \nabla F(M) = (\nabla F)(M)$, which proves H3).

Thus we have proved that φ is a homomorphism and therefore $\mathcal{A} = \varphi(L)$ is a subalgebra of $\mathcal{F}$.

<u>φ is injective.</u> Let $f, g \in L$ be such that (1) $f \neq g$, $\varphi(f) = F$ and $\varphi(g) = G$. To prove that $F \neq G$ we need to show that there exists at least a $M \in \mathbf{M}(L)$ such that $F(M) \neq G(M)$. From (1) it follows by Moisil's determination principle that (2) $\nabla f \neq \nabla g$ or (3) $\Delta f \neq \Delta g$.

If (2) occurs then (2a) $\nabla f \not\leq \nabla g$ or (2b) $\nabla g \not\leq \nabla f$. Assume (2a) holds (in the other case the proof is similar). From (2a) it follows that $D = F(\nabla f)$ is a deductive system and that $\nabla g \notin D$, therefore D is a proper deductive system and in consequence we know that D is intersection of maximal deductive systems, so there exists a maximal deductive system M such that $D \subseteq M$ and $\nabla g \notin M$, therefore $\nabla f \in M$ and $\nabla g \notin M$, so $1 = h_M(\nabla f) = \nabla(h_M(f))$ and $1 \neq h_M(\nabla g) = \nabla(h_M(g))$. Then $\nabla(h_M(f)) \neq \nabla(h_M(g))$ and $F(M) = h_M(f) \neq h_M(g) = G(M)$.

If (3) holds then (3a) $\Delta f \not\leq \Delta g$ or (3b) $\Delta g \not\leq \Delta f$. Assume (3a) holds (in the other case the proof is similar). From (3a) it follows that $D = F(\Delta f)$ is a deductive system and that $\Delta g \notin D$, therefore D is a proper deductive system. In consequence we know that D is intersection of maximal deductive systems, so there exists a maximal deductive system M such that $D \subseteq M$ and $\Delta g \notin M$. As a consequence $\Delta f \in M$ and $\Delta g \notin M$, so $1 = h_M(\Delta f) = \Delta(h_M(f))$ and $1 \neq h_M(\Delta g) = \Delta(h_M(g))$. Therefore $\Delta(h_M(f)) \neq \Delta(h_M(g))$ and then $F(M) = h_M(f) \neq h_M(g) = G(M)$.

We have proved thus that the subalgebra $\mathcal{A}$ of $\mathcal{F}$ is isomorphic to L.

To prove that φ is injective we could also proceed as follows: Let $f, g \in L$, $\varphi(f) = F$, and $\varphi(g) = G$, and assume $F = \varphi(f) = \varphi(g) = G$, this is, $F(M) = G(M)$ for all $M \in \mathbf{M}(L)$, so $h_M(f) = h_M(g)$ for all $M \in \mathbf{M}(L)$. In consequence $1 = h_M(f) \rightarrowtail h_M(g) = h_M(f \rightarrowtail g)$ and $1 = h_M(g) \rightarrowtail h_M(f) = h_M(g \rightarrowtail f)$, this is $f \rightarrowtail g, g \rightarrowtail f \in (h_M)^{-1}(1) = M$ for all $M \in \mathbf{M}(L)$ and since $\bigcap_{M \in \mathbf{M}(L)} M = \{1\}$ then $f \rightarrowtail g = 1 = g \rightarrowtail f$ and therefore $f = g$. □

Theorem 3.5.2. *If L is a finite, not simple Łukasiewicz algebra with more than one element, and $\mathbf{M}(L) = \{M_1, M_2, \ldots, M_n\}$ the set of its maximal deductive systems, then:*

$$L \cong L/M_1 \times L/M_2 \times \cdots \times L/M_n.$$

Proof. Let $\mathcal{F} = \mathbf{T}^{\mathbf{M}(L)}$, as in Theorem 3.5.1, and let $\mathcal{A} = L/M_1 \times L/M_2 \times \cdots \times L/M_n$. We know that the mapping $\varphi : L \to \mathcal{F}$, defined by: $\varphi(f) = F$, where $F(M_i) = h_{M_i}(f)$, for $i = 1, 2, \ldots, n$, and every $f \in L$, is a homomorphism from L to $\mathcal{F}$ and that since $\bigcap_{i=1}^{n} M_i = \{1\}$ then φ is injective, also as in Theorem 3.5.1. Observe that $\mathcal{A} \subseteq \mathcal{F}$, once we identify L/M_i with a subalgebra of $\mathbf{T}$.

Let us prove that the image of φ is $\mathcal{A}$. Since L is a finite, not simple Łukasiewicz algebra with more than one element, then $B(L)$ is a finite boolean algebra with more than one atom. Let $b_1, b_2, \ldots, b_n$ be the atoms of the boolean algebra $B(L)$, then by Lemma 3.1.8, $\{M_i = F(b_i) : 1 \leq i \leq n\}$ is the set of the maximal deductive systems of L and all the L/M_i, $1 \leq i \leq n$ are simple algebras. In a similar manner to that indicated in the proof of Theorem 3.5.1 we can prove that L is isomorphic to a subalgebra of the algebra $\mathcal{F}$ where the isomorphism is defined by $\varphi(x) = (h_{M_1}(x), h_{M_2}(x), \ldots, h_{M_n}(x))$ and where h_{M_i} is the natural epimorphism from L onto L/M_i. Given $y = (y_1, y_2, \ldots, y_n) \in \mathcal{A}$ then for each $y_i \in L/M_i$ there exists $x_i \in L$ such that $h_{M_i}(x_i) = y_i$. Let $x = \bigvee_{i=1}^{n}(x_i \wedge b_i)$. Since $b_i \in B(L)$ then $h_{M_j}(b_i) \in B(L_j) = \{0, 1\}$ so $h_{M_j}(b_i) = 0$ for $j \neq i$ and $h_{M_j}(b_j) = 1$. Therefore $h_{M_j}(x) = h_{M_j}(\bigvee_{i=1}^{n}(x_i \wedge b_i)) = \bigvee_{i=1}^{n}(h_{M_j}(x_i) \wedge h_{M_j}(b_i)) = h_{M_j}(x_j) \wedge h_{M_j}(b_j) = h_{M_j}(x_j) \wedge 1 = h_{M_j}(x_j) = y_j$. This proves that $\varphi(x) = y$. □

Remark 3.5.3. *Every deductive system D of L is in particular a filter and since L is a finite distributive lattice, all its filters are principal, so $D = F(x)$, for some $x \in L$ and since D is a deductive system and $x \in F(x) = D$ it follows that $\Delta x \in F(x)$ and therefore $D = F(b)$ with $b \in B(L)$. On the other hand, we know by Lemma 3.1.8 that M is a maximal deductive system of L if and only if b is an atom of $B(L)$. Furthermore (1) $L/M_i \cong \mathbf{B}$ or (2) $L/M_i \cong \mathbf{T}$ for $1 \leq i \leq n$. We can assume that $M_1, M_2, \ldots, M_k$ are deductive systems verifying (1) and $M_{k+1}, \ldots, M_n$ are deductive systems verifying (2).*

Notice that it could be the case that in L there are no deductive systems verifying (1) or (2), but there are always deductive systems verifying one of the two conditions.

Then the number of elements of $\prod_{i=1}^{n} L/M_i$ is equal to $2^k \times 3^{n-k}$.

3.6. Injective Łukasiewicz algebras

Definition 3.6.1. *A Łukasiewicz algebra C is said to be injective if for any Łukasiewicz algebra L and any S, subalgebra of L, for every homomorphism $h : S \to C$ there exists a homomorphism $H : L \to C$ extending h, this is $H(s) = h(s)$ for all $s \in S$.*

A. Monteiro in the lectures given in 1963, [36] posed to his students the problem of determining the injective Łukasiewicz algebras and conjectured that they were the complete centered Łukasiewicz algebras. L. Monteiro, published an article in 1965, [57], proving this conjecture and that this result is a consequence of an important theorem due to R. Sikorski [76].

Given that every Łukasiewicz algebra is a bounded distributive lattice, it is clear that:

Lemma 3.6.2. *The cartesian product of complete Łukasiewicz algebras is a complete Łukasiewicz algebra.*

Theorem 3.6.3. *Every Łukasiewicz algebra L is isomorphic to a subalgebra of a centered and complete Łukasiewicz algebra.*

PROOF. If L is trivial then clearly L is centered and complete. If L is simple then we know that $L \cong \mathbf{T}$ or $L \cong \mathbf{B}$ and in this latter case $\mathbf{B} \cong \{0,1\} \subset \mathbf{T}$. If L is not trivial, nor simple then by Theorem 3.5.1 we know that L is isomorphic to an L-subalgebra of the Łukasiewicz algebra $P = \mathbf{T}^{\mathbf{M}(L)}$ where $\mathbf{M}(L)$ is the set of the maximal deductive systems of L and since $P \cong \prod_{M \in \mathbf{M}(L)} T_M$ where $T_M = \mathbf{T}$ for all $M \in \mathbf{M}(L)$, and $\mathbf{T}$ is a complete Łukasiewicz algebra then by Lemma 3.6.2 P is complete. Since the cartesian product of centered Łukasiewicz algebras is a centered Łukasiewicz algebra, then P is centered. □

Theorem 3.6.4. (L. Monteiro, [57]) *A Łukasiewicz algebra C is injective if and only if C is complete and centered.*

PROOF. We will only sketch the proof. Assume C is injective, then by Theorem 3.6.3, C is isomorphic to a L-subalgebra S of a complete and centered

Łukasiewicz algebra A. Let f be an isomorphism from C to S and consider the isomorphism $h = f^{-1} : S \to C$. Then since C is injective, h can be extended to a homomorphism $H : A \to C$. Given $\{x_i\}_{i\in I} \subseteq C$ then $\{f(x_i)\}_{i\in I} \subseteq S \subseteq A$, so since A is complete, there exists $s = \bigvee_{i\in I} H(x_i) \in A$. Then it is proved that $H(s) \in C$ is the supremum of $\{x_i\}_{i\in I}$, so C is complete. If c is the center of A then one can prove that $H(c)$ is a center of C.

Now assume that C is a complete Łukasiewicz algebra with center c, A is a Łukasiewicz algebra, S an L-subalgebra of A, and $h : S \to C$ a homomorphism. Consider the boolean algebras $B(A), B(S)$ and $B(C)$. Since C is complete we know by Corollary 1.12.6 that $B(C)$ is a complete boolean algebra. Let f be the restriction of h to $B(S)$ so f is a boolean homomorphism, so by R. Sikorski's theorem [76], there exists a boolean homomorphism $F : B(A) \to B(C)$ extending f. Then we prove that the function $H : A \to C$ defined by $H(x) = (F(\Delta x) \vee c) \wedge F(\nabla x)$ is a homomorphism from A to C extending h. $\square$

This result was generalized by R. Cignoli in 1975, [14] who proved that:

Theorem 3.6.5. *A Kleene algebra is injective if and only if it is a three valued complete Post algebra.*

CHAPTER 4

Free algebras

4.1. Introduction

In this chapter we will determine the Łukasiewicz algebras L_n with a finite number n of free free generators and prove that the number of elements of this algebra is given by:

$$N(L_n) = 2^{2^n} \cdot 3^{3^n - 2^n}.$$

Definition 4.1.1. *Given a cardinal number $\alpha > 0$ we say that L is a Łukasiewicz algebra with α free generators, if*

L1) *L contains a subset G of cardinality α such that $LS(G) = L$,*

L2) *every mapping $f : G \to L'$, where L' is an arbitrary Łukasiewicz algebra, can be extended to a homomorphism h_f, necessarily unique, from L to L.'*

Under these conditions we say that G is a set of free generators of L and a Łukasiewicz algebra is said to be free if it has a set of free generators. To make explicit the cardinal number α we denote $L = L_\alpha$.

Since the notion of Łukasiewicz algebra is given through identities, we know by a result of Garret Birkhoff [8], that the Łukasiewicz algebra L_α with a set of free generators $G = \{g_i\}_{i\in I}$ for a given cardinal α exists and is unique (up to isomorphisms).

We shall study the algebras L_n, where n is a natural number ≥ 1, this is $G = \{g_1, g_2, \ldots, g_n\}$.

4.2. Determination of the Łukasiewicz algebra L_n with n free generators

The following results were indicated by A. Monteiro in 1966, [44], and the proofs were published in 1998 [49].

We saw that if L is a non trivial Łukasiewicz algebra, then L is isomorphic to a subalgebra of

$$\prod_{M\in\mathbf{M}(L)} L/M$$

and that if $\mathbf{M}(L)$ is a finite set then $L \cong \prod_{M\in\mathbf{M}(L)} L/M$. We shall prove that the set $\mathbf{M}(L_n)$ is finite, from where it follows that the set of its prime filters is finite, and we know that this set determines L_n. By Lemma 3.1.7 if $M \in \mathbf{M}(L_n)$ then $L_n/M \cong \mathbf{B}$ or $L_n/M \cong \mathbf{T}$. Identifying isomorphic algebras we have that $L_n/M = \mathbf{B}$ or $L_n/M = \mathbf{T}$

Let M be a maximal deductive system of L_n and h_M the natural homomorphism from L_n onto the quotient algebra L_n/M. Since $G = \{g_1, g_2, \ldots, g_n\}$ is the set of free generators of L_n, and h_M is the epimorphism from L_n to L_n/M, then by Lemma 2.2.9, $h_M(G)$ is a set of generators of L_n/M. Now we define $k_M = i_M \circ h_M$, where i_M is the embedding from L_n/M to $\mathbf{T}$. Since we identify L_n/M with a subalgebra of $\mathbf{T}$, we can think that k_M is h_M. The homomorphism h_M determines a mapping f from G to $\mathbf{T}$, namely the restriction, $h_{M_{|G}}$ of h_M to the set G. Since L_n is a free algebra, h_M is the only homomorphism that coincides with f when restricted to G.

Put by definition: $\psi(M) = h_{M_{|G}}$, so $\psi : \mathbf{M}(L_n) \to \mathbf{T}^G$. If f is a function from G to $\mathbf{T}$, then there exists a unique homomorphism h_f from L_n to $\mathbf{T}$ extending f. Let $M = Ker(h_f)$. By Corollary 3.1.5, M is a maximal deductive system.

We prove now that $h_M = h_f$. Let $x \in L_n$.

If $x \in M = Ker(h_f)$, then $h_f(x) = 1$. At the same time, $h_M(x) = [x]_M = [1]_M = 1$ so $h_M(x) = 1 = h_f(x)$.

If $x \in \sim Ker(h_f)$, then $x = \sim y$ for some $y \in Ker(h_f)$, so $h_f(x) = h_f(\sim y) = \sim h_f(y) = \sim 1 = 0$. On the other hand, $h_M(x) = [x]_M = [\sim y]_M = \sim [y]_M = \sim [1]_M = [0]_M$ so $h_M(x) = 0 = h_f(x)$.

If $x \notin Ker(h_f) \cup \sim Ker(h_f)$, then $h_f(x) \neq 0$ and $h_f(x) \neq 1$, so we must have $h_f(x) = c$. In a similar way, $h_M(x) = [x]_M \neq [0]_M$ and $h_M(x) \neq [1]_M$ so $h_M(x) = c$.

We have proved that $h_{Ker(h_f)} = h_f$, so $\psi(Ker(h_f)) = h_{Ker(h_f)_{|G}} = h_{f_{|G}} = f$ and therefore ψ is surjective. This already proves that the set $\mathbf{M}(L_n)$ is finite, but we will show that there exists a biunivocal correspondence between the maximal deductive systems of L_n and the mappings from G to $\mathbf{T}$. Indeed, let M_1 and M_2 be two maximal deductive systems of L_n and let $h_{M_1} : L_n \to L_n/M_1$ and $h_{M_2} : L_n \to L_n/M_2$ be the respective natural homomorphisms. Let f_1 and f_2 be the restrictions of h_{M_1} and h_{M_2}, respectively, to G, so $f_1 : G \to \mathbf{T}$, $f_2 : G \to \mathbf{T}$.

Assume that $\psi(M_1) = \psi(M_2)$ this is, that $f_1(g) = f_2(g)$, for all $g \in G$. Then $h_{M_1}(g) = f_1(g) = f_2(g) = h_{M_2}(g)$. The function $f_1 = f_2$ admits a unique extension to L_n and since h_{M_1} and h_{M_2} are both extensions of $f_1 = f_2$, $h_{M_1}(x) = h_{M_2}(x)$, for all $x \in L_n$. Therefore $M_1 = M_2$.

Thus we have proved that there exist as many maximal deductive systems as mappings from G to $\mathbf{T}$. The number of mappings from G to $\mathbf{T}$ is 3^n, therefore there exist 3^n different maximal deductive systems in L_n, this is 3^n different minimal prime filters.

Let us determine the number of maximal deductive systems M, such that $L_n/M = \mathbf{B}$. In this case, the natural homomorphism h_M goes from L_n onto $\mathbf{B}$ and therefore the restriction f of h_M to G is a mapping from G to $\mathbf{B}= \{0, 1\}$.

The number of such mappings is, evidently, 2^n. Let us see that the set of these maximal deductive systems of L_n coincides with the set of ultrafilters of L_n.

Indeed, let M be a maximal deductive system of L_n such that (1) $L_n/M = \mathbf{B}$ and let $h : L_n \to \mathbf{B}$ be the natural homomorphism. Let U be an ultrafilter such that (2) $M \subseteq U$. In Corollary 2.5.10 we proved that $M = \varphi(U)$ so $\sim M = \complement U$.

Let $S = M \cup \sim M = M \cup \complement U$. We prove now that S is a subalgebra of L_n containing G. Indeed:

(i) $\underline{G \subseteq S.}$

Let $g \in G$, if $g \in M$ then $g \in S$. If $g \notin M$ then by (1) $h_M(g) = 0$ and therefore $h_M(\sim g) = \sim h_M(g) = 1$ so $\sim g \in M$ and in consequence $g \in \sim M \subseteq S$.

(ii) S is a subalgebra.

Let $s \in S$, so $s \in M$ or $s \in \sim M$. In the former case $\sim s \in \sim M \subseteq S$. In the latter, $s = \sim m$ where $m \in M$ so $\sim s = m \in M \subseteq S$.

Let $s, t \in S$, so we can have the following cases (3) $s, t \in M$, (4) $s, t \in \sim M$ and (5) $s \in M$, $t \in \sim M$.

(3) Since M is a filter, $s \wedge t \in M \subseteq S$.

(4) $s = \sim m_1$ with $m_1 \in M$ y $t = \sim m_2$ with $m_2 \in M$, so $s \wedge t = \sim (m_1 \vee m_2)$ and since M is a filter, $m_1 \in M$ y $m_1 \leq m_1 \vee m_2$ we have that $m_1 \vee m_2 \in M$ and therefore $s \wedge t \in \sim M$.

(5) $t = \sim m$ with $m \in M$, so $s \wedge t = s \wedge \sim m = \sim (\sim s \vee m)$. Since $m \in M$, $m \leq \sim s \vee m$ and M is a filter, we have that $\sim s \vee m \in M$ and therefore $s \wedge t \in \sim M$.

To complete the proof that S is a subalgebra it only remains to show that S verifies (*) "If $s \in S$ then $\nabla s \in S$".

In order to do this we shall prove first that $B(L_n) \cap (U \setminus M) = \emptyset$. Indeed, if there is some $b \in B(L_n) \cap (U \setminus M)$ then (6) $b \in B(L_n)$, (7) $b \in U$ and (8) $b \notin M$. We know that if b is a boolean element of a Łukasiewicz algebra, its boolean complement is precisely $\sim b$ (see [11], [59]), so from (6) it follows that (9) $b \vee \sim b = 1 \in M$. By the Corollary 2.5.10, M is a prime filter of L_n, so from (9) and (8) it follows that (10) $\sim b \in M \subseteq U$. From (7) and (10) we conclude that $0 = \sim b \wedge b \in U$, a contradiction.

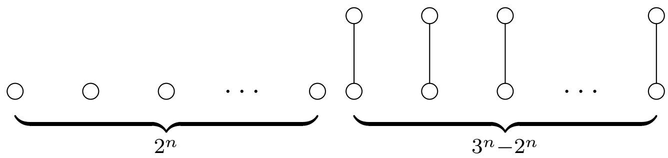
From $B(L_n) \cap (U \setminus M) = \emptyset$ it follows that $B(L_n) \subseteq \complement(U \setminus M) = \complement U \cup M = S$, and therefore (*) holds.

Therefore since S is a subalgebra of L_n containing the generators of L_n we have that $S = L_n$, this is $L_n = M \cup \complement U = M \cup (U \setminus M) \cup \complement U$ and therefore $U \setminus M \subseteq M \cup \complement U$, so $U \setminus M = (U \setminus M) \cap (M \cup \complement U) = (U \cap \complement M) \cap (M \cup \complement U) = (U \cap \complement M \cap M) \cup (U \cap \complement M \cap \complement U) = \emptyset$. From $U \setminus M = \emptyset$ it follows that $U \subseteq M$ and since $M \subseteq U$ we have finally that $M = U$ and therefore M is an ultrafilter.

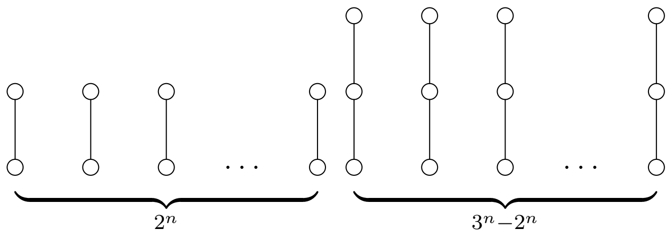
Conversely, if an ultrafilter M is a deductive system, then M is a maximal deductive system. Indeed, if M' is a deductive system such that $M \subset M' \subset L_n$ then M' would be a proper filter containing M as a proper part, which contradicts that M is an ultrafilter. Let us prove in this case that $L_n/M = \mathbf{B}$. If $L_n/M = \mathbf{T}$, and h_M is the natural epimorphism from $L_n \to \mathbf{T}$, then $h_M^{-1}(1) = M$, and there exists $a \in L_n$ such that $h_M(a) = c$. Furthermore $U = h_M^{-1}([c))$ is an ultrafilter of L_n such that $U \neq M$, so $M = h_M^{-1}(1) \subset h_M^{-1}([c)) = U$, a contradiction. Therefore $L_n/M = \mathbf{B}$.

Then, there exist 2^n maximal deductive systems that are ultrafilters and therefore $3^n - 2^n$ maximal deductive systems that are not ultrafilters.

For each maximal deductive system M that is not an ultrafilter, $L_n/M = \mathbf{T}$. We have also proved that each maximal deductive system M is properly contained in one and only one ultrafilter U. Therefore, the Hasse diagram of the set of all the prime filters of L_n is the following:



Therefore, L_n is the cartesian product of the chains in the following Hasse diagram:



From where it follows that finally

$$L_n = \mathbf{B}^{2^n} \times \mathbf{T}^{3^n-2^n}$$

and

$$\boxed{N(L_n) = 2^{2^n} \cdot 3^{3^n-2^n}.}$$

The diagram for L_1 was displayed in Example 1.3.2.

R. Cignoli and A. Monteiro presented a geometric construction of the Łukasiewicz algebra with an arbitrary set of free generators [33], which will be laid out in Chapter VI.

L. Monteiro, A. Figallo and A. Ziliani [71], presented a construction of the Łukasiewicz algebras with a given poset of free generators.

4.3. Free Moisil algebras with a finite number of free generators

Axled three-valued Łukasiewicz algebra, see section 1.3, were introduced by Gr. M. Moisil [27]. A. Monteiro called these algebras three valued Moisil algebras or Moisil algebras.

It is clear that if L, L' are Moisil algebras, e is the axis of L and $h : L \to L'$ is a homomorphism then $h(e)$ is an axis of $h(L)$.

If L is a Moisil algebra and $X \subseteq L$ we denote with $MS(X)$ the Moisil subalgebra of L generated by X. Clearly if L is a Moisil algebra and e is its axis then, $MS(X) = LS(X \cup \{e\})$.

Since the notion of Moisil algebra can be defined through identities, we know by a general result due to Garret Birkhoff [8], that there exists a Moisil algebra M_α with a set of free generators G of any given cardinality α and it is unique (up to isomorphisms).

We denote with M_n the Moisil algebra with a set $G = \{g_1, g_2, \ldots, g_n\}$ of free generators.

We will follow the method used in section 4.2.

We saw that if L is a non trivial Łukasiewicz algebra, then L is isomorphic to a subalgebra of

$$\prod_{D\in\mathbf{M}(L)} L/D,$$

and that if the set $\mathbf{M}(L)$ is finite then $L \cong \prod_{D\in\mathbf{M}(L)} L/D$.

We know that if $D \in \mathbf{M}(M_n)$ then $M_n/D \cong \mathbf{B}$ or $M_n/D \cong \mathbf{T}$. Identifying isomorphic algebras we have that $L_n/M = \mathbf{B}$ or $L_n/M = \mathbf{T}$.

Let $\mathbf{M}_1(M_n) = \{D \in \mathbf{M}(M_n) : M_n/D = \mathbf{B}\}$ and $\mathbf{M}_2(M_n) = \{D \in \mathbf{M}(M_n) : M_n/D = \mathbf{T}\}$.

If $D \in \mathbf{M}_1(M_n)$, let h_D be the natural homomorphism from M_n onto the quotient algebra M_n/D. The homomorphism h_D determines a mapping f from G to $\mathbf{B}$, the restriction, $h_{D_{|G}}$ of h_D to the set G, this is $f(g) = h_{D_{|G}}(g) = h_D(g)$, for all $g \in G$. We put by definition: $\psi_1(D) = h_{D_{|G}}$, so $\psi_1 : \mathbf{M}_1(M_n) \to \mathbf{B}^G$. In an similar way as seen in section 4.2, it follows that the mapping ψ_1 is surjective.

We prove now that ψ_1 is injective. Indeed, let $D_1, D_2 \in \mathbf{M}_1(M_n)$, $m_1 : M_n \to M_n/D_1$ and $m_2 : M_n \to M_n/D_2$ the respective natural homomorphisms. Let f_1 and f_2 be the restrictions of m_1 and m_2 respectively, to the set G, so $f_1 : G \to \mathbf{B}$, $f_2 : G \to \mathbf{B}$. Assume that $\psi_1(D_1) = \psi_1(D_2)$, this is that $f_1(g) = f_2(g)$, for all $g \in G$. Then $m_1(g) = f_1(g) = f_2(g) = m_2(g)$. The function $f_1 = f_2$ admits a unique extension to M_n and since m_1 and m_2 are extensions of f_1 and f_2 respectively, then $m_1(x) = m_2(x)$, for all $x \in M_n$. Therefore $D_1 = D_2$.

Thus we have proved that the number of elements in $\mathbf{M_1}(M_n)$ is the same as the number of functions from G to $\mathbf{B}$, and we know that this number is equal to 2^n.

In a similar manner, if $D \in \mathbf{M}_2(M_n)$ and m is the natural homomorphism from M_n onto the quotient algebra $M_n/D = \mathbf{T}$, then the homomorphism m determines in this case a mapping f from G to $\mathbf{T}$, precisely the restriction $m_{|G}$ from m to the set G, this is $f(g) = m_{|G}(g) = m(g)$, for all $g \in G$. Putting by definition: $\psi_2(D) = m_{|G}$, then $\psi_2 : \mathbf{M}_2(M_n) \to \mathbf{T}^G$. Conversely, if f is a mapping from G to $\mathbf{T}$, then there exists one and only one homomorphism h_f from M_n to $\mathbf{T}$ extending f. Since $h_f(M_n) = MS(h_f(G)) = LS(h_f(G) \cup \{c\})$ and the only subalgebra of $\mathbf{T}$ containing the center of $\mathbf{T}$ is $\mathbf{T}$ itself then $h_f(M_n) = \mathbf{T}$. In the same way as before, we conclude that ψ_2 establishes a bijection between $\mathbf{M}_2(M_n)$ and the set of all the functions from G to $\mathbf{T}$ and we know that this set has 3^n elements.

We have finished proving that:

$$M_n = \mathbf{B}^{2^n} \times \mathbf{T}^{3^n}.$$

therefore the number $N(M_n)$ of elements of M_n is:

$$\boxed{N(M_n) = 2^{2^n} \cdot 3^{3^n}}$$

A method for determining the Moisil algebra $M(\alpha)$ with a set G of free generators of cardinality α was indicated by A. Monteiro before 1976 (see [63]), but his results were published in 1998, [49]. We shall describe them here, with minor modifications introduced by L. Monteiro in [63].

Let e be the axis of $M(\alpha)$. We denote $B(\alpha) = M(\alpha)/F(\sim \nabla e)$, $P(\alpha) = M(\alpha)/F(\nabla e)$.

L. Monteiro proved, see section 3.3, that:

$$M(\alpha) \cong B(\alpha) \times P(\alpha),$$

and also that

- $B(\alpha)$ is a boolean algebra,
- $B(\alpha)$ is isomorphic to $B = \{x \in M(\alpha) : x \leq\sim \nabla e\}$,
- $P(\alpha)$ is a centered Łukasiewicz algebra,
- $P(\alpha)$ is isomorphic to $C = \{x \in M(\alpha) : x \leq \nabla e\}$,
- $h(x) = (x \wedge \sim \nabla e, x \wedge \nabla e)$ is an isomorphism from $M(\alpha)$ to $B(\alpha) \times P(\alpha)$.

It is easy to check that the function defined by $h_1(x) = x \wedge \sim \nabla e$ is an epimorphism from $M(\alpha)$ to $B(\alpha)$.

We shall prove that $B(\alpha)$ is a boolean algebra with a set of free generators of cardinality α.

Let B^* be a boolean algebra with a set of free generators G^* of cardinality α. Since G and G^* have the same cardinal, there exists a bijection $f : G \to G^*$, so since $M(\alpha)$ is a free algebra, f can be extended to a unique homomorphism $H : M(\alpha) \to B^*$. Since G is a set of generators of $M(\alpha)$ then by Lemma 2.2.9, $MS(H(G)) = H(M(\alpha))$ and since $MS(H(G)) = MS(f(G)) = MS(G^*) = B^*$ it follows that H is an epimorphism.

<u>(i) The restriction of h_1 to the set G is injective.</u>

Indeed, let $g, g' \in G$ be such that $h_1(g) = h_1(g')$ this is $g \wedge \sim \nabla e = g' \wedge \sim \nabla e$, so

$$H(g \wedge \sim \nabla e) = H(g' \wedge \sim \nabla e),$$

this is

$$H(g) \wedge \sim \nabla H(e) = H(g') \wedge \sim \nabla H(e).$$

Since H is a homomorphism, it takes the axis e of $M(\alpha)$ to the axis of B^* which we know is the element 0, so we have that

$$H(g) \wedge \sim \nabla 0 = H(g') \wedge \sim \nabla 0,$$

this is

$$H(g) = H(g')$$

and since H is an extension of f we have

$$f(g) = f(g')$$

from where it follows, since f injective, that que $g = g'$. Thus from (i) it follows that the subset $h_1(G) = G_B$ of $B(\alpha)$ has the same cardinal as G. Furthermore, since h_1 is surjective, by Lemma 2.2.9, $MS(h_1(G)) = h_1(M(\alpha)) = B(\alpha)$.

We prove now that G_B is a set of free generators of B. For this, let A be a boolean algebra, $f' : G_B \to A$ and $f_1 = f' \circ h_{1|G}$, so f_1 is a function from G to A and since $M(\alpha)$ is a free algebra, f_1 can be extended to a homomorphism $H_1 : M(\alpha) \to A$. Notice that:

<u>(ii) $Ker(h_1) \subseteq Ker(H_1)$.</u>

Indeed, if $h_1(x) = 1$ this is $1 = x \wedge \sim \nabla e$ then

$$1 = H_1(x \wedge \sim \nabla e) = H_1(x) \wedge \sim \nabla H_1(e) = H_1(x) \wedge \sim \nabla 0 = H_1(x) \wedge 1 = H_1(x).$$

From (ii) it follows by results of the theory of homomorphisms, see Lemma 2.2.6, that there exists a unique homomorphism $H_2 : B(\alpha) \to A$ such that $H_2 \circ h_1 = H_1$.

(iii) $H_2(g') = f'(g')$ for all $g' \in G_B$.

Indeed, given $g' \in G_B = h_1(G)$, there exists $g \in G$ such that $h_1(g) = g'$ so $H_2(g') = H_2(h_1(g)) = (H_2 \circ h_1)(g) = H_1(g) = f_1(g) = (f' \circ h_1)(g) = f'(h_1(g)) = f'(g')$.

Thus we have proved that $B(\alpha)$ is a boolean algebra with a set G_B of free generators of cardinality equal to α, the cardinality of G.

In a similar manner one can prove:

- The mapping defined by $h_2(x) = x \wedge \nabla e$ is an epimorphism from $M(\alpha)$ to $P(\alpha)$,
- $h_2(G) = G_C$ is a set of cardinality α,
- G_C is a set of free generators of the centered algebra $P(\alpha)$, this is, of the three valued Post algebra C.

From the result above it follows that $M(n)$ is isomorphic to the cartesian product of the boolean algebra with n free generators by the Post algebra with n free generators so $M(n) = \mathbf{B}^{2^n} \times \mathbf{T}^{3^n}$ and in consequence

$$\boxed{N(M_n) = 2^{2^n} \cdot 3^{3^n}}$$

as we pointed out before.

CHAPTER 5

Homomorphic images and further constructions

In this Chapter we shall indicate a construction of all the epimorphisms between finite Łukasiewicz algebras and determine their number (L. Monteiro (2003) [67]).

5.1. Homomorphic images of a finite boolean algebra

If B and B' are boolean algebras, we denote with $Hom(B, B')$ ($Epi(B, B')$) the set of all the homomorphisms (epimorphisms) from B to B'. We denote with B_n a boolean algebra with n atoms, where $n \in \mathbb{N}$. If $b \in B_m$ and $b' \in B_n$, let $Epi^{(b,b')}(B_m, B_n) = \{h \in Epi(B_m, B_n) : h(b) = b'\}$.

With $\mathcal{A}(B_n)$ we denote the set of the atoms of B_n, and if $b \in B_n \setminus \{0\}$ we denote $\mathcal{A}(b) = \{a \in \mathcal{A}(B_n) : a \leq b\}$.

Given B_m and B_n, if $m < n$ then $Epi(B_m, B_n) = \emptyset$.

It is well known that if $f : \mathcal{A}(B_n) \to \mathcal{A}(B_m)$ then the function $h_f : B_m \to B_n$ defined by

$$h_f(x) = \bigvee \{a \in \mathcal{A}(B_n) : f(a) \leq x\}^{1},$$

verifies:

A1) $h_f \in Hom(B_m, B_n)$,
A2) If $a \in \mathcal{A}(B_m)$ then $h_f(a) = 0$ if and only if $a \notin f(\mathcal{A}(B_n))$,
A3) h_f is surjective if and only if f is injective, [77, 78],
A4) h_f is injective if and only if f is surjective [77, 78].

If $h \in Epi(B_m, B_n)$ then given $b \in \mathcal{A}(B_n)$ we know that $[b)$ is an ultrafilter of B_n and that $h^{-1}([b))$ is an ultrafilter of B_m so $h^{-1}([b)) = [a)$ with $a \in \mathcal{A}(B_m)$. Furthermore, $Ker\,(h) \subseteq [a)$. Let $f : \mathcal{A}(B_n) \to \mathcal{A}(B_m)$ defined by $f(b) = a$, then f is injective and $h_f = h$.

There exists a bijective correspondence between the set $In(\mathcal{A}(B_n), \mathcal{A}(B_m))$ of all the injective functions from $\mathcal{A}(B_n)$ to $\mathcal{A}(B_m)$ and the set $Epi(B_m, B_n)$. For this it is enough to consider the function $\Phi(f) = h_f$.

If X is a finite set we denote with $N[X]$ its cardinality.

Let us put by definition

$$V_{m,n} = \begin{cases} \dfrac{m!}{(m-n)!}, & \text{if } m \geq n \\ \\ 0, & \text{if } m < n. \end{cases}$$

Thus:

$$N[Epi(B_m, B_n)] = V_{m,n}.$$

[1] Notice that in this setting, $\bigvee \emptyset = 0$.

Lemma 5.1.1. *Let $h \in Epi(B_m, B_n)$, where $m \geq n \geq 1$. Then*

A5) *If $a \in \mathcal{A}(B_m)$, then $h(a) = 0$ or $h(a) \in \mathcal{A}(B_n)$,*
A6) *If $b \in \mathcal{A}(B_n)$, then there exists a unique $a \in \mathcal{A}(B_m)$, such that $h(a) = b$,*
A7) *If h is injective then $h(a) \in \mathcal{A}(B_n)$, for all $a \in \mathcal{A}(B_m)$.*

Item A6) of the preceding lemma was proved by M. Abad and L. Monteiro in [3], and items A5) and A7) by the same authors in [4]. A different proof was presented by L. Monteiro and A. Kremer in [69].

Next we give another proof of $N[Epi(B_m, B_n)] = V_{m,n}$. Let $m \geq n \geq 1$. If $h \in Epi(B_m, B_n)$, we know that the quotient algebra $B_m/Ker\,(h)$ is isomorphic to B_n, and since B_m is finite $Ker\,(h) = [x)$, with $x \in B_m$. Furthermore, $B_m/[x) \cong (x]$ so $N[(x]] = N[B_n] = 2^n$. A. Monteiro proved in [66] that if B is a boolean algebra and F a filter of B then each equivalence class modulo F is coordinable with F. Therefore, if we let $N[F] = t$, $N[(x]] \cdot t = N[B_m]$ this is $2^n \cdot t = 2^m$ and therefore $t = 2^{m-n}$.

If B_n is a homomorphic image of B_m then there exists $x \in B_m$ such that $(x] \cong B_n$ and in consequence x is the supremum of n atoms of B_m. There are $\binom{m}{n}$ elements of B_m that are supremum of n atoms of B_m. Let $\mathcal{F}_n$ be the set of all the increasing sets $[x)$ where x is supremum of n atoms of B_m.

We denote with $Aut(B_n)$ the set all the automorphisms of the boolean algebra B_n. If $\alpha \in Aut(B_n)$ then α is in particular a bijection on $\mathcal{A}(B_n)$ and by item A7) of Lemma 5.1.1, α transforms atoms in atoms of B_n so clearly there exist $n!$ bijections on $\mathcal{A}(B_n)$, so $N[Aut(B_n)] = n!$.

If $h \in Epi(B_m, B_n)$ then $Ker\,(h) \in \mathcal{F}_n$. Let $\beta : Epi(B_m, B_n) \to \mathcal{F}_n$ be defined by $\beta(h) = Ker\,(h)$. It is clear that if $\alpha \in Aut(B_n)$ then $\alpha \circ h \in Epi(B_m, B_n)$. By Lemma 2.2.3, all the epimorphisms with the same kernel can be obtained this way. Given $[x) \in \mathcal{F}_n$, we consider $\beta^{-1}([x))$, so $N[\beta^{-1}([x))] = n!$ and therefore $n! \cdot \binom{m}{n} = V_{m,n} = N[Epi(B_m, B_n)]$.

If $b \in B_m \setminus \{0\}$ and $h \in Epi(B_m, B_n)$ then $h(b) = 0$ or $h(b) \neq 0$. If $b' = h(b) = 0$ then the number of elements of $Epi^{(b,0)}(B_m, B_n)$ is equal to the number of injective functions from $\mathcal{A}(B_n)$ to $\mathcal{A}(B_m) \setminus \mathcal{A}(b)$ this is:

$$N[Epi^{(b,0)}(B_m, B_n)] = V_{m-N[\mathcal{A}(b)],n},$$

so the number of epimorphisms that send b to a non-zero element of B_n is ([62], p.86):

$$V_{m,n} - V_{m-N[\mathcal{A}(b)],n}.$$

Given $b \in B_m \setminus \{0\}$ and $b' \in B_n \setminus \{0'\}$ then $h_f(b) = b'$ if and only if $f(\mathcal{A}(b')) \subseteq \mathcal{A}(b)$ and $f(\mathcal{A}(-b')) \subseteq \mathcal{A}(-b)$, so

$$N[Epi^{(b,b')}(B_m, B_n)] = V_{N[\mathcal{A}(b)],N[\mathcal{A}(b')]}.V_{m-N[\mathcal{A}(b)],n-N[\mathcal{A}(b')]}.$$

If $b \neq 0$ then $h(b) = 0$ if and only if $f(\mathcal{A}(B_n)) \subseteq \mathcal{A}(B_m) \setminus \mathcal{A}(b)$.

Lemma 5.1.2. *If $m \geq n$, and $h \in Epi^{(b,b')}(B_m, B_n)$, where $b \in B_m \setminus \{0\}$, $b' \in B_n \setminus \{0'\}$, then:*

A8) *If $a \in \mathcal{A}(b)$ and $h(a) \neq 0'$ then $h(a) \in \mathcal{A}(b')$.*
A9) *If $a \in \mathcal{A}(B_m) \setminus \mathcal{A}(b)$ and $h(a) \neq 0'$ then $h(a) \notin \mathcal{A}(b')$.*

Proof. If $a \in \mathcal{A}(b)$, this is $a \leq b$ then $h(a) \leq h(b) = b'$ so if $h(a) \neq 0'$, $h(a) \in \mathcal{A}(B_n)$ and therefore $h(a) \in \mathcal{A}(b')$.

If $a \in \mathcal{A}(B_m) \setminus \mathcal{A}(b)$, assume that $h(a) \leq b' = h(b)$ so $h(a) = h(a) \wedge h(b) = h(a \wedge b)$, therefore $a \equiv a \wedge b \pmod{Ker\,(h)}$. Since B_m is finite $Ker\,(h) = [f)$ with $f \in B_m$, and therefore (1) $a \wedge f = a \wedge b \wedge f$. From $0 \leq a \wedge f \leq a$ and $a \in \mathcal{A}(B_m)$ it follows that (2) $a \wedge f = a$ or (3) $a \wedge f = 0$. If (2) holds then by (1) we have that $a = a \wedge b \wedge f \leq b$, a contradiction. If (3) holds, since $h(f) = 1$ we have that $0 = h(0) = h(a \wedge f) = h(a) \wedge h(f) = a \wedge 1 = a$, another contradiction. Then $h(a) \not\leq b'$. □

From A3) and Lemma 5.1.2 it follows that if $h \in Epi(B_m, B_n)$ then $f = \Phi^{-1}(h)$ verifies:

DE1) $f \in In(\mathcal{A}(B_n), \mathcal{A}(B_m))$,
DE2) $f(\mathcal{A}(b')) \subseteq \mathcal{A}(b)$,
DE3) $f(\mathcal{A}(-b')) \subseteq \mathcal{A}(-b)$.

If $h \in Hom(B_m, B_n)$ then (1) $S = h(B_m)$ is a subalgebra of B_n and therefore (2) $h(B_m) \cong B_t$ with $1 \leq t \leq n$ and $m \geq t$. Let $\mathcal{A}(S) = \{s_1, s_2, \ldots, s_t\}$.

From (1) it follows that $h \in Epi(B_m, S)$, then $g = \Phi^{-1}(h) \in In(\mathcal{A}(S), \mathcal{A}(B_m))$ and (3) $h_g = h$.

Since $\{\mathcal{A}(s_1), \mathcal{A}(s_2), \ldots, \mathcal{A}(s_t)\}$ is a partition of $\mathcal{A}(B_n)$ then if $b \in \mathcal{A}(B_n)$, there exists i, $1 \leq i \leq t$ such that $b \in \mathcal{A}(s_i)$, this is $b \leq s_i$.

We define $f : \mathcal{A}(B_n) \to \mathcal{A}(B_m)$ in this manner: $f(b) = g(s_i)$ if and only if $b \in \mathcal{A}(s_i)$. Clearly if one of the sets $\mathcal{A}(s_i)$ has more than one element, f is not injective. Notice that from the definition of f we have that $f(\mathcal{A}(B_n)) = g(\mathcal{A}(S))$. By A1) $h_f = \Phi(f) \in Hom(B_m, B_n)$. Let us see that $h_f = h$. By (3), it is enough to prove that $h_f = h_g$, and for that, we need to show that $h_f(a) = h_g(a)$ for all $a \in \mathcal{A}(B_m)$.

If $a \in g(\mathcal{A}(S))$, this is $a = g(s_i)$ for some $s_i \in \mathcal{A}(S)$, then for all $b \in \mathcal{A}(s_i)$ we have $f(b) = a = g(s_i)$ so (4) $\mathcal{A}(s_i) \subseteq f^{-1}(a)$. We claim that (5) $f^{-1}(a) \subseteq \mathcal{A}(s_i)$. Indeed, if $x \in f^{-1}(a) \setminus \mathcal{A}(s_i)$ then $f(x) = a$ and $x \in \mathcal{A}(s_j)$, for some $s_j \neq s_i$. Therefore $f(x) = g(s_j) \neq a$, because g is injective, a contradiction. From (4) and (5) it follows that (6) $\mathcal{A}(s_i) = f^{-1}(a)$. Since g is injective, $h_g(a) = s_i$ (for this see [66], page 172, observation 4.12.2 (3)). Then $h_f(a) = \bigvee\{x \in \mathcal{A}(B_n) : f(x) \leq a\} = \bigvee\{x \in \mathcal{A}(B_n) : f(x) \in \mathcal{A}(a)\} = \bigvee\{x \in \mathcal{A}(B_n) : f(x) = a\} = \bigvee f^{-1}(a) =$ (by (6)) $= \bigvee \mathcal{A}(s_i) = s_i$.

If $a \notin g(\mathcal{A}(S))$, by A2) $h_g(a) = 0$. Furthermore (7) $a \notin f(\mathcal{A}(B_n))$, because if $a = f(x)$, with $x \in \mathcal{A}(B_n)$ then $x \in \mathcal{A}(s_i)$ for some i, so $a \in g(\mathcal{A}(S))$, a contradiction. From (7) it follows that $h_f(a) = \bigvee\{x \in \mathcal{A}(B_n) : f(x) \leq a\} = \bigvee\{x \in \mathcal{A}(B_n) : f(x) = a \text{ or } f(x) = 0\} = \bigvee \emptyset = 0$.

From the two cases above, it follows that $h_f(a) = h_g(a)$ for all $a \in \mathcal{A}(B_m)$.

If $m < n$ we have that $N[Epi(B_m, B_n)] = 0$, and if $h \in Hom(B_m, B_n)$ then $h(B_m)$ is a boolean subalgebra of B_n such that $1 \leq N[\mathcal{A}(h(B_m))] \leq m < n$.

It is well known that the boolean subalgebras of B_n are in bijective correspondence with the partitions of the set $\mathcal{A}(B_n)$, and that the number of subalgebras of B_n with t atoms $1 \le t \le n$ is

$$P(n,t) = \frac{\sum\limits_{i=0}^{t-1} (-1)^i \binom{t}{i} (t-i)^n}{t!}.$$

Therefore if S is a subalgebra of B_n with t atoms where $1 \le t \le m < n$ then there exist $V_{m,t}$ epimorphisms from B_m onto S, so if $m < n$,

$$N[Hom(B_m, B_n)] = \sum_{t=1}^{m} P(n,t) \cdot V_{m,t}.$$

Notice that if $m \ge n$ then

$$N[Hom(B_m, B_n)] = N[Epi(B_m, B_n)] + \sum_{t=1}^{n-1} P(n,t) \cdot N[Epi(B_m, B_t)] =$$

$$V_{m,n} + \sum_{t=1}^{n-1} P(n,t) \cdot V_{m,t}.$$

5.2. Homomorphic images of a Łukasiewicz algebra

Let L be a finite Łukasiewicz algebra, then

$$L \cong \mathbf{B}^j \times \mathbf{T}^k, \text{where } j, k \in \mathbb{Z}, j \ge 0, k \ge 0.$$

T) If $j = k = 0$ then L is trivial, this is, it has a single element,
B) If $j \ge 1, k = 0$ then L is a boolean algebra with j atoms,
P) If $j = 0, k \ge 1$ then L is a centered algebra and $B(L)$ is a boolean algebra with k atoms,
Ax) If $j \ge 1, k \ge 1$ then L is an axled Łukasiewicz algebra, that is a not a boolean algebra nor a centered algebra. The axis is the $(j+k)$-tuple

$$e = (\underbrace{0, 0, \ldots, 0}_{j}, \underbrace{c, \ldots, c}_{k})$$

Therefore

$$\nabla e = (\underbrace{0, 0, \ldots, 0}_{j}, \underbrace{1, \ldots, 1}_{k}).$$

$B(L)$ is a boolean algebra with $j+k$ atoms, and its elements are

$$(b_1, b_2, \ldots, b_j, b_{j+1}, \ldots, b_{j+k}),$$

where $b_i \in \mathbf{B} = \{0,1\}$ for $1 \le i \le j$ and $b_i \in \{0,1\} \subset \mathbf{T}$ for $j+1 \le i \le j+k$. Given $b \in B(L)$ let

$$J(b) = \{i : b_i = 1, 1 \le i \le j\}, \text{and } K(b) = \{i : b_i = 1, j+1 \le i \le j+k\},$$

so $0 \le N[J(b)] \le j$ and $0 \le N[K(b)] \le k$.

If j and k are not simultaneously zero then L is a non trivial finite Łukasiewicz algebra. We know that the homomorphic images of L are determined by the filters $[b)$ where $b \in B(L)$, and that the quotient algebra $L/[b)$ is isomorphic to the Łukasiewicz algebra $(b] = \{x \in L : x \le b\}$. Then since $B(L)$ has 2^{j+k} elements:

C) there exist 2^{j+k} homomorphic images of L.

Furthermore we have:

B) If $j \geq 1, k = 0$, then L has 2^j homomorphic images, which are boolean algebras.

P) If $j = 0, k \geq 1$, then L has 2^k homomorphic images, which are centered algebras.

Ax) If $j \geq 1, k \geq 1$, then L has 2^{j+k} homomorphic images, which are axled algebras.

Ax1) If $N[K(b)] = 0$, then $(b] \cong B^{N[J(b)]}$ therefore there are $\binom{j}{j_1}$, $0 \leq j_1 \leq j$ homomorphic images of L that are boolean algebras with j_1 atoms, and we have a total of 2^j homomorphic images which are boolean algebras. Observe that if $N[J(b)] = 0$, then $L/[b)$ is a trivial algebra.

Ax2) If $N[J(b)] = 0$, then $(b] \cong \mathbf{T}^{N[K(b)]}$ therefore there are $\binom{k}{k_1}$, $0 \leq k_1 \leq k$ homomorphic images L' of L that are centered algebras such that $B(L')$ is a boolean algebra with k_1 atoms, and we have a total of 2^k homomorphic images that are centered algebras. Notice that if $N[K(b)] = 0$, then $L/[b)$ is a trivial algebra, which coincides with the trivial algebra from Ax1).

Ax3) If $1 \leq j_1 = N[J(b)] \leq j$ and $1 \leq k_1 = N[K(b)] \leq k$, then $L/[b) \cong (b] \cong \mathbf{B}^{N[J(b)]} \times \mathbf{T}^{N[K(b)]}$ is an axled homomorphic which is not a boolean algebra nor a centered algebra.

Therefore the number of these homomorphic images is:

$$\left(\sum_{i=1}^{j}\binom{j}{i}\right) \cdot \left(\sum_{i=1}^{k}\binom{k}{i}\right) = (2^j - 1) \cdot (2^k - 1).$$

Since the trivial algebra shows up in cases Ax1) and Ax2) as a homomorphic image, if $j \geq 1$ and $k \geq 1$, we have a total of:

$$\begin{gathered} 2^j + (2^k - 1) + (2^k - 1) \cdot (2^j - 1) = \\ 2^j + (2^k - 1) \cdot (1 + (2^j - 1)) = \\ 2^j + (2^k - 1) \cdot 2^j = \\ 2^j \cdot (1 + 2^k - 1) = 2^j \cdot 2^k = 2^{j+k}, \end{gathered}$$

homomorphic images, as we had determined in C).

5.3. Epimorphisms

If L and L' are Łukasiewicz algebras, we denote with $Epi(L, L')$ the set of all the epimorphisms from L to L'.

Lemma 5.3.1. *If L and L' are axled Łukasiewicz algebras, with axis e and e' respectively and $H \in Epi(L, L')$ then $H(e) = e'$. Furthermore, if we write $h = H_{|B(L)}$, then $h \in Epi^{(\nabla e, \nabla e')}(B(L), B(L'))$.*

Proof. (1) $\Delta H(e) = H(\Delta e) = H(0) = 0'$.

Let $y \in L'$, so since H is surjective, there exists $x \in L$ such that $H(x) = y$, so (2) $\nabla y = \nabla H(x) = H(\nabla x) \leq H(\Delta x \vee \nabla e) = H(\Delta x) \vee H(\nabla e) = \Delta H(x) \vee \nabla H(e) = \Delta y \vee \nabla H(e)$.

From (1) and (2) it follows that $H(e)$ is an axis of L' and since the axis is unique $H(e) = e'$.

Furthermore, $h(\nabla e) = H(\nabla e) = \nabla H(e) = \nabla e'$. $\square$

The following lemma generalizes the results by L. Monteiro [57, 62] and also Lemma 4.1 due to L. Monteiro, M. Abad, S. Savini and J. Sewald appearing in [68].

Lemma 5.3.2. *If L and L' are axled Łukasiewicz algebras with axis e and e' respectively and $h \in Epi^{(\nabla e, \nabla e')}(B(L), B(L'))$ then the transformation $H : L \to L'$ defined by $H(x) = (h(\Delta x) \vee e') \wedge h(\nabla x)$ verifies*

a) *H is an extension of h,*
b) *$H \in Epi(L, L')$, and*
c) *H is the only extension of h.*

Proof. a) Indeed if $b \in B(L)$ then $\Delta b = \nabla b = b$, so:
$H(b) = (h(\Delta b) \vee e') \wedge h(\nabla b) = (h(b) \vee e') \wedge h(b) = h(b)$.

b1) $H(x \wedge y) = (h(\Delta(x \wedge y) \vee e') \wedge h(\nabla(x \wedge y)) =$
$(h(\Delta x \wedge \Delta y) \vee e') \wedge h(\nabla x \wedge \nabla y) = ((h(\Delta x) \wedge h(\Delta y)) \vee e') \wedge h(\nabla x) \wedge h(\nabla y) =$
$(h(\Delta x) \vee e') \wedge (h(\Delta y) \vee e') \wedge h(\nabla x) \wedge h(\nabla y) =$
$(h(\Delta x) \vee e') \wedge h(\nabla x) \wedge (h(\Delta y) \vee e') \wedge h(\nabla y) = H(x) \wedge H(y)$.

b2) Since H extends h and $\nabla x \in B(L)$ then (1) $H(\nabla x) = h(\nabla x)$.

(2) $\nabla H(x) = \nabla((h(\Delta x) \vee e') \wedge h(\nabla x)) = (\nabla h(\Delta x) \vee \nabla e') \wedge \nabla h(\nabla x) = (h(\Delta x) \vee \nabla e') \wedge h(\nabla x)$.

Since $\nabla x \leq \Delta x \vee \nabla e$, ∇x, Δx, ∇e, $\Delta x \vee \nabla e \in B(L)$ and h is a boolean homomorphism verifying $h(\nabla e) = \nabla e'$ we have that (3) $h(\nabla x) \leq h(\Delta x \vee \nabla e) = h(\Delta x) \vee h(\nabla e) = h(\Delta x) \vee \nabla e'$. From (2) and (3) it follows that (4) $\nabla H(x) = h(\nabla x)$. From (1) and (4) it follows that $\nabla H(x) = H(\nabla x)$.

b3) By definition $H(\sim x) = (h(\Delta \sim x) \vee e') \wedge h(\nabla \sim x)$, so (5) $\Delta H(\sim x) = (h(\Delta \sim x) \vee \Delta e') \wedge h(\nabla \sim x) = (h(\Delta \sim x) \vee 0') \wedge h(\nabla \sim x) =$
$h(\Delta \sim x) \wedge h(\nabla \sim x) = h(\Delta \sim x)$, and $\nabla H(\sim x) =$
$(h(\Delta \sim x) \vee \nabla e') \wedge h(\nabla \sim x) = (h(\Delta \sim x) \vee h(\nabla e)) \wedge h(\nabla \sim x) =$
$h((\Delta \sim x \vee \nabla e) \wedge \nabla \sim x)$. Since $\nabla \sim x \leq \Delta \sim x \vee \nabla e$, we have that (6) $\nabla H(\sim x) = h(\nabla \sim x)$.

Since h is a boolean epimorphism, then if $b \in B(L)$ we have that (7) $h(\sim b) = \sim h(b)$, so $\sim H(x) = (\sim h(\Delta x) \wedge \sim e') \vee \sim h(\nabla x) =$
$(h(\sim \Delta x) \wedge \sim e') \vee h(\sim \nabla x)$, and therefore:

(8) $\Delta \sim H(x) = (\Delta h(\sim \Delta x) \wedge \Delta \sim e') \vee \Delta h(\sim \nabla x) =$
$(h(\sim \Delta x) \wedge \sim \nabla e') \vee h(\sim \nabla x) = (\sim h(\Delta x) \wedge \sim h(\nabla e)) \vee \sim h(\nabla x) =$
$\sim ((h(\Delta x) \vee h(\nabla e)) \wedge h(\nabla x)) = \sim h((\Delta x \vee \nabla e) \wedge \nabla x) = \sim h(\nabla x) = h(\sim \nabla x) =$
$h(\Delta \sim x)$, and (9) $\nabla \sim H(x) = (\nabla h(\sim \Delta x) \wedge \nabla \sim e') \vee \nabla h(\sim \nabla x) =$
$(h(\sim \Delta x) \wedge \sim \Delta e') \vee h(\sim \nabla x) = (h(\sim \Delta x) \wedge \sim 0) \vee h(\sim \nabla x) =$
$h(\sim \Delta x) \vee h(\sim \nabla x) = h(\sim \Delta x) = h(\nabla \sim x)$.

From (5) and (8) it follows that (10) $\Delta H(\sim x) = \Delta \sim H(x)$ and from (6) and (9) it follows that (11) $\nabla H(\sim x) = \nabla \sim H(x)$. From (10) and (11) it follows by Moisil's determination principle that $H(\sim x) = \sim H(x)$.

b4) Given $y \in L'$, $y = (\Delta y \vee e') \wedge \nabla y$, since $\nabla y, \Delta y \in B(L')$ and h is a boolean epimorphism, there exist $b_1, b_2 \in B(L)$ such that $h(b_1) = \Delta y$ and $h(b_2) = \nabla y$.

Let $b_3 = b_1 \wedge b_2 \in B(L)$, $b_4 = b_1 \vee b_2 \in B(L)$ and $x = (b_3 \vee e) \wedge b_4 \in L$. Then $\Delta x = (\Delta b_3 \vee \Delta e) \wedge \Delta b_4 = (b_3 \vee 0) \wedge b_4 = b_3 \wedge b_4 = b_3 = b_1 \wedge b_2$, and $\nabla x = (\nabla b_3 \vee \nabla e) \wedge \nabla b_4$. So $h(\Delta x) = h(b_1 \wedge b_2) = h(b_1) \wedge h(b_2) =$
$\Delta y \wedge \nabla y = \Delta y$, and $h(\nabla x) = h((\nabla b_3 \vee \nabla e) \wedge \nabla b_4) = (h(\nabla b_3) \vee h(\nabla e)) \wedge h(\nabla b_4) = (\Delta y \vee \nabla e')) \wedge \nabla y)$ and since $\nabla y \leq \Delta y \vee \nabla e'$ we have that $h(\nabla x) = \nabla y$, so $H(x) = (h(\Delta x) \vee e') \wedge h(\nabla x) = (\Delta y \vee e') \wedge \nabla y = y$, which proves that H is surjective.

c) If $H' \in Epi(L, L')$ verifies $H'(b) = h(b)$ for all $b \in B(L)$ then

$$H'(x) = (\Delta H'(x) \vee e') \wedge \nabla H'(x) = (H'(\Delta x) \vee e') \wedge H'(\nabla x) =$$
$$(h(\Delta x) \vee e') \wedge h(\nabla x) = H(x).$$

□

Corollary 5.3.3. *If L and L' are centered Łukasiewicz algebras with centers c and c' respectively and $h \in Epi(B(L), B(L'))$ then the function $H : L \to L'$ defined by $H(x) = (h(\Delta x) \vee c') \wedge h(\nabla x)$ is the unique epimorphism from L to L' extending h.*

Proof. It is enough to notice that every center is an axis of the algebra and that $h(\nabla c) = h(1) = 1' = \nabla c'$. □

Lemma 5.3.4. *If L and L' are axled Łukasiewicz algebras with axis e and e' respectively, then there is a bijective correspondence between the sets $Epi(L, L')$ and $Epi^{(\nabla e, \nabla e')}(B(L), B(L'))$.*

Proof. If $h \in Epi^{(\nabla e, \nabla e')}(B(L), B(L'))$, then by Lemma 5.3.2, the function:

$$H(x) = (h(\Delta x) \vee e') \wedge h(\nabla x)$$

verifies $H \in Epi(L, L')$. If we put $\delta(h) = H$, then by Lemma 5.3.2, c) δ is a function.

If $H \in Epi(L, L')$, by Lemma 5.3.1 we have that $h = H_{|B(L)} \in Epi^{(\nabla e, \nabla e')}(B(L), B(L'))$ and by Lemma 5.3.2 the extension of h to L is the epimorphism H, so $\delta(h) = H$, which proves that δ is surjective.

If $h, h' \in Epi^{(\nabla e, \nabla e')}(B(L), B(L'))$ are such that $h \neq h'$ then there exists $b \in B(L)$ such that $h(b) \neq h'(b)$. If H and H' are homomorphisms extending h and h' respectively then $H(b) = h(b) \neq h'(b) = H'(b)$, so δ is injective. □

Let L and L' be non trivial finite Łukasiewicz algebras, so $L \cong \mathbf{B}^j \times \mathbf{T}^k$ and $L' \cong \mathbf{B}^{j'} \times \mathbf{T}^{k'}$, where $j, k, j', k' \geq 1$. We proved that $H \in Epi(L, L')$ if and only if $h = H_{|B(L)} \in Epi^{(\nabla e, \nabla e')}(B(L), B(L'))$ so $f = \Phi^{-1}(h) \in In(\mathcal{A}(B(L')), \mathcal{A}(B(L)))$ must satisfy the conditions DE2) and DE3), this is

$$(1) \qquad f(\mathcal{A}(\nabla e')) \subseteq \mathcal{A}(\nabla e) \text{ and } f(\mathcal{A}(\sim \nabla e')) \subseteq \mathcal{A}(\sim \nabla e).$$

Then since $N[\mathcal{A}(\nabla e)] = k$, $N[\mathcal{A}(\nabla e')] = k'$, $N[\mathcal{A}(B(L)) \setminus \mathcal{A}(\nabla e)] = j$ and $N[\mathcal{A}(B(L')) \setminus \mathcal{A}(\nabla e')] = j'$ for the set of injective functions from $\mathcal{A}(B(L'))$ to $\mathcal{A}(B(L))$ verifying (1) not to be empty it is necessary and sufficient that $k \geq k'$ and $j \geq j'$. Then by the results above:

$$(2) \qquad N[Epi(L, L')] = N[Epi^{(\nabla e, \nabla e')}(B(L), B(L'))] = V_{k,k'} \cdot V_{j,j'}.$$

Notice that:

- If L and L' are algebras with center c and c' respectively, this is $L \cong \mathbf{T}^k$ and $L' \cong \mathbf{T}^{k'}$ then
$$N[Epi(L,L')] = N[Epi^{(1,1')}(B(L),B(L'))] = N[Epi(B(L),B(L'))] = V_{N[\mathcal{A}(B(L))],N[\mathcal{A}(B(L')')]} = V_{k,k'}.$$
- If L and L' are boolean algebras, this is $L \cong \mathbf{B}^j$ and $L' \cong \mathbf{B}^{j'}$ then
$$N[Epi(L,L')] = N[Epi^{(0,0')}(B(L),B(L'))] = N[Epi(B(L),B(L'))] = V_{N[\mathcal{A}(B(L))],N[\mathcal{A}(B(L'))]} = V_{j,j'}.$$

If L is a non trivial finite Łukasiewicz algebra, let $P(L)$ be the set of its prime elements and $\varphi : P(L) \to P(L)$ the Birula-Rasiowa transformation from section 2.5. If L' is a non trivial finite Łukasiewicz algebra, a function $f : P(L') \to P(L)$ is said to be an H-function, [2] if it is biunivocal and verifies $f(\nabla p') = \nabla f(p')$, $f(\varphi(p')) = \varphi(f(p'))$. M. Abad and A. Figallo [2] proved that there exists a bijection between the H-functions and the set $Epi(L,L')$, where L and L' are axled Łukasiewicz algebras that are not boolean algebras nor centered algebras. The number of elements of $Epi(L,L')$ determined by these authors coincides with the one indicated in (2).

Clearly it is harder to construct H-functions than injectives functions from $\mathcal{A}(B_n)$ to $\mathcal{A}(B_m)$ verifying the conditions indicated in (1).

Remark 5.3.5. *If $L \cong \mathbf{B}^j \times \mathbf{T}^k$, let $b \in B(L)$ be such that $1 \leq N[J(b)] = j_1 \leq j$ and $1 \leq N[K(b)] = k_1 \leq k$. Assume for example that*
$$b = (\underbrace{\underbrace{1,1,\dots,1}_{j_1},0,0,\dots 0}_{j},\underbrace{\underbrace{1,1,\dots,1}_{k_1},0,0,\dots,0}_{k}).$$

Then the set $(b]$ has $2^{j_1} \cdot 3^{k_1}$ elements. We know that $(b]$ is a Łukasiewicz algebra and that $L/[b) \cong (b]$. Furthermore $(b]$ has as axis the element
$$e' = (\underbrace{0,0,\dots 0}_{j},\underbrace{\underbrace{c,c,\dots,c}_{k_1},0,0,\dots,0}_{k}),$$
so
$$\nabla e' = (\underbrace{0,0,\dots 0}_{j},\underbrace{\underbrace{1,1,\dots,1}_{k_1},0,0,\dots,0}_{k}).$$

If $x \in (b]$ then
$$x = (\underbrace{x_1,x_2,\dots,x_{j_1},0,0,\dots 0}_{j},\underbrace{y_1,y_2,\dots,y_{k_1},0,0,\dots,0}_{k})$$
and since the negation in $(b]$ (see section 2.6) is given by $\approx x = \sim x \wedge b$ we have that
$$\approx x = (\underbrace{\sim x_1,\sim x_2,\dots,\sim x_{j_1},0,0,\dots 0}_{j},\underbrace{\sim y_1,\sim y_2,\dots,\sim y_{k_1},0,0,\dots,0}_{k})$$

so

$$\approx \nabla e' = (\underbrace{\underbrace{1, 1, \ldots, 1}_{j_1}, 0, 0, \ldots 0,}_{j} \underbrace{0, 0, \ldots, 0}_{k}).$$

By the results in section 3.3,

$$L/[b) \cong (b] \cong (b]/[\approx \nabla e') \times (b]/[\nabla e')$$

where $(b]/[\approx \nabla e')$ *is a boolean algebra and* $(b]/[\nabla e')$ *is a centered Łukasiewicz algebra. Since* $(b]/[\approx \nabla e') \cong (\approx \nabla e']$*, and* $(b]/[\nabla e') \cong (\nabla e']$ *we have that* $N[(\approx \nabla e']] = 2^{j_1}$ *and* $N[(\nabla e']] = 3^{k_1}$*, so*

$$L/[b) \cong \mathbf{B}^{j_1} \times \mathbf{T}^{k_1}.$$

Lemma 5.3.6. (L. Monteiro [62]) *If* L *and* L' *are Łukasiewicz algebras,* $H : L \to L'$ *a homomorphism and* $h = H_{|B(L)}$ *then:*

a) $H(B(L)) \subseteq B(L')$ *and* $h : B(L) \to B(L')$ *is a boolean homomorphism,*
b) H *is completely determined by* h*,*
c) *if* $B(L') \subseteq H(L)$ *then* h *is an epimorphism from* $B(L)$ *to* $B(L')$*,*
d) *if* $H(L) \subseteq B(L')$ *then* $h = H_{|B(L)}$ *verifies* $h(\Delta x) = h(\nabla x)$ *for all* $x \in L$*. Conversely, if* $g : B(L) \to B(L')$ *is a boolean homomorphism that verifies* $g(\Delta x) = g(\nabla x)$ *for all* $x \in L$*, then* g *can be extended to a unique homomorphism* $H : L \to L'$ *such that* $H(L) \subseteq B(L')$*,*
e) *if* L *has axis* e*, then* $H(L) \subseteq B(L')$ *if and only if* $H(e) = 0$*. If* $g : B(L) \to B(L')$ *is a boolean homomorphism that verifies* $g(\nabla e) = 0$*,* g *can be extended to a unique homomorphism* $H : L \to L'$ *such that* $H(L) \subseteq B(L')$*,*
f) *if* L' *is an algebra with center* c'*, every boolean homomorphism* $g : B(L) \to B(L')$ *can be extended to a unique homomorphism* $H : L \to L'$*.*

PROOF. a) Is an immediate consequence of the properties of the homomorphisms.

b) If H and H' are homomorphisms from L to L' such that $h = H_{|B(L)} = H'_{|B(L)} = h'$ then (1) $\Delta H(x) = H(\Delta x) = h(\Delta x) = h'(\Delta x) = H'(\Delta x) = \Delta H'(x)$, and analogously (2) $\nabla H(x) = \nabla H'(x)$. From (1) and (2) by Moisil's determination principle, it follows that $H = H'$.
c) Given $b' \in B(L')$, since $B(L') \subseteq H(L)$ then $b' = H(x)$ with $x \in L$, so $\Delta x \in B(L)$ and $h(\Delta x) = H(\Delta x) = \Delta H(x) = \Delta b' = b'$.
d) If $H(L) \subseteq B(L')$ then $\Delta H(x) = H(x) = \nabla H(x)$ for all $x \in L$, so $h(\Delta x) = H(\Delta x) = \Delta H(x) = H(x) = \nabla H(x) = H(\nabla x) = h(\nabla x)$.

If $g : B(L) \to B(L')$ is a boolean homomorphism such that $g(\Delta x) = g(\nabla x)$ for all $x \in L$, then put by definition $H(x) = g(\Delta x)$, for all $x \in L$. From this definition $H(x) \in B(L')$, for all $x \in L$, so $H(L) \subseteq B(L')$. Furthermore, if $b \in B(L)$ then $H(b) = g(\Delta b) = g(b)$. Finally let us prove that H is a homomorphism.

$H(x \vee y) = g(\Delta(x \vee y)) = g(\Delta x \vee \Delta y) = g(\Delta x) \vee g(\Delta y) = H(x) \vee H(y)$.

$H(\nabla x) = g(\Delta \nabla x) = g(\nabla x) = H(x) = \nabla H(x)$.

$H(\sim x) = g(\Delta \sim x) = g(\sim \nabla x) = \sim g(\nabla x) = \sim H(x)$.

If $H' : L \to L'$ is a homomorphism extending g such that (3) $H'(L) \subseteq B(L')$ then $H'(\Delta x) = g(\Delta x) = H(x)$ so $\Delta H'(x) = H(x)$ and since by (3) $H'(x) \in B(L')$, we have that $H'(x) = H(x)$.

e) We know that $x = (\Delta x \vee e) \wedge \nabla x$, for all $x \in L$, so if $H(e) = 0$ then $H(x) = (H(\Delta x) \vee H(e)) \wedge H(\nabla x) = H(\Delta x) \wedge H(\nabla x) = H(\Delta x) = \Delta H(x) \in B(L')$ and therefore $H(L) \subseteq B(L')$. Conversely, since $H(e) \in H(L) \subseteq B(L')$ we have that $\Delta H(e) = H(e)$ so $H(e) = \Delta H(e) = H(\Delta e) = H(0) = 0$. Notice that in this case $h(\nabla e) = H(\nabla e) = \nabla H(e) = 0$.

If $g : B(L) \to B(L')$ is a boolean homomorphism such that $g(\nabla e) = 0$, since $\nabla x \leq \Delta x \vee \nabla e$ for all $x \in L$, then $g(\nabla x) \leq g(\Delta x) \vee g(\nabla e) = g(\Delta x)$, so as $g(\Delta x) \leq g(\nabla x)$, we have that $g(\Delta x) = g(\nabla x)$ for all $x \in L$. Then by (4) g extends to a unique homomorphism $H : L \to L'$ such that $H(L) \subseteq B(L')$.

f) The homomorphism H is defined by $H(x) = (g(\Delta x) \vee c') \wedge g(\nabla x)$. (L. Monteiro [57]). The uniqueness was also proved by L. Monteiro, [52]. If F is a homomorphism from L to L' extending g then $F(x) = (\Delta F(x) \vee c') \wedge \nabla F(x) = (F(\Delta x) \vee c') \wedge F(\nabla x) = (g(\Delta x) \vee c') \wedge g(\nabla x) = H(x)$.

□

Lemma 4.1 in [68] is a particular instance of the lemma above.

Lemma 5.3.7. *If L and L' are Łukasiewicz algebras with axis e and e' respectively, $h \in Epi^{(\nabla e, \nabla e')}(B(L), B(L'))$ and $H \in Epi(L, L')$ is the epimorphism extending h then:*

a) $x \in Ker\,(H) = \{x \in L : H(x) = 1\} \iff \Delta x, \nabla x \in Ker\,(h) = \{b \in B(L) : h(x) = 1\}$,

b) *If $h_1, h_2 \in Epi^{(\nabla e, \nabla e')}(B(L), B(L'))$ and $H_1, H_2 \in Epi(L, L')$ are the epimorphisms extending h_1 and h_2 respectively then:*

$$Ker\,(h_1) = Ker\,(h_2) \iff Ker\,(H_1) = Ker\,(H_2).$$

Proof.

a) $x \in Ker\,(H) \iff H(x) = 1$, so $h(\Delta x) = H(\Delta x) = \Delta H(x) = 1$ and $h(\nabla x) = H(\nabla x) = \nabla H(x) = 1$ this is $\Delta x, \nabla x \in Ker\,(h)$.

Conversely, if $\Delta x, \nabla x \in Ker\,(h)$, this is $h(\Delta x) = h(\nabla x) = 1$ then $H(x) = (h(\Delta x) \vee e') \wedge h(\nabla x) = (1 \vee e') \wedge 1 = 1$.

b) If $x \in Ker\,(H_1)$ then by a) we have that $\Delta x, \nabla x \in Ker\,(h_1) = Ker\,(h_2)$ so by a), $x \in Ker\,(H_2)$.

Conversely, if $b \in Ker\,(h_1)$ then $\Delta b = \nabla b = b \in Ker\,(h_1)$, so by a): $b \in Ker\,(H_1) = Ker\,(H_2)$ so by a), $b = \Delta b = \nabla b \in Ker\,(h_2)$.

□

Lemma 5.3.8. *If L and L' are Łukasiewicz algebras with axis e and e' respectively, and $h \in Epi^{(\nabla e, \nabla e')}(B(L), B(L'))$ then its epimorphism extension H verifies:*

a) *If* $\Delta x = 0$ *then* $H(x) \leq e'$ *and*
b) $H(x) = 0 \iff h(\nabla x) = 0$.

PROOF. a) $H(x) = (h(\Delta x) \vee e') \wedge h(\nabla x) = (h(0) \vee e') \wedge h(\nabla x) = (0 \vee e') \wedge h(\nabla x) = e' \wedge h(\nabla x) \leq e'$.
b) If $H(x) = 0$ then $h(\nabla x) = H(\nabla x) = \nabla H(x) = \nabla 0 = 0$. If $h(\nabla x) = 0$ then $H(x) = (h(\Delta x) \vee e') \wedge h(\nabla x) = (h(\Delta x) \vee e') \wedge 0 = 0$. □

5.4. Homomorphism extensions

In 1965 A. Monteiro [42], presented a theorem about the extension of boolean algebras homomorphisms which generalizes results due to R. Sikorski [76]. Using these results by A. Monteiro, in 1970 L. Monteiro [52] established similar results for Łukasiewicz algebras.

Lemma 5.4.1. *For a function h from a Łukasiewicz algebra L to a Łukasiewicz algebra L' to be a homomorphism it is necessary and sufficient that the following conditions hold:*

- $h(0) = 0$, $\quad h(1) = 1$,
- $h(x \wedge y) = h(x) \wedge h(y)$, $\quad h(x \vee y) = h(x) \vee h(y)$,
- $h(\Delta x) = \Delta h(x)$, $\quad h(\nabla x) = \nabla h(x)$.

Definition 5.4.2. *A function d from a Łukasiewicz algebra L to a Łukasiewicz algebra L' is said to be a semihomomorphism if*

Sh1) $d(1) = 1$,
Sh2) $d(x \vee y) = d(x) \vee d(y)$,
Sh3) $d(\nabla x) = \nabla d(x)$.

Lemma 5.4.3. *If d is a semihomomorphism from a Łukasiewicz algebra L to a Łukasiewicz algebra L' then*

P1) *If* $x \leq y$ *then* $d(x) \leq d(y)$,
P2) $d(\Delta x) \leq d(x)$,
P3) $d(\Delta x) \in B(L')$,
P4) $d(\Delta x) \leq \Delta d(x)$,
P5) $\sim d(x) \leq d(\sim x)$.

Theorem 5.4.4. *If L is a Łukasiewicz algebra, S a subalgebra of L, C an injective Łukasiewicz algebra, d a semihomomorphism from L to C, h a homomorphism from S to C such that* (D) $h(s) \leq d(s)$ *for all $s \in S$, then there exists a homomorphism H from L to C such that* a) *H extends h;* b) $H(x) \leq d(x)$ *for all $x \in L$.*

Theorem 5.4.5. *If d is a semihomomorphism from a Łukasiewicz algebra L to an injective Łukasiewicz algebra C and if $a_0 \in L \setminus \{0\}$ then there exists a homomorphism H from L to C such that:*

a) $H(\nabla a_0) = d(\nabla a_0)$,
b) $H(x) \leq d(x)$ *for all* $x \in L$.

L. Monteiro used the results indicated in [52] to establish a theorem of functional representation of monadic Łukasiewicz algebras in his doctoral dissertation [61], similar to the one used by P. Halmos in [21] for monadic boolean algebras.

5.5. Construction of the free boolean algebras from the free three valued Łukasiewicz algebras

The following results by A. Monteiro, were first published in 1995 in the "Informes Técnicos Internos" series of the INMABB, number 42. In 1996 they were reprinted in the "Notas de Lógica Matemática" series, volume 40, [48].

Remark 5.5.1. *If L is a Łukasiewicz algebra, $X \subseteq B(L)$, we denote with $F_B(X)$ the filter of the boolean algebra $B(L)$ generated by the set X. It is clear that $F_B(X) = F(X) \cap B(L)$.*

Let $\mathcal{L} = \mathcal{L}(\alpha)$ be the Łukasiewicz algebra with a set $G = \{g_i : i \in I\}$ of free generators of cardinality α, $\nabla G = \{\nabla g_i : i \in I\}$ and $F = F_B(\nabla G)$. Consider the quotient boolean algebra $B = B(\mathcal{L})/F$ and represent by $C_B(b)$ the equivalence class of $B(\mathcal{L})$ containing the element $b \in B(\mathcal{L})$. Then:

Theorem 5.5.2. *$B = B(\mathcal{L})/F$ is a boolean algebra that has as free generators the elements $C_B(\Delta g_i)$, $i \in I$, and the cardinal of the set $G^* = \{C_B(\Delta g_i) : i \in I\}$ is equal to α.*

Proof. Let us prove that:

(i) F is a proper filter of $B(\mathcal{L})$.

If $F = B(\mathcal{L})$, then $0 \in F$, and by Lemma 2.1.15 it follows that there exist elements $\nabla g_{i_1}, \nabla g_{i_2}, \ldots, \nabla g_{i_n} \in \nabla G$ such that:

$$0 = \bigwedge_{k=1}^{n} \nabla g_{i_k} = \nabla(\bigwedge_{k=1}^{n} g_{i_k}), \text{ and therefore} : \bigwedge_{k=1}^{n} g_{i_k} = 0.$$

Let f be the transformation from G to the Łukasiewicz algebra $\mathbf{T} = \{0, c, 1\}$, defined by:

$$f(g_i) = 1, \text{ for all } i \in I$$

then there exists a homomorphism h from $\mathcal{L}$ to T extending f so:

$$0 = h(0) = h(\bigwedge_{k=1}^{n} g_{i_k}) = \bigwedge_{k=1}^{n} h(g_{i_k}) = \bigwedge_{k=1}^{n} f(g_{i_k}) = 1.$$

This contradiction proves that F is a proper filter of $B(\mathcal{L})$.

(ii) If $j, k \in I$, and $j \neq k$ then the equivalence classes $C_B(\Delta g_j)$ and $C_B(\Delta g_k)$ are different.[2]

Indeed assume that $C_B(\Delta g_j) = C_B(\Delta g_k)$, then we have that:

$$(1) \qquad \Delta g_j \wedge t = \Delta g_k \wedge t, \text{where } t \in F.$$

Consider the transformation f from G to the Łukasiewicz algebra $\mathbf{T}$, defined for each $i \in I$ by:

$$f(g_i) = \begin{cases} c, & \text{if } i = k \\ 1, & \text{if } i \neq k. \end{cases}$$

[2] *From this point on, in this section, the original proofs by A. Monteiro have been replaced by simpler ones due to L. Monteiro.*

This transformation can be extended to a homomorphism h from $\mathcal{L}$ to $\mathbf{T}$ such that:

$$(2) \qquad h(\nabla g_i) = \nabla h(g_i) = \nabla f(g_i) = 1, \text{ for every } i \in I.$$

Let $D = h^{-1}(1)$ be the kernel of the homomorphism h, so $\nabla g_i \in D$, for all $i \in I$, and therefore

$$(3) \qquad F = F_B(\nabla G) \subseteq F(\nabla G) \subseteq D.$$

From the definition of f it follows that $f(g_k) = c$ and since by hypothesis $j \neq k$, then $f(g_j) = 1$, so

$$(4) \qquad h(\Delta g_j) = \Delta h(g_j) = \Delta f(g_j) = \Delta 1 = 1$$

and

$$(5) \qquad h(\Delta g_k) = \Delta h(g_k) = \Delta f(g_k) = \Delta(c) = 0.$$

From (1) we deduce that:

$$h(\Delta g_j \wedge t) = h(\Delta g_k \wedge t),$$

this is

$$\Delta h(g_j) \wedge h(t) = \Delta h(g_k) \wedge h(t).$$

So by (4) and (5):

$$(6) \qquad h(t) = 1 \wedge h(t) = 0 \wedge h(t) = 0.$$

Since $t \in F$ then by (3), we have that $h(t) = 1$, which contradicts (6). This contradiction proves that (ii) holds. Then we can claim:

$$\text{The set } \ G^* = \{C_B(\Delta g_i) : i \in I\} \text{ has cardinality } \ \alpha.$$

Let φ be the natural boolean homomorphism from $B(\mathcal{L})$ onto $B = B(\mathcal{L})/F$, this is, if $b \in B(\mathcal{L})$ then $\varphi(b) = C_B(b)$. By Lemma 2.2.9 the homomorphism φ transforms each set of generators of $B(\mathcal{L})$ into a set of generators of $B = B(\mathcal{L})/F$. By Corollary 1.11.9 we know that $B(\mathcal{L}) = SB(\Delta G \cup \nabla G)$, so:

$$\{C_B(\Delta g_i) : i \in I\} \cup \{C_B(\nabla g_i) : i \in I\}$$

is a set of generators of $B = B(\mathcal{L})/F$. But, since for all $i \in I$, the equivalence class $C_B(\nabla g_i) = C_B(1) = F$ is the top element of $B = B(\mathcal{L})/F$, we don't need to consider it as one of the generators and we can claim then that G^* is a set of generators of $B = B(\mathcal{L})/F$.

(iii) Every mapping f' of the set $G^* \subseteq B$ to the boolean algebra $B = \{0,1\} \subseteq \mathbf{T}$, can be extended to a boolean homomorphism h' from $B = B(\mathcal{L})/F$ to B.

Consider the mapping f from G to $\mathbf{T}$ defined by the following conditions:

$$f(g_i) = \begin{cases} 1, & \text{if } f'(C_B(\Delta g_i)) = 1 \\ c, & \text{if } f'(C_B(\Delta g_i)) = 0. \end{cases}$$

As a consequence we have:

$$(7) \qquad \nabla f(g_i) = 1 \text{ for all } i \in I$$

and

$$\Delta f(g_i) = \begin{cases} 1, & \text{if } f'(C_B(\Delta g_i)) = 1 \\ 0, & \text{if } f'(C_B(\Delta g_i)) = 0. \end{cases}$$

this is:

$$(8)\qquad \Delta f(g_i) = f'(C_B(\Delta g_i)),\ \text{for every } i \in I.$$

The mapping f from G to $\mathbf{T}$, can be extended to a homomorphism H from $\mathcal{L}$ to $\mathbf{T}$. Then by (7) we have:

$$(9)\qquad H(\nabla g_i) = \nabla H(g_i) = \nabla f(g_i) = 1.$$

The kernel $N = H^{-1}(1)$ of this homomorphism is a deductive system of $\mathcal{L}$, and by (9) we have $\nabla G \subseteq N$, so $F(\nabla G) \subseteq N$, and therefore following Remark 5.5.1,

$$F = F_B(\nabla G) = F(\nabla G) \cap B(\mathcal{L}) \subseteq N \cap B(\mathcal{L}) \subseteq N,$$

so

$$(10)\qquad H(F) = \{1\}.$$

By Lemma 5.3.6 a) we know that the homomorphism H transforms boolean elements of $\mathcal{L}$ in boolean elements of $\mathbf{T}$, and since $B(\mathbf{T}) = \{0,1\}$, then we have $H(B(\mathcal{L})) = \{0,1\}$.

Let h be the restriction of H to the set $B(\mathcal{L})$, then we can claim that h is a boolean homomorphism from $B(\mathcal{L})$ onto $B(\mathbf{T}) = \{0,1\} \subset \mathbf{T}$, and by (10) we have $h(F) = \{1\}$.

Notice that

$$\text{“ If } x, y \in B(\mathcal{L}),\ \text{and } x \in C_B(y),\ \text{then } h(x) = h(y)\text{”}.$$

Indeed, since $x \in C_B(y)$ we have: $x \wedge d = y \wedge d$, for some $d \in F$, so

$$h(x) = h(x) \wedge 1 = h(x \wedge d) = h(y \wedge d) = h(y) \wedge 1 = h(y).$$

From the result above it follows that if for each $x \in B(\mathcal{L})$, we define $h'(C_B(x)) = h(x)$, then h' is a function from $B = B(\mathcal{L})/F$ onto $B(\mathbf{T})$. It is easy to prove that h' is a boolean homomorphism. We shall prove that h' extends f', this is that:

$$h'(C_B(\Delta g_i)) = f'(C_B(\Delta g_i)),\ \text{for every } i \in I.$$

Using (8) we have that:

$$h'(C_B(\Delta g_i)) = h(\Delta g_i) = H(\Delta g_i) = \Delta H(g_i) = \Delta f(g_i) = f'(C_B(\Delta g_i)).$$

Let us prove now that G^* is a set of free generators of $B = B(\mathcal{L})/F$.

(iv) <u>Every mapping f of the set G^* to a boolean algebra A, can be extended to a boolean homomorphism from $B = B(\mathcal{L})/F$ to A.</u>

If A has a single element, it evidently verifies (iv). If the boolean algebra A is isomorphic to the boolean algebra $\{0,1\}$ then from (iii) it follows that (iv) holds.

Assume now that A has more than one element and that A is not simple, then it is well known that A is isomorphic to a boolean subalgebra A' of the boolean algebra $P = \prod_{j \in J} A_j$ where $A_j = A/M_j$, and $\{M_j : j \in J\}$ is the set of all the maximal filters of A. We also know that $A_j \cong \{0,1\}$ for all $j \in J$. If $(x_j)_{j\in J} \in P$ we know that the t-th projection of P on A_t, $t \in J$, is given by $\pi_t\left((x_j)_{j\in J}\right) = x_t$.

Let α be the isomorphism from A to A' and $f^* = \alpha \circ f$ so $f^* : G^* \to A'$, and if $g^* \in G^*$ then $f^*(g^*) = g' = (g'_j)_{j\in J}$. Let $f_j = \pi_j \circ f^*$, $j \in J$. Then $f_j : G^* \to$

$A_j \simeq \{0,1\}$ so by (iii) each f_j can be extended to a boolean homomorphism h_j from $B = B(\mathcal{L})/F$ onto A_j.

Let h be the function from $B = B(\mathcal{L})/F$ to P defined by $h(x) = (h_j(x))_{j\in J}$. Evidently h is a boolean homomorphism. Let us prove that h extends f^*. Indeed:

$$h(g^*) = (h_j(g^*))_{j\in J} = (f_j(g^*))_{j\in J} = (\pi_j(f^*(g^*)))_{j\in J} = \left(\pi_j\left(\left(g'_j\right)_{j\in J}\right)\right)_{j\in J} =$$

$$(g'_j)_{j\in J} = g' = f^*(g^*), \text{ for every } g^* \in G^*.$$

Since $h(G^*) = f^*(G^*) \subseteq A'$ then $\alpha^{-1}(h(G^*)) \subseteq \alpha^{-1}(A') = A$. Since α^{-1} and h are homomorphisms then $\alpha^{-1}\circ h$ is a homomorphism from $B = B(\mathcal{L})/F$ to A. Let us prove that $\alpha^{-1}\circ h$ is an extension of f. Indeed, $(\alpha^{-1}\circ h)(g^*) = \alpha^{-1}(h(g^*)) = \alpha^{-1}(f^*(g^*)) = \alpha^{-1}((\alpha\circ f)(g^*)) = f(g^*)$. $\square$

Remark 5.5.3. *When G is finite and $N[G] = n \in \mathbb{N}$, we know that $N[\mathcal{L}] = 2^{2^n} \times 3^{3^n-2^n}$ and that $N[B(\mathcal{L})] = 2^{3^n}$. By the preceding results,*

$$N[B(\mathcal{L})/F_B(\nabla G)] = 2^{2^n}.$$

On the other hand, we know that every equivalence class (modulo $F_B(\nabla G)$) of $B(\mathcal{L})$ has the same number of elements, so:

$$2^{2^n} = N[B(\mathcal{L})/F_B(\nabla G)] = \frac{N[B(\mathcal{L})]}{N[F_B(\nabla G)]} = \frac{2^{3^n}}{N[F_B(\nabla G)]}.$$

This proves that the number of elements of each equivalence class is: $2^{3^n-2^n}$.

5.6. Representation of a Łukasiewicz algebra by sets

Given a Łukasiewicz algebra L, let $E = \mathbf{P}(L)$ and for each $x \in L$ put $\mathcal{S}(x) = \{P \in E : x \in P\}$, then the transformation $\mathcal{S}$, which is called *Stone's transformation*, is a function from L to the set $\mathcal{P}(E) = 2^E$, of the parts of E. We know that $(2^E, \cap, \cup, \complement, E)$ is a boolean algebra and by Stone's results on distributive lattices:

SR1) $\mathcal{S}(0) = \emptyset$,
SR2) $\mathcal{S}(1) = E$,
SR3) $\mathcal{S}(x\wedge y) = \mathcal{S}(x)\cap\mathcal{S}(y)$,
SR4) $\mathcal{S}(x\vee y) = \mathcal{S}(x)\cup\mathcal{S}(y)$,

and also that the distributive lattice L is isomorphic to $L' = \mathcal{S}(L)$.

The following results by A. Monteiro, presented in a Seminar in 1966 [44] were published only in 1996[3] [48].

For each $X \in 2^E$, put $\sim X = \complement\varphi(X)$, where φ indicates the Birula-Rasiowa transformation (see section 2.5). The conditions St6) to St11) from Example 1.8.1 are then valid in 2^E, and furthermore:

SR5) $\mathcal{S}(\sim x) = \sim\mathcal{S}(x)$, for all $x \in L$.

Since $(2^E, \cap, \cup, E)$ is a distributive lattice with bottom element $\emptyset$ and top element E, then by St9), St10) and St11) the system $(2^E, \cap, \cup, \sim, E)$ is a De Morgan algebra and by SR5) it follows that $(L' = \mathcal{S}(L), \cap, \cup, \sim, E)$ is a De Morgan

[3]The students were R. Cignoli, L. Iturrioz and L. Monteiro.

subalgebra of 2^E, and since L' is a lattice isomorphic to L then L and L' are isomorphic De Morgan algebras.

Consider the operator $\boldsymbol{\nabla}$ defined on 2^E by:

St1) $\boldsymbol{\nabla}\emptyset = \emptyset$,

St2) $\boldsymbol{\nabla}\{P\} = \{P, \varphi(P)\}$, for all $P \in 2^E$.

St3) If $\emptyset \subset X \subseteq E$, $\boldsymbol{\nabla}X = \bigcup\limits_{P\in X} \boldsymbol{\nabla}\{P\}$.

As it was pointed out in Example 1.8.1 we know that $(2^E, \boldsymbol{\nabla})$ is a monadic boolean algebra and that $\boldsymbol{\nabla}X = X \cup \varphi(X)$.

If $X = \{P\}$ where $P \in E$, we write $\boldsymbol{\nabla}P$ instead of $\boldsymbol{\nabla}\{P\}$. Notice also that $\boldsymbol{\nabla}\varphi(P) = \boldsymbol{\nabla}P$.

We also proved in Example 1.8.1 that in L' the operator $\boldsymbol{\nabla}$ verifies the axioms L6) and L7). The goal of the next two lemmas is to prove L8) holds too.

Lemma 5.6.1. $\boldsymbol{\nabla}\mathcal{S}(\nabla x) = \mathcal{S}(\nabla x)$, *for all* $x \in L$.

Proof. If $x \in L$, then $\nabla x \in L$ and therefore $\mathcal{S}(\nabla x) \in L' = \mathcal{S}(L)$. Since $(2^E, \boldsymbol{\nabla})$ is a monadic boolean algebra we have that $\mathcal{S}(\nabla x) \subseteq \boldsymbol{\nabla}\mathcal{S}(\nabla x)$. Let $P \in \boldsymbol{\nabla}\mathcal{S}(\nabla x) = \bigcup\limits_{Q\in\mathcal{S}(\nabla x)} \boldsymbol{\nabla}Q = \bigcup\limits_{Q\in\mathcal{S}(\nabla x)} \{Q, \varphi(Q)\}$. Then there exists $Q \in \mathcal{S}(\nabla x)$ such that $P \in \{Q, \varphi(Q)\}$, which is equivalent to sat that there exists (*) $Q \in \mathcal{S}(\nabla x)$ such that: (1) $P = Q$, or (2) $P = \varphi(Q)$. In the former case we have that $P \in \mathcal{S}(\nabla x)$. If (2) holds, assume that $\varphi(Q) = P \notin \mathcal{S}(\nabla x)$, this is $\nabla x \notin P = \varphi(Q) = \complement \sim Q$ then $\nabla x \in \sim Q$, so $\sim \nabla x \in Q$, and by Lemma 2.5.1 (b), $\nabla x \notin Q$, which contradicts (*). $\square$

Lemma 5.6.2. $\boldsymbol{\nabla}\mathcal{S}(x) = \mathcal{S}(\nabla x)$, *for all* $x \in L$.

Proof. Since $x \leq \nabla x$, then $\mathcal{S}(x) \subseteq \mathcal{S}(\nabla x)$, so since the operator $\boldsymbol{\nabla}$ is monotonous and using Lemma 5.6.1: $\boldsymbol{\nabla}(\mathcal{S}(x)) \subseteq \boldsymbol{\nabla}\mathcal{S}(\nabla x) = \mathcal{S}(\nabla x)$.

Let $P \in \mathcal{S}(\nabla x)$, this is $\nabla x \in P$. Since L is in particular a Kleene algebra, we know that $\varphi(P)$ is comparable with P. Assume that $\varphi(P) \subseteq P$. Then we have that $Q = \varphi(P)$ verifies $Q \subseteq P = \varphi(\varphi(P)) = \varphi(Q)$ so since $\nabla x \in \varphi(Q) = P$ it follows by Lemma 2.5.6 that $x \in \varphi(Q) = P$ and therefore $P \in \mathcal{S}(x) \subseteq \boldsymbol{\nabla}\mathcal{S}(x)$, then $P \in \boldsymbol{\nabla}\mathcal{S}(x)$.

If $P \subseteq \varphi(P)$, since $\nabla x \in P$, then by Lemma 2.5.6 we have $x \in \varphi(P)$, this is $\varphi(P) \in \mathcal{S}(x)$, so $\{\varphi(P), P\} = \boldsymbol{\nabla}\varphi(P) \subseteq \boldsymbol{\nabla}\mathcal{S}(x)$, so $P \in \boldsymbol{\nabla}\mathcal{S}(x)$. $\square$

Lemma 5.6.3. L8) $\boldsymbol{\nabla}(X \cap Y) = \boldsymbol{\nabla}X \cap \boldsymbol{\nabla}Y$, *for all* $X, Y \in L' = \mathcal{S}(L)$.

Proof. Let $X, Y \in L'$, so $X = \mathcal{S}(x)$, and $Y = \mathcal{S}(y)$, where $x, y \in L$. Then
$$\boldsymbol{\nabla}(X \cap Y) = \boldsymbol{\nabla}(\mathcal{S}(x) \cap \mathcal{S}(y)) = \boldsymbol{\nabla}(\mathcal{S}(x \wedge y)) = \mathcal{S}(\nabla(x \wedge y)) = \mathcal{S}(\nabla x \wedge \nabla y) = \mathcal{S}(\nabla x) \cap \mathcal{S}(\nabla y) = \boldsymbol{\nabla}\mathcal{S}(x) \cap \boldsymbol{\nabla}\mathcal{S}(y) = \boldsymbol{\nabla}X \cap \boldsymbol{\nabla}Y.$$
$\square$

Since $(L' = \mathcal{S}(L), E, \sim, \cap, \cup)$ is a De Morgan algebra, where $\boldsymbol{\nabla}$ verifies L6), L7) and L8), we deduce that $(L' = \mathcal{S}(L), E, \sim, \boldsymbol{\nabla}, \cap, \cup)$ is a Łukasiewicz algebra and since L and L' are isomorphic De Morgan algebras, by Lemma 5.6.2, it follows that L and L' are isomorphic Łukasiewicz algebras.

We shall prove now some results to be used in section 5.7.

In the monadic boolean algebra $(2^E, \nabla)$ the *universal quantifier* is defined by:

$$\Delta X = \complement\nabla\complement X, \text{ for every } X \subseteq E.$$

Lemma 5.6.4. *If $X \subseteq E$ then:*

a) $\nabla\complement X = \nabla \sim X$.
b) $\Delta\complement X = \Delta \sim X$.

Proof. a) $\nabla \sim X = \nabla\complement\varphi(X) = \complement\varphi(X) \cup \varphi(\complement\varphi(X)) = \varphi(\complement X) \cup \complement\varphi(\varphi(X)) = \varphi(\complement X) \cup \complement X = \nabla\complement X$.
b) $\Delta \sim X = \complement\nabla\complement \sim X =$ (by part a)) $= \complement\nabla \sim\sim X = \complement\nabla X = \Delta\complement X$. $\square$

Corollary 5.6.5. *If $X \in L'$ then $\nabla\complement X \in L'$.*

Proof. If $X \in L'$, then since L' is a De Morgan subalgebra of the De Morgan algebra 2^E, we have that $\sim X \in L'$. Then since L' is a Łukasiewicz algebra $\nabla \sim X \in L'$.
By Lemma 5.6.4, $\nabla\complement X = \nabla \sim X$, and therefore we have that $\nabla\complement X \in L'$. $\square$

Lemma 5.6.6. *If $X \subseteq E$ then $\Delta X = X \cap \varphi(X) = \sim \nabla \sim X$.*

Proof. $\Delta X = \complement\nabla\complement X = \complement(\complement X \cup \varphi(\complement X)) = X \cap \complement\varphi(\complement X) = X \cap \varphi(\complement\complement X) = X \cap \varphi(X)$.
$\Delta X = \complement\nabla\complement X = \complement(\complement X \cup \varphi(\complement X)) = \complement\varphi(\complement X \cup \varphi(\complement X)) = \sim (\complement X \cup \varphi(\complement X)) = \sim (\nabla\complement X) = \sim \nabla \sim X$. $\square$

Corollary 5.6.7. *If $X \subseteq E$ then $\nabla X = \sim \Delta \sim X$.*

Proof. $\sim \Delta \sim X =$ (by Lemma 5.6.6) $= \sim\sim \nabla \sim\sim X = \nabla X$. $\square$

Corollary 5.6.8. *If $X \in L'$ then $\Delta X \in L'$.*

Proof. If $X \in L'$ then $\sim X \in L'$, so $\nabla \sim X \in L'$, and in consequence $\sim \nabla \sim X \in L'$. So by Lemma 5.6.6, $\Delta X \in L'$. $\square$

Corollary 5.6.9. *If $X \in L'$ then $\Delta\complement X \in L'$.*

Proof. If $X \in L'$, then $\sim X \in L'$, so by Corollary 5.6.8, $\Delta \sim X \in L'$, and by Lemma 5.6.4, b) : $\Delta\complement X \in L'$. $\square$

Lemma 5.6.10. *For every $X, Y \in L'$, we have:*

a) $\nabla(X \cup Y) = \nabla X \cup \nabla Y$,
b) $\Delta(X \cup Y) = \Delta X \cup \Delta Y$,
c) $\Delta(X \cap Y) = \Delta X \cap \Delta Y$.

Proof. Since $(2^E, \nabla)$ is a monadic boolean algebra, it is well known that a) and c) hold for every $X, Y \in 2^E$.
Since (L', ∇) is a Łukasiewicz algebra with necessity operator Δ by Lemma 5.6.6, then if $X, Y \in L'$, b) holds. $\square$

Lemma 5.6.11. *If $X, Y \in L'$ then*

$$\begin{aligned} \nabla(X \cap \complement Y) &= (\Delta X \cap \nabla \sim Y) \cup (\nabla X \cap \Delta \sim Y) \\ &= \nabla X \cap \nabla \sim Y \cap \Delta(X \cup \sim Y). \end{aligned}$$

Proof. $\nabla(X \cap \complement Y) =$ (by definition) $= (X \cap \complement Y) \cup \varphi(X \cap \complement Y) = (X \cap \complement Y) \cup (\varphi(X) \cap \varphi(\complement Y)) = (X \cup \varphi(X)) \cap (\complement Y \cup \varphi(\complement Y)) \cap (X \cup \varphi(\complement Y)) \cap (\varphi(X) \cup \complement Y) =$ (by definition) $= \nabla X \cap \nabla \complement Y \cap (X \cup \varphi(\complement Y)) \cap \varphi(X \cup \varphi(\complement Y)) =$ (by Lemmas 5.6.4 a) and 5.6.6) $= \nabla X \cap \nabla \sim Y \cap \Delta(X \cup \varphi(\complement Y)) =$ (by definition of $\sim$) $= \nabla X \cap \nabla \sim Y \cap \Delta(X \cup \sim Y)$.

Since $X, \sim Y \in L^{'}$ then, by Lemma 5.6.10 b), $\Delta(X \cup \sim Y) = \Delta X \cup \Delta \sim Y$, so

$$\nabla(X \cap \complement Y) = \nabla X \cap \nabla \sim Y \cap (\Delta X \cup \Delta \sim Y) = (\Delta X \cap \nabla \sim Y) \cup (\nabla X \cap \Delta \sim Y).$$

□

Corollary 5.6.12. *If $X, Y \in L^{'}$ then $\nabla(X \cap \complement Y) \in L^{'}$.*

Proof. By hypothesis: (1) $X \in L^{'}$, and (2) $Y \in L^{'}$. From (1) we deduce (3) $\nabla X \in L^{'}$. From (2) it follows that (4) $\sim Y \in L^{'}$ and therefore (5) $\nabla \sim Y \in L^{'}$.

From (1) and (4) it follows that (5) $X \cup \sim Y \in L^{'}$, so by Corollary 5.6.8: (6) $\Delta(X \cup \sim Y) \in L^{'}$. From (3), (5) and (6) it follows, by Lemma 5.6.11 that $\nabla(X \cap \complement Y) \in L^{'}$. □

5.7. Universality of the construction $\mathcal{L}$ of Łukasiewicz algebras

Let $(M, \exists)$ be a monadic boolean algebra. We saw in section 1.10 that starting from M, through construction $\mathcal{L}$, a Łukasiewicz algebra $\mathcal{L}(M)$ is obtained (A. Monteiro, [32], L. Monteiro, [70]). We shall prove now the following result by A. Monteiro which was presented in a 1966 Seminar [44] and published only in [48].

Theorem 5.7.1. (L. Monteiro, [65]) *Given a Łukasiewicz algebra L, there exists a monadic boolean algebra M such that $\mathcal{L}(M)$ is isomorphic to L, see* ([45], p. 206).

The following result is well known:

Lemma 5.7.2. *If A is a boolean algebra and R is a sublattice of A, such that $0, 1 \in R$, then the boolean subalgebra of A generated by R is* ([73], p. 74)*:*

$$BS(R) = \left\{ x \in A : x = \bigvee_{i=1}^{n} (y_i \wedge -z_i), \text{ where } y_i, z_i \in R \right\}.$$

Let L be a non trivial Łukasiewicz algebra, and $E = \mathbf{P}(L)$. We saw in section 5.6 that $(2^E, \nabla)$ is a monadic boolean algebra and that L is isomorphic to the Łukasiewicz algebra $L^{'} = \mathcal{S}(L) \subseteq 2^E$, where $\mathcal{S}$ is Stone's transformation. Since $L^{'}$ is a sublattice of the boolean algebra 2^E, and $\emptyset, E \in L^{'}$, then

$$BS(L^{'}) = \left\{ X \in 2^E : X = \bigcup_{i=1}^{n} (Y_i \cap \complement Z_i), \text{ where } Y_i, Z_i \in L^{'} \right\}.$$

We prove now that: $(BS(L^{'}), \nabla)$ *is a monadic subalgebra of the monadic boolean algebra* $(2^E, \nabla)$.

Lemma 5.7.3. *If $X \in BS(L^{'})$ then $\nabla X \in L^{'}$.*

PROOF. If $X \in BS(L^{'})$, then $X = \bigcup_{i=1}^{n}(Y_i \cap \complement Z_i)$ where $Y_i, Z_i \in L^{'}$, so $\boldsymbol{\nabla} X = \boldsymbol{\nabla}\left(\bigcup_{i=1}^{n}(Y_i \cap \complement Z_i)\right) = \bigcup_{i=1}^{n} \boldsymbol{\nabla}(Y_i \cap \complement Z_i)$. Since by Corollary 5.6.12, $\boldsymbol{\nabla}(Y_i \cap \complement Z_i) \in L^{'}$, for every $i, 1 \leq i \leq n$, and $L^{'}$ is a sublattice of 2^E we have that $\boldsymbol{\nabla} X \in L^{'}$. □

Corollary 5.7.4. $(BS(L^{'}), \boldsymbol{\nabla})$ *is a monadic subalgebra of the monadic boolean algebra* $(2^E, \boldsymbol{\nabla})$.

PROOF. Indeed, if $X \in BS(L^{'})$, then by Lemma 5.7.3, $\boldsymbol{\nabla} X \in L^{'} \subseteq BS(L^{'})$. □

Corollary 5.7.5. $\boldsymbol{\nabla}(BS(L^{'})) = \boldsymbol{\Delta}(BS(L^{'})) \subseteq L^{'}$.

PROOF. Since $BS(L^{'})$ is a monadic boolean algebra, it is well known that $\boldsymbol{\nabla}(BS(L^{'})) = \boldsymbol{\Delta}(BS(L^{'}))$, and from Lemma 5.7.3, $\boldsymbol{\nabla}(BS(L^{'})) \subseteq L^{'}$. □

If $X, Y \in 2^E$ then (see section 5.6), if we define $X \sqcup Y = \boldsymbol{\Delta} X \cup Y \cup (X \cap \boldsymbol{\Delta} \complement Y)$ and $X \sqcap Y = \boldsymbol{\nabla} X \cap Y \cap (X \cup \boldsymbol{\nabla} \complement Y)$, in [70] we proved that:

Lemma 5.7.6. *If* $X, Y \in 2^E$, *then*

a) $\boldsymbol{\nabla}(X \sqcup Y) = \boldsymbol{\nabla} X \cup \boldsymbol{\nabla} Y$.
b) $\boldsymbol{\nabla}(X \sqcap Y) = \boldsymbol{\nabla} X \cap \boldsymbol{\nabla} Y$.
c) $\boldsymbol{\Delta}(X \sqcup Y) = \boldsymbol{\Delta} X \cup \boldsymbol{\Delta} Y$.
d) $\boldsymbol{\Delta}(X \sqcap Y) = \boldsymbol{\Delta} X \cap \boldsymbol{\Delta} Y$.

Lemma 5.7.7. *If* $X, Y \in L^{'}$ *then* $X \sqcup Y \in L^{'}$.

PROOF. By Corollary 5.6.8, $\boldsymbol{\Delta} X \in L^{'}$, and by Corollary 5.6.9, $\boldsymbol{\Delta} \complement Y \in L^{'}$, so since $X \sqcup Y = \boldsymbol{\Delta} X \cup Y \cup (X \cap \boldsymbol{\Delta} \complement Y)$ and $L^{'}$ is a Łukasiewicz algebra, we have that $\boldsymbol{\Delta} X \cup Y \cup (X \cap \boldsymbol{\Delta} \complement Y) \in L^{'}$. □

Lemma 5.7.8. *If* $X, Y \in L^{'}$ *then* $\boldsymbol{\Delta}(X \cap \complement Y) = \boldsymbol{\Delta}(X \cap \sim Y) = \boldsymbol{\Delta} X \cap \boldsymbol{\Delta} \sim Y$.

PROOF. $\boldsymbol{\Delta}(X \cap \complement Y) = \boldsymbol{\Delta} X \cap \boldsymbol{\Delta} \complement Y =$ (by Lemma 5.6.4 b)) $=$
$\boldsymbol{\Delta} X \cap \boldsymbol{\Delta} \sim Y = \boldsymbol{\Delta}(X \cap \sim Y)$. □

Consider the congruence relation "$\equiv$" defined on $BS(L^{'})$, (see section 1.10) as follows:

$$X, Y \in BS(L^{'}), X \equiv Y \quad \text{if and only if} \quad \boldsymbol{\nabla} X = \boldsymbol{\nabla} Y \quad \text{and} \quad \boldsymbol{\Delta} X = \boldsymbol{\Delta} Y.$$

If $X \in BS(L^{'})$ we denote $C(X) = \{Y \in BS(L^{'}) : Y \equiv X\}$.

Let $\mathcal{L}(BS(L^{'})) = BS(L^{'})/\equiv$, then A. Monteiro, [32], and L. Monteiro, [70], proved that:

Lemma 5.7.9. $(\mathcal{L}(BS(L^{'})), C(E), \sim, \nabla, \sqcap, \sqcup)$ *is a Łukasiewicz algebra, if the operations are defined by* $\sim C(X) = C(\complement X)$, $\nabla C(X) = C(\boldsymbol{\nabla} X), C(X) \sqcap C(Y) = C(X \sqcap Y)$ *and* $C(X) \sqcup C(Y) = C(X \sqcup Y)$.

We will prove that $\mathcal{L}(BS(L^{'}))$ and L are isomorphic Łukasiewicz algebras.

Lemma 5.7.10. *If* $X, Y \in L^{'}$ *and* $X \equiv Y$ *then* $X = Y$.

Proof. From $X, Y \in L'$, it follows as (L', ∇) is a Łukasiewicz algebra, that $\nabla X, \nabla Y \in L'$ and by Corollary 5.6.8 $\Delta X, \Delta Y \in L'$. By hypothesis $\nabla X = \nabla Y$, $\Delta X = \Delta Y$, and since ∇ and Δ are the possibility and necessity operators of the Łukasiewicz algebra L', then Moisil's determination principle we have that $X = Y$. □

Lemma 5.7.11. *If $X, Y \in L'$ then there exists a unique $Z \in L'$ such that $X \cap \complement Y \equiv Z$.*

Proof. Let $Z = (\Delta X \cap \sim Y) \cup (X \cap \Delta \sim Y)$, then it is clear that $Z \in L'$.
(1) $\nabla Z = \nabla(\Delta X \cap \sim Y) \cup \nabla(X \cap \Delta \sim Y) = (\Delta X \cap \nabla \sim Y) \cup (\nabla X \cap \Delta \sim Y) =$ (by Lemma 5.6.11 b)) $= \nabla(X \cap \complement Y)$, and (2) $\Delta Z =$ (by Lemma 5.6.10 b)) $= \Delta(\Delta X \cap \sim Y) \cup \Delta(X \cap \Delta \sim Y) =$ (Lemma 5.6.10 c)) $= (\Delta X \cap \Delta \sim Y) \cup (\Delta X \cap \Delta \sim Y) = \Delta X \cap \Delta \sim Y =$ (by Lemma 5.7.8) $= \Delta(X \cap \complement Y)$, so $Z \equiv X \cap \complement Y$.
From (1) and (2) we deduce that $X \cap \complement Y \equiv Z$ and by Lemma 5.7.10, Z is unique. □

Lemma 5.7.12. *If $A, B \in BS(L')$ then $A \cup B \equiv A \sqcup B \sqcup \Delta(A \cup B)$.*

Proof. By Lemma 5.7.6:
$\nabla(A \sqcup B \sqcup \Delta(A \cup B)) = \nabla A \cup \nabla B \cup \nabla\Delta(A \cup B) = \nabla A \cup \nabla B \cup \Delta(A \cup B) = \nabla(A \cup B) \cup \Delta(A \cup B) = \nabla(A \cup B)$.
$\Delta(A \sqcup B \sqcup \Delta(A \cup B)) = \Delta A \cup \Delta B \cup \Delta\Delta(A \cup B) = \Delta A \cup \Delta B \cup \Delta(A \cup B) = \Delta(A \cup B)$. □

Corollary 5.7.13. *If $A, B \in BS(L')$, $X, Y \in L'$ and $A \equiv X$, $B \equiv Y$, then $A \cup B \equiv Z$, where $Z \in L'$.*

Proof. We saw in Lemma 5.7.12, that $A \cup B \equiv A \sqcup B \sqcup \Delta(A \cup B)$. By the hypothesis $A \equiv X$, $B \equiv Y$, we have that $A \sqcup B \equiv X \sqcup Y$ so $A \sqcup B \sqcup \Delta(A \cup B) \equiv X \sqcup Y \sqcup \Delta(A \cup B)$. Finally, observe that since $A \cup B \in BS(L')$ then by Corollary 5.7.5 $\Delta(A \cup B) \in L'$, so by Lemma 5.7.7, $Z = X \sqcup Y \sqcup \Delta(A \cup B) \in L'$. Then $A \cup B \equiv Z$, with $Z \in L'$. □

Lemma 5.7.14. *If $A \in BS(L')$, there exists $X \in L'$ such that $A \equiv X$.*

Proof. Let $A \in BS(L')$, then $A = \bigcup_{i=1}^{n} X_i$ where $X_i = Y_i \cap \complement Z_i$, and $Y_i, Z_i \in L'$, for $1 \le i \le n$. By Lemma 5.7.11, $A_i \equiv W_i$, where $W_i \in L'$, for $1 \le i \le n$.
If $n = 1$, then the lemma holds trivially. Assume that $n \ge 2$. By Lemma 5.7.11, $X_1 \equiv W_1$ and $X_2 \equiv W_2$ with $W_1, W_2 \in L'$, so by Corollary 5.7.13, we have that: (1) $X_1 \cup X_2 \equiv H_1$, where $H_1 \in L'$.
Since (2) $X_3 \equiv W_3$, with $W_3 \in L'$ then from (1) and (2) it follows by Corollary 5.7.13 that: $X_1 \cup X_2 \cup X_3 \equiv H_2$, where $H_2 \in L'$. Applying this reasoning $n - 1$ times we have that $\bigcup_{i=1}^{n} X_i \equiv H_{n-1}$, where $H_{n-1} \in L'$, which ends the proof. □

Lemma 5.7.15. *The transformation H from L to $\mathcal{L}(BS(L'))$, defined by $H(x) = C(\mathcal{S}(x))$, verifies:*

a) *H is biunivocal,*
b) *H is surjective.*

PROOF. a) If $H(x) = H(y)$, this is $C(\mathcal{S}(x)) = C(\mathcal{S}(y))$, then $\mathcal{S}(x) \equiv \mathcal{S}(y)$, and since $\mathcal{S}(x), \mathcal{S}(y) \in \mathcal{S}(L) = L'$, then by Lemma 5.7.10, $\mathcal{S}(x) = \mathcal{S}(y)$, and since $\mathcal{S}$ is biunivocal it follows that $x = y$.

b) Given $C(A) \in \mathcal{L}(BS(L'))$, where $A \in BS(L')$, by Lemma 5.7.14, we know that there exists $X \in L' = \mathcal{S}(L)$ such that $X \equiv A$, then since $X = \mathcal{S}(x)$, where $x \in L$, we have that $H(x) = C(\mathcal{S}(x)) = C(X) = C(A)$. □

Lemma 5.7.16. *The transformation H verifies:*

a) $H(x \wedge y) = H(x) \sqcap H(y)$.
b) $H(x \vee y) = H(x) \sqcup H(y)$.
c) $H(\sim x) = \sim H(x)$.
d) $H(\nabla x) = \nabla H(x)$.

PROOF. a) (1) $\boldsymbol{\nabla}(\mathcal{S}(x) \sqcap \mathcal{S}(y)) =$ (Lemma 5.7.6 b)) $= \boldsymbol{\nabla}\mathcal{S}(x) \cap \boldsymbol{\nabla}\mathcal{S}(y) =$ (by Lemma 5.6.3) $= \boldsymbol{\nabla}(\mathcal{S}(x) \cap \mathcal{S}(y))$.

(2) $\boldsymbol{\Delta}(\mathcal{S}(x) \sqcap \mathcal{S}(y)) =$ (by Lemma 5.7.6 d)) $= \boldsymbol{\Delta}\mathcal{S}(x) \cap \boldsymbol{\Delta}\mathcal{S}(y) =$ (by Lemma 5.6.10 c)) $= \boldsymbol{\Delta}(\mathcal{S}(x) \cap \mathcal{S}(y))$.

From (1) and (2) it follows that $\mathcal{S}(x) \sqcap \mathcal{S}(y) \equiv \mathcal{S}(x) \cap \mathcal{S}(y)$, so

$$\begin{aligned} H(x \wedge y) &= C(\mathcal{S}(x \wedge y)) = C(\mathcal{S}(x) \cap \mathcal{S}(y)) = C(\mathcal{S}(x) \sqcap \mathcal{S}(y)) \\ &= C(\mathcal{S}(x)) \sqcap C(\mathcal{S}(y)) = H(x) \sqcap H(y). \end{aligned}$$

b) (3) $\boldsymbol{\nabla}(\mathcal{S}(x) \sqcup \mathcal{S}(y)) =$ (by Lemma 5.7.6 a))$=$
$\boldsymbol{\nabla}\mathcal{S}(x) \cup \boldsymbol{\nabla}\mathcal{S}(y) =$ (by Lemma 5.6.10 a)) $= \boldsymbol{\nabla}(\mathcal{S}(x) \cup \mathcal{S}(y))$.
(4) $\boldsymbol{\Delta}(\mathcal{S}(x) \sqcup \mathcal{S}(y)) =$ (by Lemma 5.7.6 c)) $=$
$\boldsymbol{\Delta}\mathcal{S}(x) \cup \boldsymbol{\Delta}\mathcal{S}(y) =$ (by Lemma 5.6.10 b)) $= \boldsymbol{\Delta}(\mathcal{S}(x) \cup \mathcal{S}(y))$.
From (3) and (4) it follows that $\mathcal{S}(x) \sqcup \mathcal{S}(y) \equiv \mathcal{S}(x) \cup \mathcal{S}(y)$, so
$H(x \vee y) = C(\mathcal{S}(x \vee y)) = C(\mathcal{S}(x) \cup \mathcal{S}(y)) = C(\mathcal{S}(x) \sqcup \mathcal{S}(y)) =$
$C(\mathcal{S}(x)) \sqcup C(\mathcal{S}(y)) = H(x) \sqcup H(y)$.

c) $\boldsymbol{\nabla}\mathcal{S}(\sim x) =$ (by SR5)) $= \boldsymbol{\nabla} \sim \mathcal{S}(x) =$ (by Lemma 5.6.4 a)) $= \boldsymbol{\nabla}\complement\mathcal{S}(x)$, and $\boldsymbol{\Delta}\mathcal{S}(\sim x) =$ (by SR5)) $= \boldsymbol{\Delta} \sim \mathcal{S}(x) =$ (by Lemma 5.6.4 b)) $= \boldsymbol{\Delta}\complement\mathcal{S}(x)$. Then $\mathcal{S}(\sim x) \equiv \complement\mathcal{S}(x)$ and therefore $H(\sim x) = C(\mathcal{S}(\sim x)) = C(\complement\mathcal{S}(x)) = \sim C(\mathcal{S}(x)) = \sim H(x)$.

d) $H(\nabla x) = C(\mathcal{S}(\nabla x)) =$ (by Lemma 5.6.2)$= C(\boldsymbol{\nabla}\mathcal{S}(x)) =$ (by Lemma 5.7.9) $= \nabla C(\mathcal{S}(x)) = \nabla H(x)$.

□

By Corollary 5.7.4, $M = BS(L')$ is a monadic boolean algebra and by Lemmas 5.7.15 and 5.7.16 we have that the Łukasiewicz algebra L is isomorphic to $\mathcal{L}(M)$, which proves Theorem 5.7.1.

As we said before, this result was proved in a different way by L. Monteiro in [65].

5.8. A construction of the Łukasiewicz algebra with n free generators

Let M_n be the monadic boolean algebra with n free generators. It is well known, [22], [64], that M_n is a boolean algebra with $2^n \cdot 2^{(2^n-1)}$ atoms and that $K(M_n)$ is a boolean algebra with $2^{2^n} - 1$ atoms. Furthermore, (*) the partition of the atoms of M associated with $K(M_n)$ has $\binom{2^n}{i}$ classes with i atoms of M, $1 \leq i \leq 2^n$.

Let $L = \mathcal{L}(M_n)$, then by Remark 1.10.4, the boolean algebras $B(L)$ and $K(M_n)$ are isomorphic.

We know that the Łukasiewicz algebra L_n with n free generators has $2^{2^n} \cdot 3^{3^n - 2^n}$ elements and that $B(L_n)$ has 3^n atoms, so if $L \cong L_n$ we would have that $B(L) \cong B(L_n)$ so $2^{2^n} - 1 = 3^n$, and this holds only for $n = 1$. Then if $n > 1$ we have that the Łukasiewicz algebras $\mathcal{L}(M_n)$ and L_n are not isomorphic.

Let us see how we can determine L_n from $\mathcal{L}(M_n)$ when $n > 1$. By (*) and Remark 3.3.10 we know that

$$\mathcal{L}(M_n) \cong \mathbf{B}^{2^n} \times \mathbf{T}^{2^{2n} - 1 - 2^n}.$$

If

$$b = (\underbrace{1, 1, \ldots, 1}_{2^n} \underbrace{\underbrace{1, 1, \ldots, 1}_{3^n - 2^n}, 0, 0, \ldots, 0}_{2^{2^n} - 1 - 2^n}),$$

then by Remark 5.3.5

$$\mathcal{L}(M_n)/[b) \cong (b] \cong \mathbf{B}^{2^n} \times \mathbf{T}^{3^n - 2^n}.$$

5.9. Determinant system of a Łukasiewicz algebra

The proof of the following results about De Morgan and Kleene algebras can be found, for instance, in [51].

If M is a De Morgan algebra, and $X \subseteq M$, let

$$\sim X = \{x \in M : \sim x \in X\}.$$

If $P \in \mathbf{P}(M)$ then we know that $\varphi(P) = \complement \sim P$, is the *Birula-Rasiowa transformation* [6], [7], from $\mathbf{P}(M)$ to $\mathbf{P}(M)$, which verifies (1) $\varphi(\varphi(P)) = P$, and (2) If $P, Q \in \mathbf{P}(M)$ then $P \subseteq Q$ if and only if $\varphi(Q) \subseteq \varphi(P)$.

Lemma 5.9.1. *If $P \in \mathbf{P}(M)$ then:*
a) $\sim x \in P \iff x \notin \varphi(P)$, b) $\sim x \notin P \iff x \in \varphi(P)$.

If K is a Kleene algebra, then

$$\text{If } P \in \mathbf{P}(K) \text{ then} : P \subseteq \varphi(P) \text{ or } \varphi(P) \subseteq P.$$

Let us denote with $\mathbf{P}_1(K)$ the set of all the filters $P \in \mathbf{P}(K)$ such that $P \subseteq \varphi(P)$. Then $\mathbf{P}_1(K) \subseteq \mathbf{P}(K)$.

Since every Łukasiewicz algebra L is a Kleene algebra, then if $b \in B(L)$, the boolean complement of b, which we denote by $-b$, is equal to $\sim b$, this is $-b = \sim b$. The original proof by A. Monteiro was reproduced in [11]. A simpler proof was obtained, as announced in [45], by L. Monteiro, [59].

If R is a non trivial finite distributive lattice, we represent by $\Pi = \Pi(R)$ the poset of the prime elements of R.

Theorem 5.9.2. *If R y R' are non trivial finite distributive lattices such that $\Pi = \Pi(R)$, $\Pi' = \Pi(R')$ are isomorphic posets then R and R' are isomorphic lattices.*

Proof. Let $f : \Pi \to \Pi'$ be an order isomorphism and put by definition:

$$H(x) = \begin{cases} 0, & \text{if } x = 0 \\ \bigvee\{f(p) : p \in \Pi, p \leq x\}, & \text{if } x \neq 0 \end{cases}$$

then it is easy to check that $H : R \to R'$ is a lattice isomorphism and $H(p) = f(p)$ for all $p \in \Pi$. $\square$

Corollary 5.9.3. *Every non trivial finite distributive lattice R, is determined up to isomorphisms, by the set $\Pi = \Pi(R)$ of its prime elements.*

Let X be a poset, a subset Y of X is said to be a **lower section** of X, if $Y = \emptyset$ or if it verifies "*If $y \in Y$ and $x \leq y$ then $x \in Y$*". The subsets $(x] = \{y \in X : y \leq x\}$ are lower sections of X.

We represent by $\mathbf{S}(X)$ the set of all the lower sections of X.

Theorem 5.9.4. (G. Birkhoff.) *If X is a finite poset, there exists a finite distributive lattice R such that X and $\Pi(R)$ are isomorphics posets.*

Proof. It is well known that $(\mathbf{S}(X), \cap, \cup, \emptyset, X)$ is a bounded distributive lattice. Since X is finite then the distributive lattice $\mathbf{S}(X)$ is finite as well.

It is easy to prove that $\Pi(\mathbf{S}(X)) = \{(x] : x \in X\}$, and if we put $\beta(x) = (x]$, for all $x \in X$ then β is an order isomorphism from X to $\Pi(\mathbf{S}(X))$. $\square$

Definition 5.9.5. *Let X be a finite poset. We say that $x \in X$ is linked to $y \in X$, if there exists a finite sequence of elements of X, $a_1, a_2, \cdots, a_n$ such that $a_1 = x, a_n = y$, and a_i is comparable to a_{i+1}, $1 \leq i \leq n-1$. To denote that a is comparable to b, we write $a \parallel b$, and to indicate that x is linked to y we write $x \approx y$.*

If $x \neq y$, with $x, y \in X$, we can assume that the elements $a_i, 1 \leq i \leq n$, verify $a_i \neq a_j, i \neq j, 1 \leq i, j \leq n$.

It is well known that the relation $\approx$ is an equivalence relation defined over X. Let $K(x) = \{y \in X : y \approx x\}$ be the equivalence class containing element $x \in X$. Observe that if $y \notin K(x)$, then y is incomparable with every element in $K(x)$.

Let $K(x_1), K(x_2), \ldots, K(x_n)$ be the equivalence classes. It is well known that the subsets $K(x_i), 1 \leq i \leq n$, of X are connected posets, which we can denominate **connected components** of X, and that the poset X is the *cardinal sum* of the posets $K(x_i), 1 \leq i \leq n$, this is:

$$X = \sum_{i=1}^{n} K(x_i).$$

Notice also that the sets $K(x_i)$ are *pairwise disjoint*, and that each element (*) $a \in K(x_i)$ is incomparable with every $b \in K(x_j)$ if $i \neq j$.

We can assume that the elements $x_i, 1 \leq i \leq n$ are maximal elements of the poset X. Indeed, each $K(x_i)$ is a finite poset, so there exist $m \in K(x_i)$, m a maximal element of $K(x_i)$. Let us see that m is also a maximal element of X. If $x \in X$ is such that $m \leq x$, since $m \in K(x_i)$, then by (*) m is incomparable with every element $y \in K(x_j), j \neq i$, so $x \in K(x_i)$ and since m is a maximal element of $K(x_i)$ we have that $x_i = m$. This shows that m is a maximal element of X. Then we can write:

$$X = \sum_{i=1}^{n} K(x_i),$$

where $x_i, 1 \leq i \leq n$ are maximal elements of the poset X.

Lemma 5.9.6. *If R is a non trivial finite distributive lattice, where the poset $\Pi = \Pi(R)$ of its prime elements is isomorphic to the poset $X = X_1 + X_2$ and if $R_i,\ i = 1, 2$ is a distributive lattice whose set of prime elements is isomorphic to $X_i,\ i = 1, 2$ then R is isomorphic to $R_1 \times R_2$.*

If R is a finite distributive lattice, then we know that $P \in \mathbf{P}(R)$ if and only if $P = F(p) = \{x \in R : p \leq x\}$, where $p \in \Pi = \Pi(R)$.

Let A be a De Morgan algebra, so in this case the *Birula-Rasiowa* transformation φ from $\mathbf{P}(A)$ to $\mathbf{P}(A)$, induces a transformation ψ from $\Pi = \Pi(A)$ to Π as follows:

$$\psi(p) = q \ \text{ if and only if } \ \varphi(F(p)) = F(q). \tag{5.9.1}$$

The transformation ψ has the following properties:

Inv1) $\psi(\psi(p)) = p$, for all $p \in \Pi$,
Inv2) $p \leq q$ if and only if $\psi(q) \leq \psi(p)$, where $p, q \in \Pi$.

This means that ψ is an anti-isomorphism of the poset Π onto Π of period 2. We say that the pair $(\Pi(A), \psi)$ is the *determinant system* of the algebra A.

Definition 5.9.7. *A* Birula–Rasiowa space *is a pair (X, α) formed by a poset X and a transformation α from X to X such that:*

Inv1) *$\alpha(\alpha(x)) = x$, for every $x \in X$,*
Inv2) *$x \leq y$ if and only if $\alpha(y) \leq \alpha(x)$, where $x, y \in X$.*

It is clear that α is a bijective mapping from X to X, and that the determinant system of a De Morgan algebra is a Birula–Rasiowa space.

Definition 5.9.8. *Two Birula–Rasiowa spaces $(X, \alpha), (X', \alpha')$ are said to be isomorphic if there exists an order isomorphism f from X onto X' such that $f(\alpha(x)) = \alpha'(f(x))$ for all $x \in X$.*

Theorem 5.9.9. (A. Monteiro, [35], [38], [43], [51]). *If $(A, \sim)$ is a non trivial finite De Morgan algebra, and if $(\Pi = \Pi(A), \psi)$ is its determinant system then*

$$\sim x = \bigvee \{p \in \Pi : \psi(p) \not\leq x\}.$$

If we put $\Pi_x = \{p \in \Pi : p \leq x\}$ then L. Monteiro proved in [51] that:

Lemma 5.9.10. *If $(A, \sim)$ is a non trivial finite De Morgan algebra, then*

$$\sim x = \bigvee \{p \in \Pi : p \in \psi(\Pi \setminus \Pi_x)\} =$$

$$\bigvee \{p : p \in \Pi \setminus \psi(\Pi_x)\} = \bigvee \{\psi(q) : q \in \Pi \setminus \Pi_x\}.$$

Theorem 5.9.11. *If $(A, \sim)$ and $(A', \sim')$ are non trivial finite De Morgan algebras such that their determinant systems $(\Pi = \Pi(A), \psi), (\Pi' = \Pi(A'), \psi')$ are isomorphic Birula–Rasiowa spaces, then the De Morgan algebras $(A, \sim)$ and $(A', \sim')$ are isomorphic as well.*

Corollary 5.9.12. *Every non trivial finite De Morgan $(A, \sim)$ is determined, up to isomorphisms, by its determinant system $(\Pi = \Pi(A), \psi)$.*

The previous result was announced in 1960, [35] and its proof presented in 1962, [38], [43], [51]. According to his own words, [45] A. Monteiro's proof from 1960 was too complicated.

Theorem 5.9.13. *If (X, α) is a finite Birula–Rasiowa space, there exists a finite De Morgan algebra $(A, \sim)$, such that its determinant system $(\Pi = \Pi(A), \psi)$ is a Birula–Rasiowa space isomorphic to (X, α).* [51]

Proof. We know that $\mathbf{S}(X)$ is a distributive lattice such that $\Pi(\mathbf{S}(X))$ is a poset isomorphic to X. The transformation $\alpha : X \to X$ induces a transformation $\psi : \Pi(\mathbf{S}(X)) \to \Pi(\mathbf{S}(X))$ of the following manner: $\psi((x]) = (\alpha(x)]$, for all $x \in X$.

For each lower section Y of X let us put (see Lemma 5.9.10):

$$\sim Y = \bigcup \{\psi((x]) : (x] \in \Pi(\mathbf{S}(X)) \setminus \Pi_Y\}.$$

Then it is easy to prove that $\sim Y = X \setminus \alpha(Y)$. We prove now that $\sim Y$ is a lower section of X. If $\sim Y = X \setminus \alpha(Y) = \emptyset$, then $\sim Y$ is a lower section. If $\sim Y = X \setminus \alpha(Y) \neq \emptyset$, let (1) $p \in X \setminus \alpha(Y)$, and $x \in X$ be such that $x \leq p$. Then (2) $\alpha(p) \leq \alpha(x)$. If $x \notin X \setminus \alpha(Y)$, then $x \in \alpha(Y)$ this is $x = \alpha(y')$, where $y' \in Y$, so (3) $\alpha(x) = y' \in Y$. Since $Y \in \mathbf{S}(X)$ then from (2) and (3) we deduce that $\alpha(p) \in Y$ so $p = \alpha(\alpha(p)) \in \alpha(Y)$, which contradicts (1). Furthermore:

- $\sim X = X \setminus \alpha(X) = X \setminus X = \emptyset$,
- $\sim (\sim Y) = X \setminus \alpha(\sim Y) = X \setminus \alpha(X \setminus \alpha(Y))$. Then, as α is a period 2 bijection, we have that: $\sim (\sim Y) = X \setminus (\alpha(X) \setminus Y) = X \setminus (X \setminus Y) = X \cap Y = Y$.
- Since α is biunivocal then $\sim (Y \cap Z) = X \setminus \alpha(Y \cap Z) = X \setminus (\alpha(Y) \cap \alpha(Z)) = (X \setminus \alpha(Y)) \cup (X \setminus \alpha(Z)) = \sim Y \cup \sim Z$.

Thus $(\mathbf{S}(X), \sim)$ is a De Morgan algebra. Let us see that the Birula–Rasiowa spaces (X, α) and $(\Pi(\mathbf{S}(X)), \psi)$ are isomorphic. We already know that the transformation $\beta : X \to \Pi(\mathbf{S}(X))$, defined by $\beta(x) = (x]$, $x \in X$ is an order isomorphism. By the definition of ψ, we have that: $\beta(\alpha(x)) = (\alpha(x)] = \psi((x]) = \psi(\beta(x))$, which concludes the proof. $\square$

Let A be a non trivial finite De Morgan algebra, $\Pi = \Pi(A)$ the set of its prime elements and $\Pi = \sum_{i=1}^{n} X_i$, where X_i, $1 \leq i \leq n$, are the connected components of Π. In general, if $p \in X_i$ we cannot claim that $\alpha(p) \in X_i$, but for finite Kleene algebras we have that:

$$\psi(X_i) = X_i, 1 \leq i \leq n,$$

since $\alpha(p) \parallel p$ for all $p \in \Pi$.

Let A be a non trivial finite De Morgan algebra, $(\Pi = \Pi(A), \psi)$ its determinant system and $\Pi = \sum\limits_{i=1}^{n} K(p_i)$. It is clear that if $p, q \in \Pi, p \approx q$, then $\psi(p) \approx \psi(q)$, so:

- If $\psi(p) \in K(p)$ then $\psi(K(p)) = K(p)$.
- If $q = \psi(p) \notin K(p)$ then $\psi(K(p)) = K(q)$, and $\psi(K(p) + K(q)) = K(p) + K(q)$.

We say that in the first case $(K(p), \psi)$ and, in the second case, $(K(p)+K(q), \psi)$ are ψ-connected components of (Π, ψ), and that the respective Birula–Rasiowa spaces $(K(p), \psi)$ and $(K(p) + K(q), \psi)$ are undecomposable.

In the case in which A is a finite Kleene algebra, every connected component of $\Pi(A)$ is a ψ-connected component of $\Pi(A)$.

Lemma 5.9.14. *If A is a non trivial De Morgan algebra, and its determinant system $(\Pi = \Pi(A), \psi)$ is isomorphic to the Birula–Rasiowa space (X, α) where $X = X_1 + X_2$, $\alpha(X_1) = X_1$, $\alpha(X_2) = X_2$ and if A_i ; $i = 1, 2$ is a De Morgan algebra whose determinant system $(\Pi(A_i), \psi_i)$ is isomorphic to $(X_i, \alpha|_{X_i})$; $i = 1, 2$, then A is isomorphic to $A_1 \times A_2$.*

Definition 5.9.15. *A* Kleene space *is a Birula–Rasiowa space (X, α) such that every $x \in X$ is comparable with $\alpha(x)$.*

Lemma 5.9.16. *For a finite De Morgan algebra A to be a Kleene algebra it is necessary and sufficient that its determinant system $(\Pi = \Pi(A), \psi)$ is a Kleene space.*

Proof. It is clear that the condition is necessary. Let us see that it is sufficient. If $y \wedge \sim y = 0$ then the Kleene condition holds. Assume that $y \wedge \sim y \neq 0$. To prove that the Kleene condition holds it is enough to check that:

$$\{p \in \Pi : p \leq y \wedge \sim y\} \subseteq \{q \in \Pi : q \leq z \vee \sim z\}.$$

Let $p \in \Pi$ be such that $(1) : p \leq y \wedge \sim y$. By hypothesis (2) $\psi(p) \leq p$, or (3) $p \leq \psi(p)$. If (2) holds then by (1) we have : $\psi(p) \leq y \wedge \sim y$, then in particular $\psi(p) \leq \sim y = \bigvee\{\psi(q) : q \in \Pi \setminus \Pi_y\}$, from where it follows, since $\psi(p) \in \Pi$, that $\psi(p) \leq \psi(q_0)$, for some $q_0 \in \Pi \setminus \Pi_y$. Thus $q_0 \leq p$ and by (1) we have that $q_0 \leq y \wedge \sim y \leq y$, so $q_0 \in \Pi_y$, contradiction. Therefore condition (3) must hold. If $\psi(p) \not\leq z$ then $\psi(p) \in \Pi \setminus \Pi_z$, so $(4) : \psi(p) \leq \sim z \leq z \vee \sim z$. From (3) and (4) we have $p \leq z \vee \sim z$. If $\psi(p) \leq z$, then $p \leq z \leq z \vee \sim z$. □

Theorem 5.9.17. *If $(A, \sim)$ and $(A', \sim')$ are non trivial finite Kleene algebras such that their determinant systems $(\Pi = \Pi(A), \psi), (\Pi' = \Pi(A'), \psi')$ are isomorphic Kleene spaces, then the Kleene algebras $(A, \sim)$ and $(A', \sim')$ are isomorphic as well.*

Corollary 5.9.18. *Every non trivial finite Kleene algebra, $(A, \sim)$ is determined, up to isomorphisms, by its determinant system $(\Pi = \Pi(A), \psi)$.*

Theorem 5.9.19. *If (X, α) is a finite Kleene space, there exists a finite Kleene algebra $(A, \sim)$ such that its determinant system $(\Pi = \Pi(A), \psi)$ is a Kleene space isomorphic to (X, α).*

Remark 5.9.20. a) *If* $X = \{x\}$, *then* $\alpha(x) = x$, *and* $A = \{0, 1\}$, *where* $0 < 1$, $\sim 0 = 1$ *and* $\sim 1 = 0$.

b) *If* $X = \{x, y\}$, *where* $x < y$, $\alpha(x) = y$ *y* $\alpha(y) = x$, *then* $A = \{0, c, 1\}$, *where* $0 < c < 1$, $\sim 0 = 1, \sim 1 = 0$, *and* $\sim c = c$.

Let L be a finite Łukasiewicz algebra. Then $(\mathbf{P}(L), \subseteq)$ is a finite poset and

$$\mathbf{P}(L) = \sum_{i=1}^{n} K(P_i),$$

where $P_i, 1 \leq i \leq n$ are the maximal elements of the poset $(\mathbf{P}(L), \subseteq)$. We shall prove that $K(P) = \{P, \varphi(P)\}$, for every maximal element P of $\mathbf{P}(L)$, this is $P \in \mathbf{U}(L)$. Since $U \parallel \varphi(U)$, for all $U \in \mathbf{U}(L)$, then $\{U, \varphi(U)\} \subseteq K(U)$.

It is clear that $\mathcal{V} = \{U \in \mathbf{U}(L) : U \in \mathbf{p}(L)\}$, $\mathcal{W} = \{U \in \mathbf{U}(L) : U \notin \mathbf{p}(L)\}$, is a bipartition of the set $\mathbf{U}(L)$.

If $U \in \mathcal{V}$, (1) $U \in \mathbf{U}(L)$ and (2) $U \in \mathbf{p}(L)$. Since L is a Kleene algebra then (3) $U \subseteq \varphi(U)$ or (4) $\varphi(U) \subseteq U$. From (1) and (3) or from (2) and (4) we have: $U = \varphi(U)$. Let $P \in K(U)$, where $U \in \mathcal{V}$, then there exists a sequence $P_1, P_2, \ldots, P_n$ of elements of $\mathbf{P}(L)$ such that: $P_1 = U$, $P_n = P$, and $P_i \parallel P_{i+1}$, $P_i \neq P_{i+1} 1 \leq i \leq n-1$. By hypothesis $U \parallel P_2$, so if $U \subseteq P_2$, since U is an ultrafilter we have: $P_2 = U$. If $P_2 \subseteq U$, since U is a minimal prime filter minimal we have: $P_2 = U$. Then $P = U$, for all $P \in K(U)$, so $K(U) = \{U\}$.

If $U \in \mathcal{W}$, (1) $U \in \mathbf{U}(L)$ and (2) $U \notin \mathbf{p}(L)$, then there exists (3) $M \in \mathbf{p}(L)$ such that (4) $M \subseteq U$, then $M \in K(U)$. By (2) we have that (5) $M \notin \mathbf{U}(L)$. By Corollary 2.5.16 we know that (6) $\varphi(M)$ is the unique proper filter containing M as a proper part. From (5) and (6) we deduce that $U = \varphi(M)$, this is $M = \varphi(U)$.

Let us see that in this case:

Lemma 5.9.21. *Let* U *be such that* $U \in \mathbf{U}(L)$, $U \notin \mathbf{p}(L)$ *and* $M = \varphi(U) \subseteq U$. *Then:*

a) *If* $Q \parallel U$ *and* $Q \neq U$ *then* $Q = M$.
b) *If* $Q \parallel M$ *and* $M \neq Q$ *then* $Q = U$.
c) *If* $U \in \mathcal{W}$ *then* $K(U) = \{U, \varphi(U)\}$.

Proof.

a) Since $Q \parallel U$, $Q \neq U$ and U is an ultrafilter of L then $Q \subseteq U$.

If $Q \notin \mathbf{p}(L)$, then there exists $P_1 \in \mathbf{p}(L)$ such that $P_1 \subseteq Q \subseteq U$, which is impossible by Corollary 2.5.14. So we have that $Q \in \mathbf{p}(L)$, then: $Q, M \subseteq U$, $Q, M \in \mathbf{p}(L)$, $Q, M \notin \mathbf{U}(L)$ and by Corollary 2.5.16, $Q = M = \varphi(M)$.

b) From $Q \parallel M$, it follows that $Q \subseteq M$ or $M \subseteq Q$. Since $Q \neq M$, and M is a minimal prime filter we must have $M \subseteq Q$. We deduce then that $M \notin \mathbf{U}(L)$ and since $M \in \mathbf{p}(L)$ it follows by Corollary 2.5.16 that $Q = U$.

c) Since $U \in \mathcal{W}$ and $P \in K(U)$, then there exists a sequence $P_1, P_2, \ldots, P_n$ of elements of $\mathbf{P}(L)$ such that $P_1 = U$, $P_n = P$, and $P_i \parallel P_{i+1}$, $P_i \neq P_{i+1}$, $1 \leq i \leq n-1$. Since $P_2 \parallel U$, $P_2 \neq U$, then by a), $P_2 = M$. Since

$M = P_2 \parallel P_3$ and $P_2 \neq P_3$, then by b) we have $P_3 = U$. Then $K(U) = \{U, \varphi(U)\}$.

□

Remark 5.9.22. *From the lemma above, if L is a non trivial finite Łukasiewicz algebra, the connected components of $\mathbf{P}(L)$ are of the form (A) $\{F(p)\} = \{\varphi(F(p))\}$ or (B) $\{F(p), \varphi(F(p))\}$, where $p \in \Pi$, is a minimal element of Π. Then the poset Π is the cardinal sum of posets Π_i, $1 \leq i \leq n$ where each Π_i is a chain with one or two elements. Notice furthermore that in case (A), since $F(p) = \varphi(F(p))$ then $\psi(p) = p$, and in case (B) $F(q) = \varphi(F(p)) \subsetneq F(p)$ if $p < q$ and $\psi(q) = p$, with both $p, q \in \Pi$.*

Notice that if L is a Łukasiewicz algebra, we have that for all $x \in L$:

$$\nabla x = \bigwedge \{b \in B(L) : x \leq b\} \ ; \ \Delta x = \bigvee \{b \in B(L) : b \leq x\}.$$

Theorem 5.9.23. *If $L, L^{'}$ are non trivial finite Łukasiewicz algebras such that their determinant systems $(\Pi = \Pi(L), \psi), (\Pi^{'} = \Pi(L^{'}), \psi^{'})$ are isomorphic Kleene spaces, then the Łukasiewicz algebras L and $L^{'}$ are isomorphic as well.*

Proof. We know that the function $H : L \to L^{'}$ defined in Theorem 5.9.2 is a Kleene algebra isomorphism from L to the Kleene algebra $L^{'}$ (see Theorems 5.9.11 and 5.9.17).

Let us prove that H verifies (*) $H(\nabla x) = \nabla H(x)$, for all $x \in L$. It is clear that if $x = 0$, (*) holds. Assume that $x \neq 0$. $H(\nabla x) = H(\bigwedge\{b : b \in B(L), x \leq b\}) = \bigwedge\{H(b) : b \in B(L), x \leq b\}$ and $\nabla H(x) = \bigwedge\{b^{'} : b^{'} \in B(L'), H(x) \leq b'\}$. Let us prove that $\{H(b) : b \in B(L), x \leq b\} = \{b^{'} : b^{'} \in B(L'), H(x) \leq b'\}$, from where (*) follows.

Let $y \in \{H(b) : b \in B(L), x \leq b\}$, so $y = H(b)$, where $b \in B(L), x \leq b$. Since H is a lattice isomorphism and $b \in B(L)$, then $y = H(b) \in B(L')$ and $H(x) \leq H(b) = y$. Conversely, if $b^{'} \in B(L')$ and $H(x) \leq b'$, since H is surjective $b^{'} = H(b)$, for some $b \in B(L)$. We have thus that $H(x) \leq H(b)$, from where it follows that $x \leq b$. □

Corollary 5.9.24. *Every non trivial finite Lukasiewicz algebra is determined, up to isomorphisms, by its determinant system.*

Recall the following definition and result (see [11], [16]) : If K is a Kleene algebra, we say that the set $B(K)$ of boolean elements of K, which is a boolean algebra, is:

- *relatively upward complete* if it verifies: If $x \in K$ then there exists $\bigwedge\{b \in B(K) : x \leq b\}$ in $B(K)$ and $\nabla x = \bigwedge\{b \in B(K) : x \leq b\}$.
- *separating* if it verifies: If $x, y \in K$ and $y \not\leq x$, then there exists $b \in B(K)$ such that $x \leq b$ and $y \not\leq b$, or there exists $b' \in B(K)$ such that $b' \leq y$ and $b' \not\leq x$.

Theorem 5.9.25. *If K is a Kleene algebra such that the set $B(K)$ of its boolean elements is relatively upward complete and separating, then there exists a unique Łukasiewicz algebra structure one K,* [11].

The operators ∇ and Δ are defined by:

$$\nabla x = \bigwedge\{b \in B(K) : x \leq b\} \ ; \ \Delta x = \bigvee\{b \in B(K) : b \leq x\}$$

for all $x \in K$.

Lemma 5.9.26. *If K is a non trivial finite Kleene algebra and its determinant system $(\Pi(K), \psi)$ is isomorphic to the Kleene space (X, α) where $X = X_1 + X_2$, (with $\alpha(X_i) = X_i$, $i = 1, 2$) and if K_i, $i = 1, 2$ is a Kleene algebra such that its determinant system $(\Pi(K_i), \psi_i)$ is isomorphic to (X_i, α), $i = 1, 2$, then K is isomorphic to $K_1 \times K_2$.*

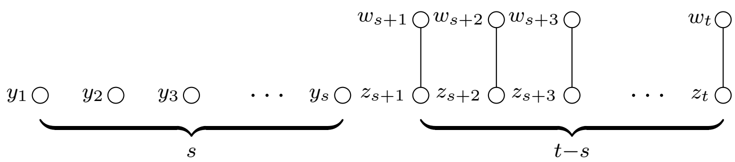
Lemma 5.9.27. *If K_1, K_2 are non trivial finite Kleene algebras, and $B(K_1)$, $B(K_2)$ are relatively upward complete and separating, then $K_1 \times K_2$ is a Kleene algebra and $B(K_1 \times K_2)$ is relatively upward complete.*

Theorem 5.9.28. *If (X, α) is a non trivial finite Kleene space such that $X = \sum_{i=1}^{t} Y_i$, where*

$$Y_i = \begin{cases} \{y_i\}, \text{ and } \alpha(y_i) = y_i, & \text{if } 1 \leq i \leq s, \\ \\ \{z_i, w_i\}, \ z_i < w_i, \ \ \alpha(z_i) = w_i, \text{ and } \alpha(w_i) = z_i, & \text{if } s+1 \leq i \leq t. \end{cases}$$

then there exists a non trivial finite Łukasiewicz algebra A such that its determinant system $(\Pi(A), \psi)$ is a Kleene space isomorphic to (X, α).

Proof. It is clear that $N[X] = 2t - s$.



Let A_i, $1 \leq i \leq t$, be a Kleene algebra such that $\Pi(A_i)$ is isomorphic to Y_i, then we can let (see Remark 5.9.20):

$$A_i = \begin{cases} \{0_i, 1_i\}, \ \sim 0_i = 1_i, & \text{if } 1 \leq i \leq s, \\ \\ \{0_i, c_i, 1_i\}, \ \sim 0_i = 1_i, \ \sim c_i = c_i, & \text{if } s+1 \leq i \leq t. \end{cases}$$

Then,

$$\Pi(A_i) = \begin{cases} \{1_i\}, & \text{if } \ 1 \leq i \leq s, \\ \\ \{c_i, 1_i\}, & \text{if } s+1 \leq i \leq t, \end{cases}$$

and if h_i, $1 \leq i \leq t$ are isomorphisms from Y_i to $\Pi(A_i)$ then

- $h_i(y_i) = 1_i$, for $1 \leq i \leq s$,
- $h_i(z_i) = c_i$, for $s+1 \leq i \leq t$,
- $h_i(w_i) = 1_i$, for $s+1 \leq i \leq t$.

We now let A be the Kleene algebra $\prod_{i=1}^{t} A_i$.

$$\text{Since } B(A_i) = \begin{cases} A_i, & \text{if } 1 \leq i \leq s, \\ \\ \{0_i, 1_i\}, & \text{if } s+1 \leq i \leq t, \end{cases}$$

and both are relatively upward complete and separating sets, then by Lemma 5.9.27, $B(A)$ is a relatively upward complete and separating set as well, so by Theorem 5.9.25, there is a unique Łukasiewicz algebra structure defined on A.

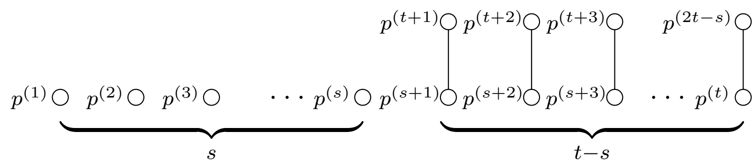
We prove now that (X, α) and $(\Pi(A), \psi)$ are isomorphic Kleene spaces. It is well known that $\Pi(A) = \{p^{(j)}\}_{j=1}^{2t-s}$, where

$$p^{(j)} = (p_1^{(j)}, \ldots, p_s^{(j)}, p_{s+1}^{(j)}, \ldots, p_t^{(j)}),\ 1 \leq j \leq 2t - s.$$

and $p_i^{(j)}$ is defined as follows:

- if $1 \leq j \leq s$, then
$$p_i^{(j)} = \begin{cases} 1_i & \text{if } i = j \\ 0_i & i \neq j, \end{cases}$$
- if $s+1 \leq j \leq t$, then
$$p_i^{(j)} = \begin{cases} c_i & \text{if } i = j \\ 0_i & i \neq j, \end{cases}$$
- if $t+1 \leq j \leq 2t - s$, then
$$p_i^{(j)} = \begin{cases} 1_i & \text{if } i = j - t + s \\ 0_i & \text{otherwise.} \end{cases}$$

Then $p^{(s+r)} < p^{(t+r)}$ for $1 \leq r \leq t - s$.



By (5.9.1) and Remark 5.9.22,

(6) $\psi(p^{(j)}) = p^{(j)},\ 1 \leq j \leq s,$
(7) $\psi(p^{(j)}) = p^{(j+t-s)},\ s+1 \leq j \leq t$, and
(8) $\psi(p^{(j)}) = p^{(j-t+s)},\ t+1 \leq j \leq 2t - s.$

If $x \in X$, we define $H : X \to \Pi(A)$ as follows:

(9) $H(y_j) = p^{(j)}$, for $1 \leq j \leq s$,
(10) $H(z_j) = p^{(j)}$, for $s+1 \leq j \leq t$, and
(11) $H(w_j) = p^{(j+t-s)}$, for $s+1 \leq j \leq t$.

It is easy to see that H is an order isomorphism from X to $\Pi(A)$.
By hypothesis, we have that
Furthermore $H(\alpha(x)) = \psi(H(x))$. Indeed:

- If $1 \leq j \leq s$, then $H(\alpha(y_j)) \overset{(1)}{=} H(y_j) \overset{(9)}{=} p^{(j)}$ and $\psi(H(y_j)) = \psi(p^{(j)}) \overset{(6)}{=} p^{(j)}$,

- If $s+1 \leq j \leq t$, then $H(\alpha(z_j)) \stackrel{(2)}{=} H(w_j) \stackrel{(11)}{=} p^{(j+t-s)}$ and $\psi(H(z_j)) = \psi(p^{(j)}) \stackrel{(8)}{=} p^{(j+t-s)}$,
- If $s+1 \leq j \leq s$, then $H(\alpha(w_j)) \stackrel{(3)}{=} H(z_j) \stackrel{(10)}{=} p^{(j)}$ and $\psi(H(w_j)) = \psi(p^{(j+t-s)}) \stackrel{(7)}{=} p^{(j)}$.

$\square$

CHAPTER 6

Geometric construction of free algebras

In the annual meeting of the U.M.A. of 1964, A. Monteiro and R. Cignoli [33] presented the following results. The details of this work have never been published before. The method used is similar to the one employed in 1961, for free De Morgan algebras, by O. Chateubriand and A. Monteiro, [10] (this work was developed during A. Monteiro's stay in the Universidad de Buenos Aires).

In 1979 R. Cignoli publishes an article [15] generalizing the results from [33].

6.1. Introduction

Let T be a non-empty set and φ an involution on T. For each $X \subseteq T$ put (see Example 1.8.1):

$$\sim X = \complement\varphi(X) \tag{6.1.1}$$

and

$$\nabla X = X \cup \varphi(X) \tag{6.1.2}$$

The operations defined above verify (see Example 1.8.1):

St1) $\nabla\emptyset = \emptyset$; St4) $X \subseteq \nabla X$

St5) $\nabla(X \cap \nabla Y) = \nabla X \cap \nabla Y$; St9) $\sim\sim X = X$

St10) $\sim(X \cap Y) = \sim X \cup \sim Y$; St11) $\sim T = \emptyset$

It is easy to verify also that:

St12) $\sim\emptyset = T$; St13) $\sim(X \cup Y) = \sim X \cap \sim Y$

St14) $\nabla T = T$; St15) $\nabla(X \cup Y) = \nabla X \cup \nabla Y$

St16) $\sim X \cup \nabla X = T$; St17) $\sim X \cap \nabla X = \sim X \cap X$

By St9), St12) and St13) it follows that the system $(2^T, \cap, \cup, \sim, T)$ is a De Morgan algebra.

From St15) it follows that

St18) If $X \subseteq Y$ then $\nabla X \subseteq \nabla Y$.

Then since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$ by St18) it follows that

$$\nabla(X \cap Y) \subseteq \nabla X \text{ and } \nabla(X \cap Y) \subseteq \nabla Y$$

therefore:

St19) $\nabla(X \cap Y) \subseteq \nabla X \cap \nabla Y$.

A. Monteiro proved (see Example 1.8.1) that the inclusion:

$$\nabla X \cap \nabla Y \subseteq \nabla(X \cap Y) \tag{6.1.15}$$

holds if and only if φ is the identity function on T. Then, the system $(2^T, \cap, \cup, \sim, \nabla, T)$ is a Łukasiewicz algebra if and only if the involution φ which defines $\sim$ and ∇ through the formulas (6.1.1) and (6.1.2) is the identity function on T. In this case $\sim$ coincides with the complement (on T) and ∇ is the identity operator on 2^T, so it is a boolean algebra, and therefore a Łukasiewicz algebra. There can exist subalgebras S of the De Morgan algebra $(2^T, \cap, \cup, \sim, T)$ such that the relation (6.1.15) holds for every pair of elements $X, Y \in S$, and in this case S is a non trivial Łukasiewicz algebra, as we saw in Example 1.8.1.

Lemma 6.1.1. *Let T be a non-empty set, φ an involution on T and $\{G_i\}_{i\in I}$ a family of subsets of T. Let $\mathcal{L}$ be a subalgebra of the De Morgan algebra $(2^T, \cap, \cup, \sim, T)$ with the operations $\sim$ and ∇ defined using the involution φ and the formulas (6.1.1) and (6.1.2), and containing the sets G_i for all $i \in I$.*

Then $\mathcal{L}$ is a Łukasiewicz algebra if and only if for every $i, j \in I$:

$$\nabla(G_i \cap \sim G_i) \cap \nabla(G_j \cap \sim G_j) \subseteq \nabla(G_i \cap \sim G_i \cap G_j \cap \sim G_j).$$

Proof. It is clear that the condition is necessary. We shall prove that it is sufficient. In order to do that, consider for each $i \in I$, the following sets:

$$G_i^1 = G_i \cap \sim G_i \quad ; \quad G_i^2 = \nabla(G_i \cap \sim G_i)$$

$$G_i^3 = \sim \nabla \sim G_i \quad ; \quad G_i^4 = \sim \nabla G_i$$

Using (6.1.1) and (6.1.2), it is easy to see that:

$$\begin{aligned} & G_i^1 = G_i \cap \complement\varphi(G_i), \\ (1)\quad & G_i^2 = (G_i \cap \complement\varphi(G_i)) \cup (\varphi(G_i) \cap \complement G_i), \\ & G_i^3 = G_i \cap \varphi(G_i), \\ & G_i^4 = \complement\varphi(G_i) \cap \complement G_i. \end{aligned}$$

Now we prove that:

$$G_i = G_i^1 \cup G_i^3 \quad (2) \quad ; \quad \sim G_i = G_i^1 \cup G_i^4 \quad (3)$$

$$\sim G_i^1 = G_i^1 \cup G_i^3 \cup G_i^4 \quad (4) \quad ; \quad \sim G_i^2 = G_i^3 \cup G_i^4 \quad (5)$$

$$\sim G_i^3 = G_i^2 \cup G_i^4 \quad (6) \quad ; \quad \sim G_i^4 = G_i^2 \cup G_i^3 \quad (7)$$

$$\nabla G_i^1 = G_i^2 \quad (8) \quad ; \quad \nabla G_i^2 = G_i^2 \quad (9)$$

$$\nabla G_i^3 = G_i^3 \quad (10) \quad ; \quad \nabla G_i^4 = G_i^4 \quad (11)$$

(2) $G_i^1 \cup G_i^3 = (G_i \cap \complement\varphi(G_i)) \cup (G_i \cap \varphi(G_i)) = G_i \cap (\complement\varphi(G_i) \cup \varphi(G_i)) = G_i \cap T = G_i$,

(3) $G_i^1 \cup G_i^4 = (G_i \cap \complement\varphi(G_i)) \cup (\complement\varphi(G_i) \cap \complement G_i) = \complement\varphi(G_i) \cap (G_i \cup \complement G_i) = \complement\varphi(G_i) \cap T = \complement\varphi(G_i) = \sim G_i$,

(4) $G_i^1 \cup G_i^3 \cup G_i^4 =$ (by (21)) $= G_i^3 \cup \sim G_i = (G_i \cap \varphi(G_i)) \cup \sim G_i = (G_i \cup \sim G_i) \cap (\varphi(G_i) \cup \sim G_i) = \sim G_i^1 \cap (\varphi(G_i) \cup \complement\varphi(G_i)) = \sim G_i^1 \cap T = \sim G_i^1$.

(5) $G_i^3 \cup G_i^4 = (G_i \cap \varphi(G_i)) \cup (\complement\varphi(G_i) \cap \complement G_i) = (G_i \cup \complement\varphi(G_i)) \cap (\varphi(G_i) \cup \complement G_i)$.
$\sim G_i^2 = \complement\varphi((G_i \cap \complement\varphi(G_i)) \cup (\varphi(G_i) \cap \complement G_i)) = (\complement\varphi(G_i) \cup G_i) \cap (\complement G_i \cup \varphi(G_i))$.

(6) $G_i^2 \cup G_i^4 = (G_i \cap \complement\varphi(G_i)) \cup (\varphi(G_i) \cap \complement G_i) \cup (\complement\varphi(G_i) \cap \complement G_i) =$
$(\complement\varphi(G_i) \cap T) \cup (\varphi(G_i) \cap \complement G_i) = \complement\varphi(G_i) \cup \complement G_i$.
$\sim G_i^3 = \complement\varphi(G_i \cap \varphi(G_i)) = \complement\varphi(G_i) \cup \complement G_i$.

(7) $G_i^2 \cup G_i^3 = (G_i \cap \complement\varphi(G_i)) \cup (\varphi(G_i) \cap \complement G_i) \cup (G_i \cap \varphi(G_i)) =$
$(G_i \cap (\complement\varphi(G_i) \cup \varphi(G_i))) \cup (\varphi(G_i) \cap \complement G_i) = (G_i \cap T) \cup (\varphi(G_i) \cap \complement G_i) =$
$G_i \cup (\varphi(G_i) \cap \complement G_i) = G_i \cup \varphi(G_i)$.
$\sim G_i^4 = \complement\varphi(\complement\varphi(G_i) \cap \complement G_i) = G_i \cup \varphi(G_i)$.

(8) $\nabla G_i^1 = G_i^1 \cup \varphi(G_i^1) = (G_i \cap \complement\varphi(G_i)) \cup (\varphi(G_i) \cap \complement G_i) =$ (by (1)) $= G_i^2$.

(9) $\nabla G_i^2 = G_i^2 \cup \varphi(G_i^2) =$ (by (1)) $= G_i^2 \cup (\varphi(G_i) \cap \complement G_i) \cup (G_i \cap \complement\varphi(G_i)) = G_i^2 \cup G_i^2 = G_i^2$.

(10) $\nabla G_i^3 = G_i^3 \cup \varphi(G_i^3) = (G_i \cap \varphi(G_i)) \cup (\varphi(G_i) \cap G_i) = G_i \cap \varphi(G_i) = G_i^3$.

(11) $\nabla G_i^4 = G_i^4 \cup \varphi(G_i^4) = (\complement\varphi(G_i) \cap \complement G_i) \cup (\complement G_i \cap \complement\varphi(G_i)) = \complement G_i \cap \complement\varphi(G_i) = G_i^4$.

With the sets G_i^p, $i \in I$, $1 \le p \le 4$ we define the family of subsets C_{ij} of T, in the following manner:

$$C_{ij} = (\varepsilon_j^1 \cap G_i^1) \cup (\varepsilon_j^2 \cap G_i^2) \cup (\varepsilon_j^3 \cap G_i^3) \cup (\varepsilon_j^4 \cap G_i^4)$$

where ε_j^p is either $\emptyset$ or T. Then for each $i \in I$ we have at most 2^4 different sets C_{ij}. Moisil ([25], p. 441) proved that for each $i \in I$ there exist exactly 12 different sets C_{ij}, so $1 \le j < 2^4$.

Let $\mathcal{L}$ be the sublattice of the distributive lattice $(2^T, \cap, \cup)$ generated by the elements C_{ij}. We shall prove that the system $(\mathcal{L}, T, \sim, \nabla, \cap, \cup)$ is a Łukasiewicz algebra of subsets of T (determined by φ) containing all the G_i, $i \in I$.

(i) $G_i^p \in \mathcal{L}$, for $1 \le p \le 4$.
Indeed:
(12) If $\varepsilon_j^1 = T$ and $\varepsilon_j^2 = \varepsilon_j^3 = \varepsilon_j^4 = \emptyset$ then $C_{ij} = G_i^1 \in \mathcal{L}$.
(13) If $\varepsilon_j^2 = T$ and $\varepsilon_j^1 = \varepsilon_j^3 = \varepsilon_j^4 = \emptyset$ then $C_{ij} = G_i^2 \in \mathcal{L}$.
(14) If $\varepsilon_j^3 = T$ and $\varepsilon_j^1 = \varepsilon_j^2 = \varepsilon_j^4 = \emptyset$ then $C_{ij} = G_i^3 \in \mathcal{L}$.
(15) If $\varepsilon_j^4 = T$ and $\varepsilon_j^1 = \varepsilon_j^2 = \varepsilon_j^3 = \emptyset$ then $C_{ij} = G_i^4 \in \mathcal{L}$.

(ii) $G_i, \sim G_i \in \mathcal{L}$.
By (12), (14) and (2) we have that $G_i = G_i^1 \cup G_i^3 \in \mathcal{L}$.
By (12), (15) and (3) we can also claim that $\sim G_i = G_i^1 \cup G_i^4 \in \mathcal{L}$.

(iii) $\sim G_i^p \in \mathcal{L}$, for $1 \le p \le 4$.
This is an immediate consequence of (i), (4), (5), (6) and (7).

(iv) $\emptyset, T \in \mathcal{L}$.
If $\varepsilon_j^1 = \varepsilon_j^2 = \varepsilon_j^3 = \varepsilon_j^4 = \emptyset$ then $\emptyset = C_{ij} \in \mathcal{L}$.
By (8) $\nabla G_i^1 = G_i^2$, so by St16) $T = \nabla G_i^1 \cup \sim G_i^1 = G_i^2 \cup \sim G_i^1$ and by (i) and (iii), $G_i^2 \cup \sim G_i^1 \in \mathcal{L}$, this is $T \in \mathcal{L}$.

(v) $\sim C_{ij} \in \mathcal{L}$.

Since $C_{ij} = \bigcup_{p=1}^{4} (\varepsilon_j^p \cap G_i^p)$ from St10) and St13) it follows that:

$$(16) \qquad \sim C_{ij} = \bigcap_{p=1}^{4} (\sim \varepsilon_j^p \cup \sim G_i^p).$$

By (iii), we know that $\sim G_i^p \in \mathcal{L}$, for $1 \leq p \leq 4$, then for each $i \in I$ and each $p, 1 \leq p \leq 4$ there exists an index j_p such that:

$$\sim G_i^p = C_{i\,j_p}.$$

Therefore, by (16) we have:

$$(17) \qquad \sim C_{ij} = \bigcap_{p=1}^{4} (\sim \varepsilon_j^p \cup C_{i\,j_p}).$$

From St11), St12) and the definition of ε_j^p, we have that $\sim \varepsilon_j^p \cup C_{i\,j_p}$ is equal to T or $C_{i\,j_p}$, and since $T, C_{i\,j_p} \in \mathcal{L}$ from (17) it follows that $\sim C_{ij}$ is the intersection of a finite number of elements of $\mathcal{L}$, so $\sim C_{ij} \in \mathcal{L}$.

(vi) If $X \in \mathcal{L}$ then $\sim X \in \mathcal{L}$.

If $X \in \{\emptyset, T\}$ then by St11) and St12) $\sim X \in \{\emptyset, T\}$.

It is well known (see for instance [8]) that every $X \in \mathcal{L} \setminus \{\emptyset, T\}$ is of the form:

$$X = \bigcup_{r=1}^{m} \bigcap_{s=1}^{n_r} C_{i(r,s)j(r,s)}$$

so, by St10) and St13) we have that:

$$\sim X = \bigcap_{r=1}^{m} \bigcup_{s=1}^{n_r} \sim C_{i(r,s)j(r,s)}$$

from where it follows using (v) that $\sim X \in \mathcal{L}$.

(vii) The system $(\mathcal{L}, T, \cap, \cup, \sim)$ is a De Morgan algebra of subsets of T.

By construction $(\mathcal{L}, \cap, \cup)$ is a distributive lattice of subsets of T and by (v) $\mathcal{L}$ is closed under the operation $\sim$, so by (6.1.1), $T \in \mathcal{L}$ and since St9) and St13) hold, (vii) follows.

(viii) $\nabla C_{ij} \in \mathcal{L}$.

Since $C_{ij} = \bigcup_{p=1}^{4} (\varepsilon_j^p \cap G_i^p)$, by St15) we have that

$$(18) \qquad \nabla C_{ij} = \bigcup_{p=1}^{4} \nabla(\varepsilon_j^p \cap G_i^p).$$

By St1) and St14) $\varepsilon_j^p = \nabla \varepsilon_j^p$ so by St5) we have that:

$$(19) \qquad \nabla(\varepsilon_j^p \cap G_i^p) = \nabla \varepsilon_j^p \cap \nabla G_i^p = \varepsilon_j^p \cap \nabla G_i^p.$$

Then from (18) and (19) we have that:

$$\nabla C_{ij} = (\varepsilon_j^1 \cap \nabla G_i^1) \cup (\varepsilon_j^2 \cap \nabla G_i^2) \cup (\varepsilon_j^3 \cap \nabla G_i^3) \cup (\varepsilon_j^4 \cap \nabla G_i^4)$$

and by (8) to (11) it follows that:

$$\nabla C_{ij} = (\varepsilon_j^1 \cap G_i^2) \cup (\varepsilon_j^2 \cap G_i^2) \cup (\varepsilon_j^3 \cap G_i^3) \cup (\varepsilon_j^4 \cap G_i^4)$$

this is

$$\nabla C_{ij} = (\emptyset \cap G_i^1) \cup ((\varepsilon_j^1 \cup \varepsilon_j^2) \cap G_i^2) \cup (\varepsilon_j^3 \cap G_i^3) \cup (\varepsilon_j^4 \cap G_i^4)$$

and therefore $\nabla C_{ij} \in \mathcal{L}$.

(ix) $\nabla(G_i^p \cap G_j^q) = \nabla G_i^p \cap \nabla G_j^q$, $i, j \in I, 1 \leq p, q \leq 4$.

If $2 \leq q \leq 4$, then by (9) to (11) we have that $\nabla G_i^q = G_i^q$ and therefore using St5) we have

$$\nabla(G_i^p \cap G_j^q) = \nabla(G_i^p \cap \nabla G_j^q) = \nabla G_i^p \cap \nabla G_j^q,\ 2 \leq p, q \leq 4. \tag{20}$$

Interchanging G_i^p and G_j^q we have that (20) holds for all $i, j \in I$ and p, q not simultaneously equal to 1.

By St19) we know that:

$$\nabla(G_i^1 \cap G_j^1) \subseteq \nabla G_i^1 \cap \nabla G_j^1.$$

On the other hand, using the hypothesis of the lemma

$$\nabla G_i^1 \cap \nabla G_j^1 = \nabla(G_i \cap \sim G_i) \cap \nabla(G_j \cap \sim G_j) \subseteq$$

$$\nabla(G_i \cap \sim G_i \cap G_j \cap \sim G_j) = \nabla(G_i^1 \cap \sim G_j^1),$$

so (20) also holds for $p = q = 1$.

(x) $\nabla(C_{ij} \cap C_{kl}) = \nabla C_{ij} \cap \nabla C_{kl}$.

Let $C_{ij} = \bigcup\limits_{p=1}^{4} (\varepsilon_j^p \cap G_i^p)$ and $C_{kl} = \bigcup\limits_{q=1}^{4} (\varepsilon_l^q \cap G_k^q)$ then we have

$$C_{ij} \cap C_{kl} = \bigcup_{p,q=1}^{4} (\varepsilon_{jl}^{pq} \cap G_i^p \cap G_k^q), \text{where } \varepsilon_{jl}^{pq} = \varepsilon_j^p \cap \varepsilon_l^q.$$

Since $\nabla \varepsilon_{jl}^{pq} = \varepsilon_{jl}^{pq}$ then using St5), St15) and (ix) we have

$$\nabla(C_{ij} \cap C_{kl}) = \bigcup_{p,q=1}^{4} (\varepsilon_{jl}^{pq} \cap \nabla(G_i^p \cap G_k^q)) = \bigcup_{p,q=1}^{4} (\varepsilon_{jl}^{pq} \cap \nabla G_i^p \cap \nabla G_k^q) =$$

$$\bigcup_{p=1}^{4} (\varepsilon_j^p \cap \nabla G_i^p) \cap \bigcup_{q=1}^{4} (\varepsilon_l^q \cap \nabla G_k^q) = \nabla C_{ij} \cap \nabla C_{kl}.$$

(xi) If $X \in \mathcal{L}$ then $\nabla X \in \mathcal{L}$.

If $X \in \{\emptyset, T\}$ then by St1) and St14) we have that $\nabla X \in \{\emptyset, T\}$. If $X \in \mathcal{L} \setminus \{\emptyset, T\}$ then $X = \bigcup\limits_{r=1}^{m} \bigcap\limits_{s=1}^{n_r} C_{i(r,s)j(r,s)}$ so by St15) and (x),

$$\nabla X = \bigcup_{r=1}^{m} \bigcap_{s=1}^{n_r} \nabla C_{i(r,s)j(r,s)}$$

then using (viii) we have $\nabla X \in \mathcal{L}$.

(xii) The system $(\mathcal{L}, T, \sim, \nabla, \cap, \cup)$ is a Łukasiewicz algebra.

By (vii) we have that the system $(\mathcal{L}, T, \sim, \cap, \cup)$ is a De Morgan algebra and by (xi) $\mathcal{L}$ is closed under the operation ∇. Axioms L6) and L7) are a consequence of St5) and St16).

To prove L8) we proceed as follows: if X, $Y \in \mathcal{L}$, suppose first that $X \in \{\emptyset, T\}$. Then L8) follows immediately from St1) and St14).

If X, $Y \in \mathcal{L} \setminus \{\emptyset, T\}$, then we proceed as in the proof of part (xi), using the identity we proved in part (x) for the sets C_{ij}.

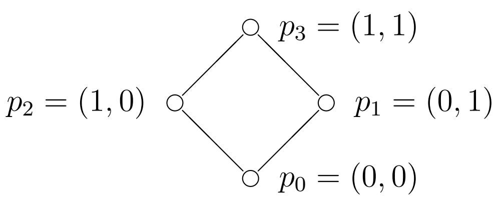
Items (ii) and (xii) prove the lemma. $\square$

Remark 6.1.2. *We shall prove that in the setting of the previous lemma, we also have that* $\mathcal{L} = LS(\mathcal{G})$.

If $\mathcal{L}'$ *is a Łukasiewicz algebra of subsets of* T *determined by* φ *such that* (1) $G_i \in \mathcal{L}'$ *for all* $i \in I$, *then* (2) $\mathcal{L} \subseteq \mathcal{L}'$. *Indeed, by* (1) *it follows that* $G_i^p \in \mathcal{L}'$, $1 \leq p \leq 4$ *and therefore* (3) $C_{ij} \in \mathcal{L}'$. *Thus if* $X \in \mathcal{L}$ *and* $X \in \{\emptyset, T\}$ *then* $X \in \mathcal{L}'$. *If* $X \in \mathcal{L} \setminus \{\emptyset, T\}$, *we have that* (4) $X = \bigcup_{r=1}^{m} \bigcap_{s=1}^{n_r} C_{i(r,s)j(r,s)}$. *From* (3) *and* (4) *it follows that* $X \in \mathcal{L}'$, *which proves* (2). *So* $\mathcal{L}$ *is the least subalgebra containing* $\mathcal{G} = \{G_i\}_{i \in I}$ *and therefore* $\mathcal{L} = LS(\mathcal{G})$.

6.2. Geometric construction of the free Łukasiewicz algebras

Given the poset $D = \{0, 1\}$, where $0 < 1$, let $B = \{0, 1\} \times \{0, 1\}$, [10], so the poset B has the following diagram:



Let $\psi : B \to B$ be defined by the following table:

x	p_0	p_1	p_2	p_3
$\psi(x)$	p_3	p_1	p_2	p_0

Observe that if $b = (x, y) \in B$ then $\psi(b) = \psi((x, y)) = (1 - y, 1 - x)$.

Let I be a set of cardinality $\mathbf{c}$ ($\mathbf{c} \neq 0$) and consider the set $E = \prod_{i \in I} B_i$, where $B_i = B$ for all $i \in I$.

Let E' be the set of all the elements $(x_i)_{i \in I} \in E$ that do not have simultaneously coordinates with the values p_0 and p_3.

If $(x_i)_{i \in I} \in E'$ we define a function $\varphi : E' \to E'$ as follows:

$$\varphi(x) = (\psi(x_i))_{i \in I},$$

where ψ is the involution on B indicated before. Clearly $\varphi(x) \in E'$ and φ is an involution on E'. Then by the previous section, the formulas (6.1.1) and (6.1.2) let us define the operations $\sim X$ and ∇X for every subset X of E'.

For each $i \in I$, let

$$G_i = \{x = (x_j)_{j \in I} \in E' : x_i \in \{p_2, p_3\}\}, \tag{6.2.1}$$

and $\mathcal{G} = \{G_i\}_{i \in I}$.

Consider the following element of E': $e = (e_i)_{i \in I}$ where $e_i = p_3$ for all $i \in I$, so $e \in G_i$ for all $i \in I$, and therefore all the sets G_i are non-empty. Analogously, the element $f = (f_i)_{i \in I}$ where $f_i = p_2$ for all $i \in I$, of E' belongs to every $G_i, i \in I$.

Let $x^{(j)} = (x_i^{(j)})_{i\in I}$ be such that $x_j^{(j)} = p_1$ and $x_i^{(j)} = p_3$ for all $i \neq j$, so $x^{(j)} \notin G_j$ and therefore $x^{(j)} \in G_i$ for all $i \neq j$. This proves that $\mathcal{G}$ and I have the same cardinality.

Remark 6.2.1. *From the definition of φ we have that $x \in G_i$, this is $x_i \in \{p_2, p_3\}$ if and only if $\psi(x_i) \in \{\psi(p_2), \psi(p_3)\} = \{p_2, p_0\}$, so:*
$\varphi(G_i) = \{x \in E' : x_i \in \{p_2, p_0\}\}$ and $\sim G_i = \complement\varphi(G_i) = \{x \in E' : x_i \in \{p_1, p_3\}\}$.

We define, as in Lemma 6.1.1, $G_i^2 = \nabla(G_i \cap \sim G_i)$. We also saw in that lemma that (1) $G_i^2 = (G_i \cap \complement\varphi(G_i)) \cup (\varphi(G_i) \cap \complement G_i)$. Let $x \in X = \nabla(G_i \cap \sim G_i) \cap \nabla(G_j \cap \sim G_j) = G_i^2 \cap G_j^2$ so by (1) we have that:
(2) $x \in (G_i \cap \sim G_i) \cup (\varphi(G_i) \cap \complement G_i)$ and (3) $x \in (G_j \cap \sim G_j) \cup (\varphi(G_j) \cap \complement G_j)$.
From (2) it follows that (4) $x \in G_i \cap \sim G_i$ or (5) $x \in \varphi(G_i) \cap \complement G_i$.
From (4) it follows that $x_i \in \{p_2, p_3\}$ and by Remark 6.2.1 $x_i \in \{p_1, p_3\}$, so $x_i = p_3$.
From (5) it follows by Remark 6.2.1 that $x_i \in \{p_2, p_0\}$ and since $x \in \complement G_i$ we have that $x_i \in \{p_0, p_1\}$, so $x_i = p_0$.
Analogously from (3) it follows that $x_j = p_3$ or $x_j = p_0$.
Therefore if $x \in X$ we have that (6) $x_i = p_3$ or (7) $x_i = p_0$ and (8) $x_j = p_3$ or (9) $x_j = p_0$. Since $x \in E'$, (6) and (9) cannot hold simultaneously, and the same is true for (7) and (8).
Then if $x \in X$ and (6) and (8) hold, then $x_i = x_j = p_3$, and if (7) and (9) hold, then $x_i = x_j = p_0$.
Therefore if $x \in X$ we have that $x_i = x_j = p_0$ or $x_i = x_j = p_3$.
Now let

$$Y = \nabla(G_i \cap \sim G_i \cap G_j \cap \sim G_j) =$$

$$= (G_i \cap \sim G_i \cap G_j \cap \sim G_j) \cup (\varphi(G_i) \cap \complement G_i \cap \varphi(G_j) \cap \complement G_j)$$

then $y \in Y$ is equivalent to

$$y \in G_i \cap \sim G_i \cap G_j \cap \sim G_j \text{ or } y \in \varphi(G_i) \cap \complement G_i \cap \varphi(G_j) \cap \complement G_j$$

and by Remark 6.2.1, this is equivalent to:

$$y_i \in \{p_2, p_3\} \cap \{p_1, p_3\} \text{ and } y_j \in \{p_2, p_3\} \cap \{p_1, p_3\}$$

or

$$y_i \in \{p_2, p_0\} \cap \{p_0, p_3\} \text{ and } y_j \in \{p_2, p_0\} \cap \{p_0, p_3\}$$

this is $y_i = y_j = p_3$ or $y_i = y_j = p_0$, so

$$\nabla(G_i \cap \sim G_i) \cap \nabla(G_j \cap \sim G_j) \subseteq \nabla(G_i \cap \sim G_i \cap G_j \cap \sim G_j).$$

Then, by Lemma 6.1.1, there exists a Łukasiewicz algebra $\mathcal{L}$ of subsets of E', determined by φ, which contains all the sets $G_i, i \in I$.
By Remark 6.1.2, $\mathcal{L} = LS(\mathcal{G})$, where $\mathcal{G} = \{G_i\}_{i\in I}$.

Theorem 6.2.2. *$\mathcal{L}$ is a Łukasiewicz algebra with a set of free generators of cardinality* $\mathbf{c}$.

Proof. We need to prove that given an arbitrary Łukasiewicz algebra A and a function $f : \mathcal{G} \to A$, f can be extended to a homomorphism $h : \mathcal{L} \to A$.

If A has a single element, $A = \{0\}$, then $f(G_i) = 0$ for all $i \in I$ and therefore the function $h(X) = 0$ for all $X \in \mathcal{L}$ is a homomorphism extending f.

If A has more than one element, we know (see sections 5.6 and 5.9) that A is isomorphic to a Łukasiewicz algebra $\mathcal{A}$, which is a subalgebra of subsets of a certain set T, determined by an involution α on T, where $\approx X = \complement_T \alpha(X)$ and $\nabla X = X \cup \alpha(X)$, for all $X \subseteq T$.

Let $f : \mathcal{G} \to \mathcal{A}$. Then for each $i \in I$ we write $f(G_i) = H_i$, where $H_i \in \mathcal{A}$ this is $H_i \subseteq T$.

Given $X \subseteq T$, let $K_X : T \to D = \{0, 1\}$ be the function defined by:

$$K_X(t) = \begin{cases} 1, & \text{if } t \in X \\ \\ 0, & \text{if } t \notin X. \end{cases}$$

Notice that (1) $\underline{K_{\complement_T X}(t) = 1 - K_X(t).}$

Consider now the function $K_i : T \to B$ defined by:

$$K_i(t) = (K_{H_i}(t), K_{\approx H_i}(t)),$$

so by the definition of $\approx$ and (1) we have that

$$(2) \qquad K_i(t) = (K_{H_i}(t), K_{\complement_T \alpha(H_i)}(t)) = (K_{H_i}(t), 1 - K_{\alpha(H_i)}(t)).$$

Let us prove that (3) $\underline{K_{H_i}(\alpha(t)) = K_{\alpha(H_i)}(t).}$

Indeed, $K_{H_i}(\alpha(t)) = 1 \iff \alpha(t) \in H_i \iff t = \alpha(\alpha(t)) \in \alpha(H_i) \iff K_{\alpha(H_i)}(t) = 1$.

From (3) it follows that replacing t by $\alpha(t)$ we get: (4) $\underline{K_{H_i}(t) = K_{\alpha(H_i)}(\alpha(t)).}$

As in [10] let us define $K : T \to E'$ by:

$$K(t) = (K_i(t))_{i \in I}.$$

Notice that:

$$(K_i(t))_{i\in I} = K(t) \in G_i \iff K_i(t) \in \{(1, 0), (1, 1)\} \iff$$
$$(K_{H_i}(t), 1 - K_{\alpha(H_i)}(t)) = (1, 0) \text{ or } (K_{H_i}(t), 1 - K_{\alpha(H_i)}(t)) = (1, 1).$$

Then:

$$K_{H_i}(t) = 1 \text{ and } 1 - K_{\alpha(H_i)}(t) = 0$$

or

$$K_{H_i}(t) = 1 \text{ and } 1 - K_{\alpha(H_i)}(t) = 1$$

and therefore

$$(5) \qquad K(t) \in G_i \iff K_{H_i}(t) = 1.$$

It is clear that K is a function from T to E. To prove that K is a function from T to E' we need to prove that for each $t \in T$, the element $K(t) = (K_i(t))_{i\in I}$ does not have coordinates with values p_0 and p_3 simultaneously. Assume that there exists $t \in T$ and two elements $i, j \in T$ such that $K_i(t) = p_0$ and $K_j(t) = p_3$, then by the definition of the functions K_i we have that:

$$(6) \qquad t \notin H_i \text{ and } t \notin \alpha(H_i)$$

$$(7) \qquad t \in H_j \text{ and } t \notin \alpha(H_j).$$

But the conditions (6) and (7) are contradictory. Indeed, from (7) it follows that $t \in H_j$ and $t \in \approx H_j$, so $t \in H_j \cap \approx H_j$ and since every Łukasiewicz algebra is a Kleene algebra, we have that $H_j \cap \approx H_j \subseteq H_i \cup \approx H_i$ and therefore $t \in H_i \cup \approx H_i$, which contradicts (6).

Since K is a function from T to E', if we define $h : 2^{E'} \to 2^T$ by $h(X) = K^{-1}(X)$ for all $X \subseteq E'$, then for all $X, Y \subseteq E'$ the following hold:

$$(8) \qquad h(X \cap Y) = h(X) \cap h(Y),$$

$$(9) \qquad h(X \cup Y) = h(X) \cup h(Y),$$

$$(10) \qquad h(E') = T,$$

$$(11) \qquad h(\complement_{E'} X) = K^{-1}(\complement_{E'} X) = \complement_T K^{-1}(X) = \complement_T h(X).$$

Thus h is a De Morgan homomorphism from $2^{E'}$ to 2^T if and only if

$$h(\sim X) = \approx h(X), \text{ for every } X \subseteq E'.$$

This is, if and only if, for every $X \subseteq E'$,

$$h(\complement_{E'} \varphi(X)) = K^{-1}(\complement_{E'} \varphi(X)) = \complement_T \alpha(K^{-1}(X)) = \complement_T \alpha(h(X)),$$

so by (11) this is equivalent to prove that

$$\complement_T K^{-1}(\varphi(X)) = \complement_T \alpha(K^{-1}(X)), \text{ for every } X \subseteq E'$$

this is

$$(12) \qquad K^{-1}(\varphi(X)) = \alpha(K^{-1}(X)), \text{ for every } X \subseteq E'.$$

Let us prove now that (12) holds. Indeed $t \in K^{-1}(\varphi(X)) \iff K(t) \in \varphi(X) \iff$ (13) $\varphi(K(t)) \in \varphi(\varphi(X)) = X$. But since $\varphi(K(t)) = \varphi((K_i(t))_{i\in I}) = (\psi(K_i(t)))_{i\in I}$ then (13) is equivalent to (14) $(\psi(K_i(t)))_{i\in I} \in X$.

As by (2) $K_i(t) = (K_{H_i}(t), 1 - K_{\alpha(H_i)}(t))$, then

$$\psi(K_i(t)) = (1 - (1 - K_{\alpha(H_i)}(t)), 1 - K_{H_i}(t)) = (K_{\alpha(H_i)}(t), 1 - K_{H_i}(t))$$

and therefore using (3), (4) and (2) we have that

$$\psi(K_i(t)) = (K_{H_i}(\alpha(t)), 1 - K_{\alpha(H_i)}(\alpha(t))) = K_i(\alpha(t)).$$

So (14) is equivalent to $K(\alpha(t)) = (K_i(\alpha(t)))_{i\in I} \in X$ which in turn is equivalent to $\alpha(t) \in K^{-1}(X)$ this is $t \in \alpha(K^{-1}(X))$.

We have proved (12), which gives us

$$(15) \qquad h(\varphi(X)) = \alpha(h(X)), \text{ for every } X \subseteq E'.$$

From (11) and (15) we deduce

$$(16) \qquad h(\sim X) = h(\complement_{E'} \varphi(X)) = \complement_T h(\varphi(X)) = \complement_T \alpha(h(X)) = \approx h(X).$$

Thus we have proved that (i) <u>h is a De Morgan homomorphism from $2^{E'}$ to 2^T.</u>

Furthermore, from the definition of ∇, (9), (15) and the definition of $\boldsymbol{\nabla}$ we have that:

$$(17) \qquad h(\nabla X) = h(X \cup \varphi(X)) = h(X) \cup h(\varphi(X)) = h(X) \cup \alpha(h(X)) = \boldsymbol{\nabla} h(X).$$

From (8), (9), (10), (16) and (17) it follows that:

<u>(ii) h is a homomorphism from $\mathcal{L}$ to $\mathcal{A}$.</u>

Let us prove now that (iii) h extends f, this is, that

$$K^{-1}(G_i) = h(G_i) = f(G_i) = H_i, \text{ for every } G_i \in \mathcal{G}.$$

Indeed $t \in H_i \iff K_{H_i} = 1$ so by (5) this is equivalent to $K(t) \in G_i$, which is equivalent to $t \in K^{-1}(G_i) = h(G_i)$.

Finally, let us prove that (iv) $h(\mathcal{L}) \subseteq \mathcal{A}$. By (ii), $h(\mathcal{L})$ is a subalgebra of the algebra 2^T and since $LS(\mathcal{G}) = \mathcal{L}$, then $h(\mathcal{G})$ generates $h(\mathcal{L})$, this is $LS(h(\mathcal{G})) = h(\mathcal{L})$. Since $h(G_i) = f(G_i) = H_i \in \mathcal{A}$, for all $i \in I$ then $h(\mathcal{G}) \subseteq \mathcal{A}$ so $h(\mathcal{L}) = LS(\mathcal{G}) \subseteq \mathcal{A}$, which concludes the proof. $\square$

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